A. Conditional Expectation. Let \( B(t), t > 0 \) represent Brownian motion, with \( B(0) = 0 \). Compute

1. \( E[B((t)|(B(s))^3 \quad (s < t)]. \)
2. \( E\left[\int_0^T B(t)dt|B(u), u \leq s\right] \quad (s < t < T). \)
3. \( E[sin(B(t))|B(s)] \quad (s < t). \)
4. \( E[B(t)|\min \{B(s), s \leq t\} > -1 ] \).

B. Brownian Scale Invariance. Let \( B(t) \) be Brownian motion.

1. If \( \lambda > 0 \) then \( X_{\lambda}(t) = \lambda B(t/\lambda^2) \) is a Brownian motion.
2. \( X(t) = tB(1/t), \ t > 0 \) is a Brownian motion. Verify that \( X(0) = 0 \). (“Time inversion.”)

C. AR processes (i) Let \( X_n \) be a discrete time AR(1) process, such that

\[ X_{n+1} = \frac{1}{2}X_n + \frac{1}{8}\nu_{n+1} \]

where \( \nu_k \) are independent, identically distributed \( N(0,1) \). Show that \( X_n \) has a long-term, or equilibrium distribution for \( n \to \infty \). Also, compute the correlation between \( X_n \) and \( X_{m+n} \) as \( n \to \infty \). (ii) Consider the AR(2) process

\[ X_{n+1} = \frac{1}{2}X_n + \frac{1}{3}X_{n-1} + \frac{1}{8}\nu_{n+1} \]

Does \( X_n \) admit an equilibrium distribution for \( n \gg 1 \)? If so, compute it. [Hint: write the AR(2) equation as a “vector AR(1)” equation introducing the vector variable \( Y_n = (X_n, X_{n-1}) \). Solve the latter equation iteratively.]

D. Wiener’s construction of Brownian motion

Consider the following family of functions

\[ X_N(t) = \frac{\nu_0t}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{0}^{N} \frac{1}{n} (\nu_n(1 - \cos nt) + \nu'_n \sin nt) \]  
(1)

where \( \nu_k, \nu'_k \) are i.i.d. \( N(0,1) \). Show that \( X_N(t) \) is a Gaussian process and calculate \( E(X_N(t)), E(X_N(t)X_N(s)) \). Take the limit as \( N \to \infty \) of the covariance and argue, heuristically, that
\[ X(t) = \frac{\nu_0 t}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_0^\infty \frac{1}{n} (\nu_n (1 - \cos nt) + \nu'_n \sin nt) \]  

represents a Brownian path. This construction was proposed by N. Wiener in 1923.

**E. Exponential Martingale.** Let \( B(t) \) be a Brownian motion. Show that

\[ M_\sigma(t) = e^{\sigma B(t) - \frac{\sigma^2 t}{2}} \]  

is a martingale. Deduce that

\[ X^{(m)}(t) = \left[ \frac{d^m}{d\sigma^m} \right]_{\sigma=0} M_\sigma(t) \]  

is a martingale for all \( m \geq 1 \). Compute explicitly \( X^{(m)}(t) \) for \( m = 1, 2, 3 \).