Reconstruction of volatility: Pricing index options using the steepest-descent approximation

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Abstract: We propose a formula for calculating the implied volatility of index options based on the volatility skews of the options on the underlying stocks and on a given correlation matrix for the basket. The derivation uses the steepest-descent approximation for the multivariate probability distribution function of forward prices. A simple financial justification is provided. We apply the formula to compute the implied volatilities of liquidly-traded options on exchange-traded funds (ETFs) across different strikes. Our theoretical results are found to be in very good agreement with contemporaneous quotes on the Chicago Board of Options Exchange (CBOE) and American Stock Exchange (AMEX).

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1 Local volatility of indices and the method of steepest descent

Quantitative modeling in finance and option pricing theory have focused, by and large, on models with few underlying risk-factors. This shortcoming is apparent when considering the subject of this paper: the pricing of index options on baskets of 20 to 100 stocks in relation to the values of the options on the component stocks. Despite the theoretical and practical importance of the question, few techniques are available in the current literature (for a review, see [Ro]). Due to the large number of stocks involved in a typical exchange traded fund (ETF) or index, this problem lies beyond the scope of PDE techniques and requires sophisticated Monte Carlo simulation. In this paper, we analyze the valuation of equity index options in relation to the volatilities of the components using the method of steepest-descent for diffusion kernels (see [V] [A] [B]). In this approximation, the calculation of certain conditional expectations – a key step needed to characterize the local volatility function of the index – is replaced by the evaluation of a function at the *most likely price configuration associated with a given index level*. This procedure is based on mathematically rigorous asymptotics. It provides, in our opinion, powerful new insight on valuation of index products. Furthermore, it gives excellent results in terms of matching market quotes by "reconstructing" the implied volatility of an option on a basket using information on the underlying stocks and their options.

We consider a basket of n stocks described by their price processes $S_i = S_i(t)$, $i = 1, \ldots, n$ and an index or ETF on these stocks which consists of w_i shares of the ith stock. The price of the index is

$$B = \sum_{i=1}^{n} w_i S_i,$$

with the w_i's constant.

Adopting a standard one-factor model for pricing options on stocks in the presence of a volatility skew (see [Ru] [D] [DKK]), we assume a risk-neutral diffusion measure for each component,

$$\frac{dS_i}{S_i} = \sigma_i(S_i, t)\, dZ_i + \mu_i dt,$$

where $\sigma_i(S_i, t)$ is a local volatility function associated with the i^{th} stock and

μ_i is the drift associated with the cost of carry. We assume that $Z_i = Z_i(t)$ are standard Brownian motions which satisfy

$$E(dZ_i dZ_j) = \rho_{ij} dt$$

where ρ_{ij} is given. For simplicity, we assume that ρ_{ij} is constant, since this is the most likely situation when correlations are estimated using historical data[3].

An important element of our analysis is the stochastic volatility function associated with the index *viz.*, $\sigma_B = \sigma_B(\mathbf{S}, t)$, $\mathbf{S} = (S_1, ..., S_n)$, which is given by

$$\sigma_B^2 = \frac{1}{B^2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j w_i w_j S_i S_j.$$

This expression depends on the individual stock prices and not just on the index level. A local volatility function for the index (i.e. which depends only on the spot price of the index) can be obtained by calculating the expectation of σ_B^2 conditional on the value of the index. More precisely, we claim that the function $\sigma_{B,loc} = \sigma_{B,loc}(B, t)$, given by

$$\begin{aligned}
\sigma_{B,loc}^2 &= \mathbb{E}\left\{\sigma_B^2 \mid B(t) = B\right\} \\
&= \mathbb{E}\left\{ \frac{1}{B^2} \sum_{ij=1}^{n} \rho_{ij} \sigma_i \sigma_j w_i w_j S_i(t) S_j(t) \;\middle|\; \sum_{i=1}^{n} w_i S_i(t) = B \right\},
\end{aligned}$$

is such that the one-dimensional diffusion process

$$\frac{dB}{B} = \sigma_{B,loc}(B, t)\, dW + \mu_B dt$$

(with μ_B representing the cost-of-carry of the ETF), returns the same prices for European-style index options as the n-dimensional model based on the dynamics for the entire basket.

To see this, we note that $\sigma_B(\mathbf{S}, t)$ can be viewed as a stochastic volatility process that drives the index price $B(t)$. The above formula then expresses

[3] The results presented here apply to more general correlation/volatility structures, including, for instance the case of multivariate stochastic volatility/stochastic correlation models.

a well-known relation between the stochastic volatility of a process and its local volatility (see [DKK] [BN] [G] [L]).

To provide a tractable expression for $\sigma_{B,loc}$, we rewrite the latter equation formally as

$$\sigma_{B,loc}^2 = \frac{\mathbb{E}\left\{\sigma_B^2 \ \delta\left(B\left(t\right) - B\right)\right\}}{\mathbb{E}\left\{\delta\left(B\left(t\right) - B\right)\right\}} \tag{1}$$

where $\delta(x)$ is the Dirac delta function, and use asymptotic analysis to determine an approximate value for $\sigma_{B,loc}^2$ in the limit $\overline{\sigma}^{\,2} t \ll 1$.[4]

The main mathematical tool for carrying out this calculation is Varadhan's Formula [V] [A] [B]. We introduce the change of variables $x^i = \ln \dfrac{S_i}{S_i\left(0\right) e^{\mu_i t}}$, and introduce the diffusion matrix of the process $\mathbf{x} = (x^1, ..., x^n)$, $a^{ij} = \sigma_i \sigma_j \rho_{ij}$. Notice that $F_i \doteq S_i\left(0\right) e^{\mu_i t}$ is the forward price of i^{th} the stock for delivery at time t.We consider the inverse of a^{ij}, which we denote by g_{ij}, and the associated Riemmanian metric

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j. \tag{2}$$

Let $\pi\left(\mathbf{x}_0, t_0; \mathbf{x}, t\right)$ denote the probability density function associated with the process $(x^1, ..., x^n)$, *i.e.*

$$\pi\left(\mathbf{x}_0, t_0; \mathbf{x}, t\right) dx = \mathbb{P}\left\{\mathbf{x}\left(t\right) \in \mathbf{B}\left(x; \left|dx\right|\right) \left|\mathbf{x}\left(t_0\right) = \mathbf{x}_0\right.\right\}$$

where $\mathbf{B}\left(\mathbf{x}; r\right)$ is the Euclidean ball with center x and radius r.

Varadhan's Formula states that

$$\pi\left(\mathbf{0}, 0; \mathbf{x}, t\right) \sim e^{-\frac{d^2(\mathbf{0},\mathbf{x})}{2t}} = e^{-\frac{(\overline{\sigma})^2 d^2(\mathbf{0},\mathbf{x})}{2(\overline{\sigma})^2 t}} \tag{3}$$

where

$$d^2\left(\mathbf{0}, \mathbf{x}\right) = \inf_{\mathbf{x}(0)=0, \mathbf{x}(1)=\mathbf{x}} \int_0^1 \sum_{i,j=1}^{n} g_{ij}\left(\mathbf{x}\left(s\right), 0\right) \dot{x}^i \dot{x}^j ds. \tag{4}$$

[4]Here $\overline{\sigma}$ denotes a characteristic volatility level associated to the equity basket. For example, in the case of DJX (Dow Industrial Average), the index volatility in 2001 ranged between 20% and 25% approximately. Taking $\overline{\sigma} = 0.20$ as the typical level, we find that a 6-month option has $(\overline{\sigma})^2\, t \simeq 0.02$. This regime produces distributions that have very low variance in dimensionless units.

Here $\dot{\mathbf{x}}$ is the time-derivative of $\mathbf{x}$. The asymptotics in Varadhan's Formula are understood in the sense that the ratio of the logarithms of the two terms tends to 1 as $\overline{\sigma}^2 t \ll 1$. The expression $d^2(\mathbf{0}, \mathbf{x})$ can be interpreted as the distance between the points $\mathbf{0}$ and $\mathbf{x}$ in the metric (2).

According to the method of steepest descent, the probability density function $\pi(\mathbf{0}, 0; \mathbf{x}, t)$ is strongly peaked near the points $\mathbf{x}$ where $d^2(\mathbf{0}, \mathbf{x})$ is minimal, for $\overline{\sigma}^2 t \ll 1$. Therefore, we can obtain an approximate expression for the ratio of expectations in (1) by setting

$$\sigma_{B,loc}^2 \simeq \sum_{\mathbf{x}^* \in M} q(\mathbf{x}^*)\, \sigma_B^2(\mathbf{x}^*, t), \tag{5}$$

where M is the set of points on the hypersurface

$$\Gamma_B = \left\{ \mathbf{x} : \sum_{i=1}^{n} w_i F_i e^{x_i^*} = B \right\}$$

which have the shortest distance to the origin in the metric d.[5] The numbers $q(\mathbf{x}^*)$ are positive weights that satisfy $\sum_{\mathbf{x}^* \in M} q(\mathbf{x}^*) = 1$. They are proportional to the curvature of the metric at the minimizer. Generically, i.e. barring symmetries and isolated points, there is a unique minimizer $\mathbf{x}^*$ with $q(\mathbf{x}^*) = 1$.

Formula (5) expresses a relation between the local volatility of the index, the correlation matrix and the local volatilities of the component stocks. It admits a simple interpretation. Notice that the surfaces

$$\Theta_\delta = \{\mathbf{x} : d(\mathbf{0}, \mathbf{x}) = \delta\}$$

correspond approximately to the level sets of the probability density function of the multivariate price process. The vector(s) $\mathbf{x}^*$ which produces the minimum value of the distance to the "price manifold" Γ_B correspond therefore to the most probable vector(s) $\mathbf{S} = (S_1, ..., S_n)$ such that $\sum_{i=1}^{n} w_i S_i = B$. *Thus, the method of steepest-decent equates the local volatility of the index at a given level to its stochastic volatility evaluated at the most probable price configuration* $\mathbf{S}$ *conditional on reaching that level.*

[5] We committed a slight abuse of notation in (5), writing $\sigma_B(\mathbf{x}^*, t)$ instead of $\sigma_B(\mathbf{S}^*, t)$.

2 Computing the most likely configuration

Let us concentrate on the case where the minimizer $\mathbf{x}^*$ is unique, since this is the generic case. Abusing somewhat the notations, for simplicity we denote by $\sigma_i(x^i) = \sigma_i(F_i e^{x^i}, 0)$ and $\sigma_{B,loc}(\overline{x}) = \sigma_{B,loc}(B(0)e^{\overline{x}}, 0)$, $\overline{x} = \ln\left(B/\left(B(0)e^{\mu_B t}\right)\right)$ the local volatilities of the underlying assets (resp. the basket), at time to maturity 0, as a function of log-moneyness.

With these notations (5) reads

$$\sigma_{B,loc}^2 = \sum_{i,j=1}^{n} \rho_{ij}\sigma_i\left(x_i^*\right)\sigma_j\left(x_j^*\right)p_i\left(\mathbf{x}^*\right)p_j\left(\mathbf{x}^*\right), \tag{6}$$

where

$$p_i\left(\mathbf{x}\right) = \frac{F_i e^{x_i} w_i}{\sum_{k=1}^{n} F_k e^{x_k} w_k} \tag{7}$$

represents the percentage of stock i represented in the index when $S_i = F_i e^{x_i}$.

In order to obtain useful formulas, we need to characterize $\mathbf{x}^*$. Notice that the metric satisfies

$$\sum_{i,j} g_{ij}\left(\mathbf{x}, 0\right)\dot{x}^i\dot{x}^j = \sum_{i,j}\left(\rho^{-1}\right)_{ij}\frac{\dot{x}^i}{\sigma_i\left(x^i\right)}\frac{\dot{x}^j}{\sigma_j\left(x^j\right)}$$

Introducing the change of variables

$$y^i = \int_0^{x^i}\frac{du}{\sigma_i\left(u\right)},$$

we obtain from (4) the simple problem of calculus of variations

$$\int_0^1 \sum_{i,j=1}^{n}\left(\rho^{-1}\right)_{ij}\dot{y}^i\dot{y}^j ds = \text{ minimum}$$

subject to the nonlinear constraint

$$\sum_{i=1}^{n} w_i F_i e^{x^i\left(y^i\right)} = B. \tag{8}$$

Given the simple structure of this problem, the solution is such that $\dot{\mathbf{y}}$ is constant. The Euler-Lagrange first order conditions for a minimum can be expressed in the form

$$\begin{aligned} \sum_{j=1}^{n} \left(\rho^{-1}\right)_{ij} \dot{y}^j &= \lambda w_i F_i e^{x^i\left(y^i\right)} \frac{\partial x^i\left(y^i\right)}{\partial y^i} \\ &= \lambda w_i F_i e^{x^i(\mathbf{y})} \sigma_i\left(x^i\right), \end{aligned}$$

where λ is a Lagrange multiplier associated with the price constraint (8). At this point, it is convenient to recast the Lagrange multiplier λ as λ/B since it then becomes dimensionless. Using this redefinition and multiplying both sides of the equation by the correlation matrix ρ, we obtain

$$\begin{aligned} \dot{y}^i &= \lambda \frac{\sum_{j=1}^{n} w_j F_j e^{x^j(\mathbf{y})} \rho_{ij} \sigma_j\left(x^j\right)}{B} \\ &= \lambda \sum_{j=1}^{n} p_j\left(\mathbf{x}\right) \rho_{ij} \sigma_j\left(x^j\right). \end{aligned}$$

Now, since $\dot{y}^i$ is constant, it is equal to its average over the interval $(0,1)$, so that $\dot{y}^i = y^i$.

We have just established the following result.

Link between index and stocks local volatilities

- *In the limit $\overline{\sigma}^{\,2} t \ll 1$ the local volatility of the index is given by*

$$\sigma^2_{B,loc}(\overline{x}) = \sum_{i,j=1}^{n} \rho_{ij} \sigma_i\left(x_i^*\right) \sigma_j\left(x_j^*\right) p_i\left(\mathbf{x}^*\right) p_j\left(\mathbf{x}^*\right), \tag{9}$$

where p_i is defined in (7) and x^* is the solution of the nonlinear system

$$\begin{cases} \displaystyle\int_0^{x_i^*} \frac{du}{\sigma_i(u)} &= \displaystyle\lambda \sum_{j=1}^n \rho_{ij} p_j(x^*) \sigma_j(x_j^*), \quad \forall i = 1, \ldots, n \\ \displaystyle\sum_{i=1}^n p_i(x) e^{x_i^* - x^i} &= 1. \end{cases} \tag{10}$$

3 From local volatilities to Black-Scholes implied volatilities

We translate formulas (9)-(10) into a relation between the Black-Scholes implied volatilities of the index option and the implied volatilities of the options on individual stocks. For this, we take advantage of a recent result by Berestycki, Busca and Florent [BBF1] [BBF2] which states that the implied volatility is the "harmonic mean" of the local volatility function in the steepest-descent approximation. More precisely we have:

$$\sigma_B^I(\overline{x}) = \left(\frac{1}{\overline{x}} \int_0^{\overline{x}} \frac{du}{\sigma_{B,loc}(u)} \right)^{-1} \tag{11}$$

and, conversely,

$$\sigma_i(x^i) = \left(\frac{d}{dy} \left(\frac{y}{\sigma_i^I(y)} \right)_{|y=x^i} \right)^{-1}, \quad \forall i = 1, \ldots, n. \tag{12}$$

Equations (9), (10) together with (11) and (12) provide a direct link between the Black-Scholes implied volatilities of the index and the underlying assets.

To derive tractable formulas, we consider a linear approximation to the price constraint (8), namely

$$\sum_{i=1}^n p_i(0) \left(1 + \lambda \sigma_i(0) \sum_{j=1}^n \rho_{ij} p_j(\mathbf{0}) \sigma_j(0) - x^i \right) \simeq 1, \tag{13}$$

which is exact to first order in $|x_i|$. Noting that $\overline{x} \simeq \sum_{i=1}^{n} p_k(0)x_k$, this yields

$$\lambda \simeq \frac{\overline{x}}{\sigma_B(0)^2} \tag{14}$$

and, from (10)

$$x_i^* \simeq \frac{\overline{x}}{\sigma_B(0)^2} \sum_{j=1}^{n} \rho_{ij} p_j(\mathbf{0}) \sigma_i(0) \sigma_j(0). \tag{15}$$

for all $i = 1, \ldots, n$.

To simplify the relations (9)-(10)-(11)-(12), we use the fact that the harmonic-mean relation between implied and local volatilities (11) and (12) give rise to the approximations

$$\sigma_B^I(\overline{x}) \simeq \frac{\sigma_{B,loc}(\overline{x}) + \sigma_B(0)}{2}, \tag{16}$$

and

$$\sigma_i\left(x^i\right) \simeq 2\sigma_i^I\left(x^i\right) - \sigma_i^I(0), \quad \forall i = 1, \ldots, n, \tag{17}$$

which, again, are valid to first-order for $|x_i| << 1$. Relation (16) is referred sometimes as the "1/2-slope rule", see [G]. Thus (9) reduces to

$$\sigma_{B,loc} = \sqrt{\sum_{i,j=1}^{n} \rho_{ij} p_i(\mathbf{x}^*) p_j(\mathbf{x}^*) \left(2\sigma_i^I(x_i^*) - \sigma_i^I(0)\right) \left(2\sigma_j^I\left(x_j^*\right) - \sigma_j^I(0)\right)}, \tag{18}$$

which, together with (16) and (15) provides a convenient approximate link between implied volatilities.[6]

4 The formula in terms of relation between Black-Scholes Deltas

In this section, we express the correspondence between index volatilities and component volatilities in a more transparent way. To do this, we express

[6] Since these calculations are exact to first order in $|x_i|$ for small values of the parameter, it follows that the above relation gives an exact formula for the "slope" of the index volatility function at $|\bar{x} = 0$.

the most probable configuration for stock prices at time t in terms of their log-moneyness normalized by volatility.

Notice that (with all functions evaluated at $s = 1$), the components of the vector $\mathbf{y}^*$satisfy

$$y^i = \frac{x^i}{\sigma_i^I(\mathbf{x})} = \frac{1}{\sigma_i^I(S^*, t)} \ln \frac{S^*}{F_i}.$$

Recall also that the Black-Scholes formula for the Delta of a call option with strike K is $\Delta = N(d_1)$, with $N(x) = (2\pi)^{-1/2} \int_{-\infty}^x \exp(-y^2/2)dy$ where

$$d_1 = \frac{1}{\sigma\sqrt{t}} \ln \frac{F}{K} + O(\sigma\sqrt{t}).$$

We conclude that the Delta of a call option on the i^{th} underlying stock with strike $S^* = F_i e^{x_i^*}$ is given, to leading order as $\sigma\sqrt{t} \to 0$, by

$$\Delta_i = N\left(-\frac{y^i}{\sqrt{t}}\right),$$

which provides an alternative interpretation for the vector $\mathbf{y}^*$. A similar approximation holds for the Delta of a call option with strike B. Using equation (14) we find that

$$\Delta_B \simeq N\left(-\frac{1}{\sigma_B^I\sqrt{t}} \ln \frac{B}{F_B}\right) = N\left(-\frac{\lambda\sigma_B^I}{\sqrt{t}}\right).$$

Applying the inverse-normal distribution function to both sides and using the first-order optimality conditions (15) for the most probable configuration, we conclude that

$$N^{-1}(\Delta_i) \simeq N^{-1}(\Delta_B) \times \sum_{j=1}^n \rho_{ij} \left(\frac{\sigma_j^I}{\sigma_B^I}\right) p_j$$

or, finally,

$$\Delta_i \simeq N\left[N^{-1}(\Delta_B) \times \sum_{j=1}^n \rho_{ij} \left(\frac{\sigma_j^I}{\sigma_B^I}\right) p_j\right], \tag{19}$$

which gives a simple relation between the Delta of an index call option and the deltas of the call options on the components that are used in the volatility reconstruction formula.

To shed light on equations (16) - (18) using equation (19), we can assume that the expression $\sum_{j=1}^{n}\rho_{ij}\left(\frac{\sigma_j^I}{\sigma_B^I}\right)p_j$ in (19) is evaluated using at-the-money implied volatilities. Equation (19) then gives, for a given value of Δ_B, the corresponding vector $(\Delta_1, ..., \Delta_n)$ of call-option deltas corresponding to different strikes $(S_1^*, ..., S_n^*)$ used in connection with the volatility skews of the components. In fact, the implied volatilities $\left(\sigma_1^I\left(S_1^*,t\right), ..., \sigma_n^I\left(S_n^*,t\right)\right)$ can then be substituted directly into equations (16) - (18) to generate a curve of implied volatilities $\sigma_B^I\left(B,t\right)$ for index options. This represents a useful shortcut for reconstructing the implied volatility skew of an ETF option, without having to solve problem (10).

Two limiting cases seem noteworthy. Consider first the case of perfectly correlated stocks. In this case, from (18), we have $\sigma_B^I \simeq \sum_{j=1}^{n} p_j\sigma_j^I$, so our results imply that the most probable point corresponds to strikes that all have approximately the same delta as the index option. This *equal-delta approximation* for pricing index options in terms of the component volatility skews is apparently well-known to professionals[7] . Our analysis suggests that an equal-delta approximation is indeed appropriate for ETFs which exhibit high correlation among the components, but also indicates that it may not be optimal for low-correlation indices.

Consider next the (more unlikely) case when all stocks are uncorrelated. Equation (19) shows that the most probable price configuration corresponds to a set of deltas such that

$$\begin{aligned}\Delta_i &\simeq N\left(N^{-1}\left(\Delta_B\right)\times\left(\frac{\sigma_i^I}{\sigma_B^I}\right)p_i\right)\\ &= N\left(N^{-1}\left(\Delta_B\right)\times\frac{p_i\sigma_i^I}{\sqrt{\sum_j p_j^2\left(\sigma_j^I\right)^2}}\right).\end{aligned}$$

For example, assume that the index option is an out-of-the-money call with $\Delta_B < 0.5$. (In this case, $N^{-1}\left(\Delta_B\right) < 0$). Since the fraction $\frac{p_i\sigma_i^I}{\sqrt{\sum_j p_j^2\left(\sigma_j^I\right)^2}}$

[7] From a private communication with a former head of AMEX specialist firm.

is less than unity, the deltas of the individual options associated with the steepest-descent approximation will be *higher* than Δ_B, *i.e.* the relevant strikes are closer to the money.

Finally, consider the case of an *uncorrelated* basket with a single, exceptionally volatile stock $\sigma_1^I \gg \sigma_i^I, i \neq 1$. In this case, the most probable configuration will have $x_i \approx 0, i \geq 2$, corresponding to 50-delta implied volatilities $i \neq 1$ for the low-volatilitiy stocks and a delta (or strike) for the high-volatility stock which coincides roughly with that of the index.

5 Experimental results

We compared the results of the method of steepest descent with contemporaneous market quotes taken from two indices traded on U.S. markets on March 20, 2002.

We considered options on the Dow Jones Industrial Average (CBOE, symbol:DJX) as well as options on the Merrill Lynch Biotech HOLDR (AMEX, symbol:BBH). Two short-term expirations were considered: April (front month) and May. The experiment used historical estimates for correlations between the index components and implied volatilities for call options downloaded simultaneously with the implied volatilities for the index. We used equation (19) to derive the deltas associated with different strikes and formulas (16) and (18) to compute the theoretical implied volatilities of the index call options. We used mid-market implied volatilities for the components of the index. The results were then compared with the actual market quotes on index options available at the same time. The results presented here are representative of other indices that we analyzed as well.

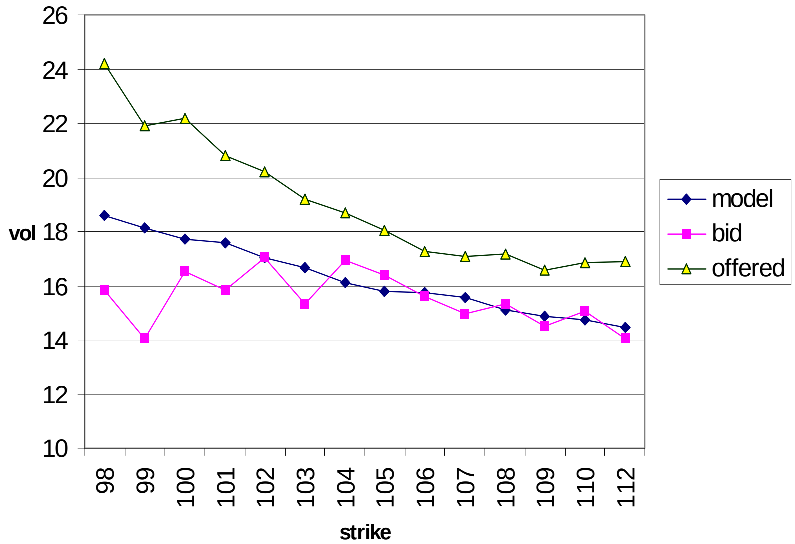

Figure 1: DJX April 2002 Index Implied Volatilities: Model vs. Market. Market quotes are expressed in terms of implied volatilities (bid, offered) and compared with the predictions of the steepest-descent method using (7a)-(7b) and the approximation (8) for calculating the most probable configuration.

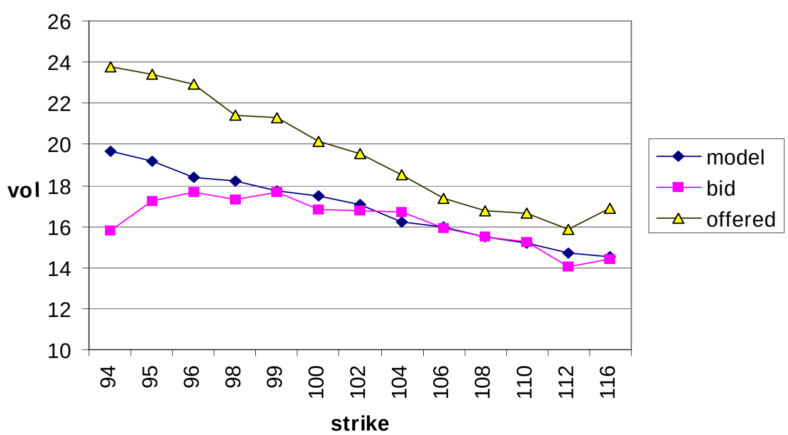

Figure 2: DJX May 2002 Implied Volatilities: Model vs. Market

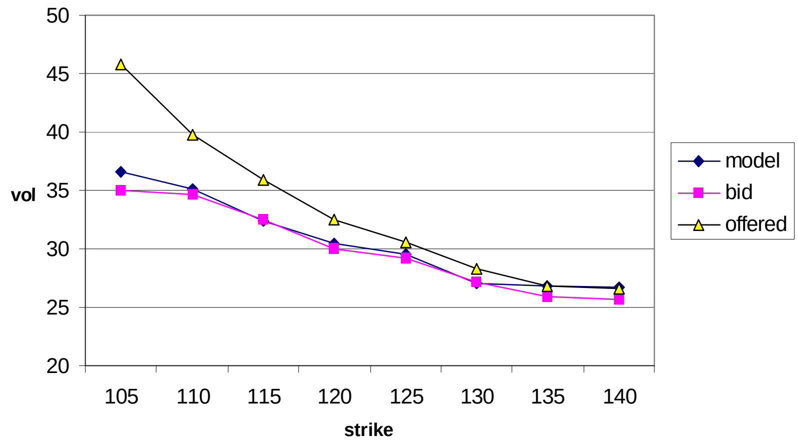

Figure 3: BBH April Index Implied Volatilities: Model vs. Market

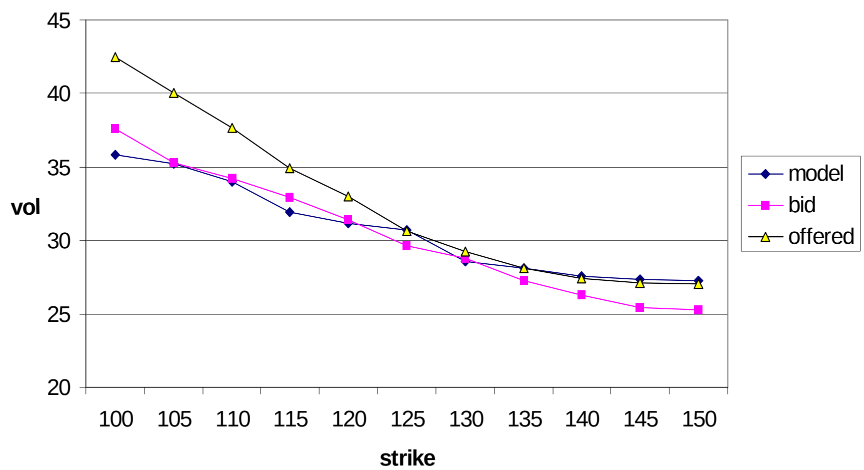

Figure 4: BBH May Index Volatilities: Model vs. Market.

Strike	Bid	Offered	Midmarket	SD
98	15.84	24.20	20.76	18.57
99	14.06	21.90	18.66	18.14
100	16.53	22.20	19.57	17.73
101	15.85	20.82	18.46	17.59
102	17.05	20.23	18.67	17.02
103	15.36	19.22	17.32	16.66
104	16.92	18.68	17.80	16.13
105	16.37	18.05	17.21	15.80
106	15.60	17.26	16.43	15.76
107	14.97	17.10	16.03	15.57
108	15.34	17.16	16.26	15.12
109	14.50	16.57	15.55	14.89
110	15.05	16.83	15.96	14.74
112	14.07	16.89	15.58	14.45

DJX April 2002 Implied Volatility (Spot DJX$\simeq$105)

Strike	Bid	Offered	Midmarket	SD
94	15.81	23.76	20.64	19.65
95	17.24	23.43	20.75	19.15
96	17.67	22.90	20.53	18.42
98	17.28	21.38	19.44	18.19
99	17.69	21.27	19.54	17.73
100	16.81	20.12	18.51	17.48
102	16.74	19.51	18.14	17.07
104	16.67	18.54	17.61	16.20
106	15.89	17.38	16.63	15.96
108	15.51	16.75	16.13	15.51
110	15.26	16.66	15.97	15.18
112	14.05	15.86	14.98	14.73
116	14.41	16.91	15.76	14.53

DJX May 2002 Implied Volatility

Strike	Bid	Offered	Midmarket	SD
105	34.95	45.75	41.23	36.65
110	34.60	39.73	37.31	35.06
115	32.45	35.87	34.20	32.44
120	30.00	32.50	31.26	30.44
125	29.18	30.59	29.89	29.49
130	27.14	28.24	27.69	27.00
135	25.86	26.86	26.37	26.78
140	25.70	26.56	26.14	26.67

BBH April 2002 Implied Volatility (Spot BBH$\simeq$125)

Strike	Bid	Offered	Midmarket	SD
100	37.56	42.50	40.19	35.79
105	35.31	40.01	37.77	35.18
110	34.25	37.67	36.00	34.01
115	32.90	34.91	33.91	31.91
120	31.37	33.03	32.20	31.14
125	29.64	30.65	30.15	30.67
130	28.76	29.26	29.01	28.56
135	27.27	28.14	27.71	28.12
140	26.26	27.39	26.83	27.60
145	25.44	27.10	26.30	27.30
150	25.26	27.04	26.19	27.27

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6 Conclusions

We derived a simple formula, based on the method of steepest-descent, that links the local volatility function of an index with the local volatility functions for the index components and a given correlation matrix. The intuition behind the steepest descent approximation is that, if the dimensionless time scale $(\overline{\sigma})^2 t$ is sufficiently small, the local volatility of the index should be determined from the most likely configuration of stock prices conditional on arriving at a given index level.

In a second step, we propose a second approximation which operates at the level of the implied volatilities, *i.e.* of option prices. This approximation uses an estimate for the most likely configuration arising from the Euler-Lagrange equations together with an asymptotic relation between local and implied volatilities derived by Berestycki-Busca-Florent in [BBF1] [BBF2] (also in the limit $\overline{\sigma}^2 t \ll 1$).

Finally, we characterized the most likely price configuration to a vector of Black-Scholes deltas which determine which points on the volatility skews of the component stocks contribute the most to the index implied volatility (in this approximation). The resulting formula is useful because it can be tested directly on market data.

The predictions of the formula, which is based on historical correlations and the volatility skews of the components, are in very good agreement with contemporaneous quotes for index options.

We note, however, that we have not undertaken at this point an extensive statistical study in this direction, leaving it for a future publication. Despite observing good agreement with the data, discrepancies from market quotes may arise due to effects not contemplated in the model. First, we expect that the short-dimensionless-time asymptotics will break down, or at least be less accurate, for longer expirations. Discrepancies may also arise from an oversimplification of the risk-neutral probability distribution governing stock prices and from our simplistic treatment of correlations.

In this last regard, we believe that the steepest-descent model could be useful to analyze correlation "risk-premia". By this we mean that observed differences between the shape of the actual index volatility and the one predicted by the steepest-descent approximation (using, say, historical correlations) can be attributed to expectations about future correlations which are dependent on the index level – i.e to a "correlation skew" which might thus be observable through index option prices.

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