Let $\mathbf{X} = (X_1(t), \ldots, X_m(t))$ be $m$ independent Brownian motions $m(\theta, t)$. Let $\{\sigma_{i,k}(t)\}_{i=1,\ldots,n, \ k=1,\ldots,m}$ be non-anticipative for and let $\mu_i(t)$ $i=1,\ldots,n$ be non-anticipative with respect to $(X_1,\ldots,X_m)$.

Consider the process

$$Y_i(t) = Y_{i,0} \int_0^t \sigma_{i,k}(s) \, dX_k(s) + \int_0^t \mu_i(s) \, ds.$$

or,

$$dY_i(t) = \sum_{k=1}^m \sigma_{i,k}(t) \, dX_k(t) + \mu_i(t) \, dt \quad i=1,\ldots,n.$$

This process is known as an Itô process.
Example: Let $\sigma_i(t)$ be given functions (processes) and $R_{ij}(t)$ be a process representing the correlation function of the returns of $n$ stock.

$$E[r_i(t) r_j(t)] = \sigma_i(t) \sigma_j(t) R_{ij}(t) dt$$

Assume that $R_{ij}$ has $m$ non-zero eigenvalues

$$R_{ij} = [V_{ik}] [\lambda_{i1} \ldots \lambda_{im}] [V^T]$$

$$r_i(t) = \sigma_i(t) \left[ \sum_{k=1}^{m} V_{ik}(t) dX_k(t) \right] + \mu_i(t) dt$$

$$\frac{dS_i(t)}{S_i(t)} = \sigma_i(t) \left[ \sum_{k=1}^{m} V_{ik}(t) dX_k(t) \right] + \mu_i(t) dt$$

$$r(t) = \frac{S_t(t+\delta t) - S_t(t)}{S_t(t)}$$
\[ dS_i(t) = S_i(t) \cdot \sigma_i(t) \left( \sum_{n=1}^{m}N_n(t) dX_n(t) \right) + S_i(t) \mu_i(t) dt \]

Thus, It\'s processes provide a natural class of models for representing the evolution of a group of stocks, for example.

The parameters \( \sigma_i(t) \), \( \mu_i(t) \), and \( \Delta \sigma_i(t) / \sigma_i(t) \) can be estimated from data.

Example 2: Stock return are conditionally normal, with stochastic volatility

\[ \mu(t) = \sigma(t) \cdot dX(t) \]

\[ \Delta \sigma(t) / \sigma(t) = \kappa(t) \cdot dW(t) \]
\[
\frac{dS_t}{S_t} = \sigma_t \ dX_t
\]
\[
\frac{d\sigma_t}{\sigma_t} = k_t \ dW_t
\]
\[
E(dX_t \ dW_t) = \rho \ dt
\]

**Example:** Let \( S_t = S&P \ 500 \) index, \( \sigma_t = \) volatility of \( S&P \ 500 \) index.

![Graph of S&P 500 index and VIX implied volatility over time.](image)
Take VIX as proxy for $\sigma_t$.

\[
\begin{align*}
\frac{\Delta S_t}{S_t} &\sim N(0, \sigma_t^2) \\
\frac{\Delta \sigma_t}{\sigma_t} &\sim N(0, \kappa^2)
\end{align*}
\]

\[
\frac{\Delta \sigma}{\sigma} = \beta \frac{\Delta S}{S} + \varepsilon
\]

\[
\frac{dS}{S} = \sigma \cdot dW_t
\]

\[
\frac{d\sigma}{\sigma} = \beta \frac{dS}{S} + \kappa \cdot dW_2
\]

\[
\begin{cases}
\frac{dS_t}{S_t} = \sigma_t \cdot dW_{1,t} \\
\frac{d\sigma_t}{\sigma_t} = \beta_t \cdot \sigma_t \cdot dW_{1,t} + \kappa_2 \cdot dW_2
\end{cases}
\]

\[
\beta_t = \frac{\kappa \varphi}{\sigma}
\]

\[
\begin{cases}
\frac{dS_t}{S_t} = \sigma_t \cdot dW_{1,t} \\
\frac{d\sigma_t}{\sigma_t} = \kappa \left( p \cdot dW_{1,t} + \sqrt{1-p^2} \cdot dW_{2,t} \right)
\end{cases}
\]
Both the multivariate model for stock prices and the model for a shock price with stochastic volatility are widely used in mathematical finance.

Returning to Itô Processes. Consider a general Itô process

$$dX_t = \sum_{k=1}^{m} \sigma_k \cdot dW_k + \mu_t \cdot dt$$

- \(\{W_k, k=1, \ldots, n\}\) independent Brownian motions.
- \(\sigma_k\) non-anticipative with respect to \(\mathbb{E}dW_k \sum_{k=1}^{m}\)
- \(\mu_t\) non-anticipative with respect to \(\mathbb{E}W_k, k=1\)
In integral form:

\[ X_t = X_0 + \int_0^t \sum_{k=1}^m \gamma_k \cdot dW_k(t) + \int_0^t \mu(t) \, dt. \]

Its processes are processes with continuous paths.

**Generalized Ito's Lemma:** let

\[ dX = \sigma(t, X) \, dW(t) + \mu(t, X) \, dt \]

Then, if \( f(x, t) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) is a function,

\[
\begin{align*}
 df(X(t), t) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(t), t) \, dX_i(t) + \frac{\partial f}{\partial t}(X(t), t) \, dt \\
&\quad + \frac{1}{2} \sum_{i,j=1}^n \sigma_{ij}(X(t), t) \, dt \sum_{k=1}^m \rho_{ik}(X(t), t) \, dW_k(t) \\
&\quad - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(t), t) \mu_i(t) \, dt + \frac{\partial f}{\partial t}(X(t), t) \, dt.
\end{align*}
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} (x(t), t) \, dt
\]

where
\[
a_{ij} = \sum_{k=1}^{m} \sigma_{ik} \sigma_{jk}
\]

Stochastic:

If \( dx = \sigma \, dw + \mu \, dt \)
\( w = (w_1, \ldots, w_m) \) are i.i.d. BM.

\[
df(x, t) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \left( \sum_{k=1}^{m} \sigma_{ik} \, dw_k \right) + \]
\[
\left( \frac{\partial f}{\partial t} + \sum_{k=1}^{m} \mu_k \frac{\partial f}{\partial x_k} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \, dt
\]
"Generalised Itô's rule:"

\[
    dX_i = \sum_k \sigma_{ik} dw_k + \mu_i dt
\]

\[
dX_i dX_j = \sigma_{ij} dt
\]

**Diffusion Processes.** A diffusion process is an Itô process such that

\[
\begin{align*}
    \sigma_{ik}(t) &= \tilde{\sigma}_{ik}(X(t), t) \\
    \mu_i(t) &= \tilde{\mu}_i(X(t), t)
\end{align*}
\]

In other words, Diffusion processes are Itô processes for which the local characteristics depend on the process (and not on the past for example).
Example:
\[ dX(t) = -\alpha X(t) \cdot dW_t + \mu X(t) \cdot dt \]

The local characteristics are
\((\sigma X(t), \mu X(t))\), which are functions of position.

Note:
\[ X(t) = X(0) e \sigma W_t - \frac{1}{2} \sigma^2 t + \mu t \]

Multivariate stochastic model:
\[ \frac{dX_i(t)}{X_i(t)} = \sum_{b=1}^{m} \sigma_{ib} \cdot dW_b + \mu_i \cdot dt \]

\[ \sigma_{ib} = \sigma_{ib} X_i \quad \mu_i = \mu_i X_i \]

The dynamics depend only on the current position. Thus \((X_1, \ldots, X_n)\) is a multidimensional