Stochastic Calculus Fall 2009
Homework 3

1. Multivariate models. Download end-of-day data for the components of the Dow Jones Industrials Average for the last year. Using this data, specify a model for the joint dynamics of the component stocks of the type

\[
\frac{dS_i}{S_i} = \sum_{k=1}^{m} \sigma_{ik} dZ_k + \mu_i dt
\]  

(1)

2. Brownian reflection principle. Let \( W(t) \) be a standard Brownian motion, let \( a > 0 \), and define \( \tau_a \) as the first-passage time of the path through level \( a \):

\[
\tau_a = \inf \{ t : W(t) = a \} .
\]

We define the reflected path around \( a \) as

\[
W_a(t) = \begin{cases} 
W(t) & \text{if } t < \tau_a \\
2a - W(t) & \text{if } t \geq \tau_a 
\end{cases}
\]  

(2)

(a) Show that \( W_a(t) \) is also a Brownian motion. (b) Show that the following equation holds

\[
\text{Prob} \left\{ \max_{0 \leq t \leq T} W(t) > a \right\} = \text{Prob} \{ \tau_a < T, W(T) < a \} + \text{Prob} \{ \tau_a < T, W(T) \geq a \} = \text{Prob} \{ W_a(T) > a \} + \text{Prob} \{ W(T) \geq a \} = 2 \text{Prob} \{ W(T) \geq a \} = 2 \left( 1 - N \left( \frac{a}{\sqrt{T}} \right) \right)
\]  

(3)

where \( N(x) \) is the cumulative distribution function of a standard normal. (c) Using Monte Carlo simulation, compute \( \text{Prob} \{ \tau_x < 1 \} \) for \( x = 0.5, 1, 1.5 \) and 2, and corroborate the result (b).

3. Ito’s Lemma and polynomials. Let \( W(t) \) be standard Brownian motion and let \( P(x) \) be a polynomial of degree \( n, n \geq 1 \). Find values of the constants \( a, b \), and a differential equation satisfied by \( P(x) \), so that the process

\[
X_t = t^a \frac{W(t)}{t^b}
\]

is a martingale. [Hint: apply Itô’s Lemma to \( t^a P(x/t^b) \).] (b) List polynomials that have this property for \( n = 2, 3, 4 \).
4. **2-D Brownian motion.** (a) Let $W_1(t)$ and $W_2(t)$ be two independent Brownian motions and let $f_t$ and $g_t$ be two non-anticipative functions with respect to $(W_1, W_2)$. Show that if $f_t^2 + g_t^2 = 1$ then

$$Y_t = \int_0^t (f_t \, dW_1(t) + g_t \, dW_2(t))$$

is a Brownian motion. What if $f_t^2 + g_t^2 \neq 1$? Is there a time-change under which this process is a Brownian motion? (b) With the notation of (a), define

$$\theta_t = \int_0^t \frac{W_1(s) \, dW_2(s) - W_2(s) \, dW_1(s)}{W_1^2(s) + W_2^2(s)}.$$ 

where $(W_1, W_2)$ is a BM and $W_1(0) + W_2(0) \neq 0$. Show that $\theta(t)$ is a martingale and that it can be viewed as a Brownian motion after a change of time. [Extra credit for geometrically inclined students: give a geometric interpretation of $\theta_t$. Based on this interpretation, show an easy way to compute $\theta$ from the 2-d Brownian path. Hint: apply Itô’s Lemma to $\arctan\left(\frac{W_2(t)}{W_1(t)}\right)$].

5. **Bessel processes.** Let $(W_1(t), \ldots, W_n(t))$ be $n$ independent Brownian motions. Define the new process

$$B_n(t) = \sqrt{W_1^2(t) + \ldots + W_n^2(t)}.$$  \hspace{1cm} (4)

This process describes the distance to the origin of an $n$-dimensional Brownian motion. Show that $B_n$ satisfies the equation

$$dB_n(t) = dZ(t) + \frac{n - 1}{2B_n(t)} \, dt$$

where $Z(t)$ is a suitable Brownian motion. Assume that $B_n(0) \neq 0$. Argue that $B_n(t)$ can never hit the level zero.