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LARGE DEVIATIONS FROM THE CIRCULAR LAW

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Abstract. We prove a full large deviations principle, in the scale N^2, for the empirical measure of the eigenvalues of an $N \times N$ (non self-adjoint) matrix composed of i.i.d. zero mean random variables with variance N^{-1}. The (good) rate function which governs this rate function possesses as unique minimizer the circular law, providing an alternative proof of convergence to the latter. The techniques are related to recent work by Ben Arous and Guionnet, who treat the self-adjoint case. A crucial role is played by precise determinant computations due to Edelman and to Lehmann and Sommers.

1. Introduction

Let $X^N = \{X_{ij}^N\}$ be an $N \times N$ matrix whose entries are independent centered normal random variables of variance N^{-1}. We denote the (complex) eigenvalues of X^N by $Z_i, i = 1, \cdots, N$, and form the empirical measure

$$\hat{\mu}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{Z_i}.$$

The law of $\hat{\mu}^N$ is denoted by $\overline{Q}^N$.

The celebrated circular law (c.f., e.g., Edelman A. (1997),Girko V.L. (1984),Mehta M.L. (1991)) states that $\hat{\mu}^N$ converges in distribution to the uniform law U on the disc $D = \{Z : |Z| \leq 1\}$. Our goal in this paper is to study the corresponding large deviations. We follow a similar study which was carried out in Ben Arous G., Guionnet A. (1997) for the case of self-adjoint matrices. In that case, the large deviations fluctuation have speed N^{-2} and a rate function related to Voiculescu's non commutative entropy. More precisely, if $Y^N = \{Y_{ij}^N\}$ is an $N \times N$ self-adjoint matrix with independent (for $j > i$) centered normal random variables of variance $N^{-1}/2$ (variance N^{-1} if $i = j$), and eigenvalues λ_i^y, and if $\tilde{\mu}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^y}$, then $\tilde{\mu}^N$ satisfies the large deviation principle with speed N^{-2} and rate function

$$I_R(\mu) = \frac{1}{2} \int_{\mathbb{R}} x^2 \mu(dx) - \Sigma_R(\mu) - \frac{3}{8} - \frac{1}{4} \log 2$$

where

$$\Sigma_R(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} \log |x - y| \mu(dx) \mu(dy).$$

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Our main result is an analogous theorem in the non self-adjoint case. Let $M_1^S(\mathbb{C})$ denote the space of symmetric probability measures on $\mathbb{C}$ (symmetric with respect to complex conjugation, that is $\mu(A) = \mu(A^*)$ for A^* denoting the complex conjugate of A), equipped with the weak topology. We denote the Lévy metric on $M_1^S(\mathbb{C})$ by $\rho(\cdot,\cdot)$, and recall that it is compatible with the weak topology. For $\mu \in M_1^S(\mathbb{C})$, and Borel-measurable function $f : \mathbb{C}^2 \to \mathbb{R}$, define

$$\langle f\mu, \mu\rangle := \int_{\mathbb{C}}\int_{\mathbb{C}} f(x,y)\mu(dx)\mu(dy).$$

Define next

$$\Sigma(\mu) = \int_{\mathbb{C}}\int_{\mathbb{C}} \log|x-y|\mu(dx)\mu(dy) := \langle \log|x-y|\mu, \mu\rangle,$$

and let

$$I(\mu) = \frac{1}{2}\left(\int_{\mathbb{C}} |x|^2\mu(dx) - \Sigma(\mu)\right) - K$$

where

$$K := \frac{1}{2}\left(\int_{\mathbb{C}} |x|^2 U(dx) - \Sigma(U)\right).$$

(Lemma 2.1 below shows that actually $K = 3/8$). The LDP for $\hat{\mu}^N$ is characterized in the following theorem:

THEOREM 1.1.

1. *I is a good convex rate function on $M_1^S(\mathbb{C})$.*
2. *$I(\mu)$ is infinite as soon as μ satisfies one of the following conditions:*
   a) $\int |x|^2\mu(dx) = \infty$
   b) *There exists an $A \subset \mathbb{C}$ of positive μ-mass but null logarithmic capacity, i.e.*

$$\gamma(A) = \exp\left\{ - \inf_{\nu \in M_1^S(\mathbb{C}), \operatorname{supp}\nu \subset A} \iint \log\frac{1}{|x-y|}\nu(dx)\nu(dy)\right\} = 0$$

3. *$I(\mu)$ achieves its (unique) minimum value in $M_1^S(\mathbb{C})$ at U.*
4. *$\hat{\mu}^N$ satisfies the large deviation principle (LDP) with speed N^{-2} and rate function I, that is*

$$\begin{aligned} -\inf_{\mu\in A^o} I(\mu) &\le \liminf_{N\to\infty} \frac{1}{N^2}\log \overline{Q}^N(\hat{\mu}^N \in A) \\ &\le \limsup_{N\to\infty} \frac{1}{N^2}\log \overline{Q}^N(\hat{\mu}^N \in A) \le -\inf_{\mu\in\bar{A}} I(\mu). \end{aligned}$$

(Compare with Theorem 1.1 in Ben Arous G., Guionnet A. (1997)).

Technically, the main difference between Theorem 1.1 and Ben Arous G., Guionnet A. (1997) is the lack of ordering in $\mathbb{C}$, which prevents us from using, in the proof of the lower bound, the approximation procedure presented in Ben Arous G., Guionnet A. (1997). Instead, we present an appropriate smoothing procedure (c.f., Lemma 2.2).

Our results extend naturally to allow for perturbations of the spectral measure $\hat{\mu}^N$, c.f. the remark at the end of Section 2. However, unlike in the self-adjoint or unitary cases, there does not appear to be a "natural" class of Gaussian ensembles fitting these extensions. As in the self-adjoint case, the non-Gaussian situation remains largely open.

After this work was completed, we learnt of recent work of Hiai F., Petz D. (1998), where similar questions for matrices with Gaussian complex entries are treated.

2. Proof of Theorem 1.1

We prove the theorem in a sequence of Lemmas.

Lemma 2.1. *$K = 3/8$ and I is a good convex rate function.*

Proof. Note first that, by a direct computation,

$$\begin{aligned}\pi^2\Sigma(U) &= \int_{\mathbb{C}}\int_{\mathbb{C}} \log|x-y|dxdy = \int_{|x|>|y|} \log|x-y|dxdy \\ &= 4\pi\int_0^1 rdr\int_{\rho=0}^r \rho d\rho \int_{\theta=0}^{2\pi} d\theta \log|r-\rho e^{j\theta}|. \end{aligned} \tag{2.1}$$

Because $\log|x|$ is harmonic, for $\rho < r$,

$$\int_{\theta=0}^{2\pi} d\theta \log|r-\rho e^{j\theta}| = 2\pi \log r,$$

hence, substituting back in (2.1),

$$\pi^2\Sigma(U) = 8\pi^2\int_0^1 r\log r\int_0^r \rho d\rho dr = -\frac{\pi^2}{4}. \tag{2.2}$$

The conclusion $K = 3/8$ follows from (2.2) and the fact that $\int_{\mathbb{C}}|x|^2U(dx) = 1/2$.

The proof that the level sets of $I(\cdot)$ are compact follows the proof of 3) in property 2.1 of Ben Arous G., Guionnet A. (1997). Indeed, the closeness of the level sets and the boundedness below of $I(\cdot)$ follow by truncating the integrands in the definition of $\Sigma(\mu)$ and using monotone convergence. The compactness of the level set then follows by noting that, for any $r > r_0$ with r_0 large enough (such that $\alpha - \log(\alpha) \geq 2r^2 - \log(2r^2)$ for any $\alpha \geq 2r^2$), and with B_r denoting the centered disc of radius r in the complex plane,

$$\begin{aligned}&(\mu(B_r^c))^2 \\ &= \mu\otimes\mu(|X|^2 \geq r, |Y|^2 \geq r) \\ &= \mu\otimes\mu(|X|^2+|Y|^2 - \log(|X|^2+|Y|^2) \geq 2r^2 - \log(2r^2)) \\ &\leq \frac{1}{2r^2-\log(2r^2)}\iint (|x|^2+|y|^2-\log(|x|^2+|y|^2))\mu(dx)\mu(dy) \\ &\leq \frac{1}{2r^2-\log(2r^2)}\iint\left((|x|^2+|y|^2-\log(|x-y|^2)+\log(\frac{|x-y|^2}{|x|^2+|y|^2})\right)\mu(dx)\mu(dy) \\ &\leq \frac{2I(\mu)+1}{r^2-\log(2r^2)/2}.\end{aligned}$$

This immediately implies the compactness of the level set, for

$$\{\mu : I(\mu) \leq L\} \subset \cap_{n\geq\lceil r_0\rceil}\{\mu : \mu(B_n^c) \leq \sqrt{\frac{2L+1}{n^2-\log(2n^2)/2}}\}.$$

That $I(\cdot)$ is convex is proved exactly as the proof of 4) in property 2.1 of Ben Arous G., Guionnet A. (1997). Since we do not need this property,

we do not reproduce the proof. Thus one needs only show that $I \geq 0$. This follows from the lower bound below (Lemma 2.5), since $K = 3/8 = K_1$ where K_1 is defined in (2.7). □

LEMMA 2.2. *Assume that $\mu \in M_1^S(\mathbb{C})$ does not possess atoms, and that $I(\mu) < \infty$. Let $\mu_\varepsilon = \mu * \gamma_\varepsilon$, where γ_ε is the standard centered Gaussian law on $\mathbb{C}^2$ of εI covariance ($*$ denotes the convolution). Then $\mu_\varepsilon \longrightarrow_{\varepsilon\to 0} \mu$ and*

$$I(\mu_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} I(\mu). \tag{2.3}$$

Proof. The first assertion is obvious, for if (Z, n_ε) denote independent random variables distributed according to $\mu \times \gamma_\varepsilon$, then $Z_\varepsilon = Z + n_\varepsilon$ is distributed according to μ_ε.

Turning to the proof of (2.3), note that by the lower semicontinuity of $I(\cdot)$ and the convergence of the second moment of n_ε, it suffices to prove that

$$\limsup_{\varepsilon\to 0} \; -\Sigma(\mu_\varepsilon) \leq -\Sigma(\mu). \tag{2.4}$$

Toward this end, note that

$$\begin{aligned} -\Sigma(\mu_\varepsilon) &= \int_{\mathbb{C}}\int_{\mathbb{C}} \log\frac{1}{|x-y|}\mu_\varepsilon(dx)\mu_\varepsilon(dy) \\ &= E\log\frac{1}{|Z - Z' + n_\varepsilon - n'_\varepsilon|} \\ &= E\log\frac{1}{|Z-Z'|} + E\log\frac{1}{\left|1+\frac{n_\varepsilon - n'_\varepsilon}{Z-Z'}\right|} \\ &= -\Sigma(\mu) + E\log\frac{1}{\left|1+\frac{\tilde{n}_\varepsilon}{Z-Z'}\right|} \end{aligned} \tag{2.5}$$

where (Z, Z') are independent copies of Z, $(n_\varepsilon, n'_\varepsilon)$ are independent copies of n_ε, and $\tilde{n}_\varepsilon = \sqrt{2}n_\varepsilon$. (Note that we have used here the fact that μ possesses no atoms). Thus, the lemma follows from (2.5) as soon as we show that

$$\limsup_{\varepsilon\to 0} E\left(\mathbf{1}_{\{\left|1+\frac{\tilde{n}_\varepsilon}{Z-Z'}\right|\leq 1\}}\log\frac{1}{\left|1+\frac{\tilde{n}_\varepsilon}{Z-Z'}\right|}\right) \leq 0. \tag{2.6}$$

Define now, for $a \in \mathbb{C}$,

$$J_a = E\left(\mathbf{1}_{\{\left|1+\frac{\tilde{n}_\varepsilon}{a\sqrt{\varepsilon}}\right|\leq 1\}}\log\frac{1}{\left|1+\frac{\tilde{n}_\varepsilon}{a\sqrt{\varepsilon}}\right|}\right).$$

Fix $\alpha > 0$ (eventually, we take $\alpha \to \infty$). Then, for some universal constant C_1 independent of ε, α, a, and all α large enough,

$$\begin{aligned} J_a &= \frac{1}{2\pi}\int \mathbf{1}_{\{|1+y/a|\le 1\}} e^{-|y^2|/2} \log\frac{1}{|1+y/a|}dy \\ &= a^2\frac{1}{2\pi}\int \mathbf{1}_{\{|1+z|\le 1\}} e^{-a^2|z^2|/2} \log\frac{1}{|1+z|}dz \\ &\le \mathbf{1}_{\{\alpha>|a|\}} \quad |a|^2 \int_{|z+1|\le 1} \log\frac{1}{|z+1|}dz \\ &\quad + \mathbf{1}_{\{\alpha\le|a|\}} \quad |a|^2 \int_{\substack{|z+1|\le 1\\ |z|\le|a|^{-3/4}}} \log\frac{1}{|z+1|}dz \\ &\quad + \mathbf{1}_{\{\alpha\le|a|\}} \quad |a|^2 e^{-\sqrt{|a|}/4} \int_{\substack{|z+1|\le 1\\ |z|>|a|^{-3/4}}} \log\frac{1}{|z+1|}dz \\ &\le C_1\left(\mathbf{1}_{\{\alpha>|a|\}} \quad \alpha^2 + \mathbf{1}_{\{\alpha\le|a|\}}\left(\frac{1}{|a|^{1/4}} + |a|^2 e^{-\sqrt{|a|}/4}\right)\right) \end{aligned}$$

where we have used that $|\log|1+z|\,| \le 2|z|$ for $|z|$ small enough in bounding the second term. Hence,

$$\begin{aligned} &\limsup_{\varepsilon\to 0} E\left(\mathbf{1}_{\{\left|1+\frac{\tilde n_\varepsilon}{Z-Z'}\right|\le 1\}} \log\frac{1}{\left|1+\frac{\tilde n_\varepsilon}{Z-Z'}\right|}\right) \\ &= \limsup_{\varepsilon\to 0} E\left(J_{\frac{Z-Z'}{\sqrt{\varepsilon}}}\right) \\ &\le C_1 \limsup_{\varepsilon\to 0}\left(\alpha^2\mu\times\mu(\{Z,Z' : |Z-Z'|<\alpha\sqrt{\varepsilon}\}) + \alpha^{-1/4} + \sup_{\theta>\alpha}\theta^2 e^{-\sqrt{\theta}/4}\right) \\ &= C_1\left(\alpha^{-1/4} + \sup_{\theta>\alpha}\theta^2 e^{-\sqrt{\theta}/4}\right) \end{aligned}$$

where we used in the last equality the fact that μ possesses no atoms and therefore $\mu\times\mu(\{Z,Z' : Z = Z'\}) = 0$. Taking now $\alpha\to\infty$ completes the proof of the lemma. □

The key to the asymptotics of $\overline{Q}^N$ lies in an exact Jacobian computation. For $\mu \in M_1^S(\mathbb{C})$, let $k(\mu) = N\mu\Big(\{z : \mathrm{Im}(z) = 0\}\Big)$. Clearly, $0 \le k(\hat\mu^N) \le N$ is an integer. Define $\ell(\mu) = \frac{N-k(\mu)}{2}$, then $0 \le \ell(\hat\mu^N) \le \lfloor\frac{N}{2}\rfloor$ is again an integer.

For any sequence $\underline{\lambda} = \{\lambda_j\}_{j=1}^k$, $\underline{X} = \{X_j\}_{j=1}^\ell$, $\underline{Y} = \{Y_j\}_{j=1}^\ell$, define $\mu^{\underline{\lambda},\underline{X},\underline{Y}}(dz)$ in $M_1^S(\mathbb{C})$ by

$$\mu^{\underline{\lambda},\underline{X},\underline{Y}} = \frac{1}{N}\left(\sum_{j=1}^k \delta_{\lambda_j} + \sum_{j=1}^\ell\Big(\delta_{X_j+iY_j} + \delta_{X_j-iY_j}\Big)\right).$$

Fix

$$C_N = \frac{2^{-N(N+1)/4}N^{N(N-1)/4}}{\prod_{i=1}^N \Gamma\left(\frac{i}{2}\right)},\ K_1 := \lim_{N\to\infty}\frac{1}{N^2}\log C_N = 3/8, \qquad (2.7)$$

where the equality $K_1 = 3/8$ is obtained by directly evaluating the limit.

LEMMA 2.3. *For any measurable $A \subset M_1^S(\mathbb{C})$,*

$$\overline{Q}^N(\hat{\mu}^N \in A) =$$
$$C_N \sum_{\ell=0}^{\lfloor \frac{N}{2} \rfloor} (\sqrt{N})^N \int \cdots \int \prod_{j=1}^{k} d\lambda_j \prod_{j=1}^{\ell} dX_j \prod_{j=1}^{\ell} dY_j \mathbf{1}_{\{\mu^{\underline{\lambda},\underline{X},\underline{Y}} \in A, \ell(\mu^{\underline{\lambda},\underline{X},\underline{Y}})=\ell\}}$$
$$\cdot 2^{2\ell} \exp\left(-\frac{N^2}{2} \int_{\mathbb{C}} |x|^2 \mu^{\underline{\lambda},\underline{X},\underline{Y}}(dx)\right) \prod_{j=1}^{\ell} \left(\sqrt{N} Y_j \operatorname{erfc}\left(Y_j \sqrt{2N}\right) e^{2Y_j^2 N}\right)$$
$$\cdot \exp\left(\frac{N^2}{2} \iint_{\substack{x \neq y \\ x \neq y^*}} \log|x-y| \mu^{\underline{\lambda},\underline{X},\underline{Y}}(dx) \mu^{\underline{\lambda},\underline{X},\underline{Y}}(dy)\right).$$

Here,

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \int_x^{\infty} e^{-\theta^2} d\theta.$$

Proof. see Edelman A. (1997) or Lehmann N, Sommers H.J. (1991).

We next turn to the proof of a preliminary upper bound. Define $f(x,y) = \frac{|x|^2+|y|^2}{2} - \log|x-y|$.

LEMMA 2.4. *Let $\nu \in M_1^S(\mathbb{C})$ be given, and denote by $B(\nu,\delta)$ the (Lévy) ball in $M_1^S(\mathbb{C})$ of radius δ centered at ν. Then,*

$$\lim_{\delta \searrow 0} \limsup_{N \to \infty} N^{-2} \log \overline{Q}^N\left(B(\nu,\delta)\right) \leq -\left(\frac{1}{2}\langle f\nu,\nu\rangle - K_1\right).$$

Proof. The proof mimics the argument in Section 3.1 of Ben Arous G., Guionnet A. (1997). Note that

$$\iint_{\text{or } \substack{x=y \\ x=y*}} \hat{\mu}^N(dx)\hat{\mu}^N(dy) = \frac{N + 2\ell(\hat{\mu}^N)}{N^2} \leq \frac{2}{N}.$$

For $M \in \mathbb{R}^+$, define $f_M(x,y) = \min(M, f(x,y))$. Using the bound $x \operatorname{erfc}(\sqrt{2}x) e^{2x^2} \leq C_1$ for some universal constant C_1, one concludes from Lemma 2.3 that

$$\overline{Q}^N\left(B(\nu,\delta)\right) \leq C_N (\sqrt{N})^N B_N e^{2MN}$$
$$\cdot \sum_{\ell=0}^{\lfloor \frac{N}{2} \rfloor} \int \cdots \int \prod_{j=1}^{k} d\lambda_j \prod_{j=1}^{\ell} dX_j \prod_{j=1}^{\ell} dY_j \exp\left(-\frac{1}{4}\sum_{j=1}^{k} \lambda_j^2 - \frac{1}{4}\sum_{j=1}^{\ell}(X_j^2 + Y_j^2)\right)$$
$$\cdot \mathbf{1}_{\{\mu^{\underline{\lambda},\underline{X},\underline{Y}} \in B(\nu,\delta) \cap \ell(\mu^{\underline{\lambda},\underline{X},\underline{Y}})=\ell\}} \exp\left(-\frac{N^2}{2}\left(1-\frac{2}{N}\right)\left\langle f_M \mu^{\underline{\lambda},\underline{X},\underline{Y}}, \mu^{\underline{\lambda},\underline{X},\underline{Y}}\right\rangle\right) e^{2MN}$$

where $\log B_N = O(N)$. Hence, with $\log B_N^{(1)} = o(N^2)$,

$$\overline{Q}^N\Big(B(\nu,\delta)\Big)$$
$$\leq C_N B_N^{(1)} \sum_{\ell=0}^{\frac{N}{2}} \int\cdots\int \prod_{j=1}^{k} d\lambda_j \prod_{j=1}^{\ell} dX_j \prod_{j=1}^{\ell} dY_j \exp\left(-\frac{1}{4}\sum_{j=1}^{k}\lambda_j^2 - \frac{1}{4}\sum_{j=1}^{\ell}(X_j^2+Y_j^2)\right)$$
$$\cdot \mathbf{1}_{\{\underline{\mu_\lambda},\underline{X},\underline{Y}\in B(\nu,\delta)\cap\ell(\underline{\mu_\lambda},\underline{X},\underline{Y})=\ell\}} \exp\left(-\frac{N^2}{2}\left(1-\frac{2}{N}\right)\inf_{\mu\in B(\nu,\delta)}\langle f_M\mu,\mu\rangle\right),$$

implying that

$$\limsup_{N\to\infty}\frac{1}{N^2}\log\overline{Q}^N\Big(B(\nu,\delta)\Big) \leq \lim_{N\to\infty}\frac{1}{N^2}\log C_N - \frac{1}{2}\inf_{\mu\in B(\nu,\delta)}\langle f_M\mu,\mu\rangle\,.$$

Since $\mu\mapsto\langle f_M\mu,\mu\rangle$ is continuous in the weak topology, we obtain that

$$\lim_{\delta\searrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\overline{Q}^N\Big(B(\nu,\delta)\Big) \leq K_1 - \frac{1}{2}\langle f_M\nu,\nu\rangle$$

and the lemma follows by monotone convergence, taking $M\to\infty$. □

The complementary lower bound is given by:

LEMMA 2.5. *Let $\nu\in M_1^S(\mathbb{C})$ be given, and denote by $B(\nu,\delta)$ the (Lévy) ball of radius δ centered at ν. Then*

$$\lim_{\delta\searrow 0}\liminf_{N\to\infty} N^{-2}\log\overline{Q}^N\Big(B(\nu,\delta)\Big) \geq -\left(\frac{1}{2}\langle f\nu,\nu\rangle - K_1\right).$$

Proof. By considering $\nu_\varepsilon = \nu * \gamma_\varepsilon$ and using Lemma 2.2, it is clear that we may consider only $\nu\in M_1^S$ which possesses no atoms and whose density g_ν with respect to Lebesgue's measure is continuous and positive everywhere.

Let

$$\begin{aligned} X_- &= \max\Big\{x:\ \nu\Big([x,\infty)\times(-\infty,\infty)\Big)\geq 1-\frac{\eta}{8}\Big\}\\ X_+ &= \min\Big\{x:\ \nu\Big((-\infty,x]\times(-\infty,\infty)\Big)\geq 1-\frac{\eta}{8}\Big\}\\ \overline{Y} &= \min\Big\{y:\ \nu\Big((-\infty,\infty)\times[0,y]\Big]\geq 1-\frac{\eta}{8}\Big\}. \end{aligned}$$

Let $D=[X_-,X_+]\times[0,\overline{Y}]$, $D^F=[X_-,X_+]\times[-\overline{Y},\overline{Y}]$, and note that $\nu(D)\geq \frac{1}{2}-\frac{\eta}{2}$. Let C be such that, on D, the density g_ν of ν with respect to Lebesgue's measure satisfies $C\geq g_\nu\geq 1/C$.

For simplicity, we assume in the sequel that N is even, the case of N odd requiring only obvious modifications. Fix $\delta'>0$. Modifying slightly X_-, X_+ and $\overline{Y}$ if necessary, partition D into disjoint squares $\{B_\ell\}_{\ell=1}^L$ of length $\sqrt{C/N\delta'}$ and centers Z_ℓ. Note that $C^2/N\delta'\geq\nu(B_\ell)\geq 1/N\delta'$ and hence $N\delta'/2\geq L\geq N\delta'/2C^2$. Fix $\alpha_\ell=\lfloor N\nu(B_\ell)\rfloor$, then $\lceil C^2/\delta'\rceil\geq\alpha_\ell\geq\lfloor 1/\delta'\rfloor$, and $1/2-\delta'-\eta\leq 1/N\sum_{\ell=1}^L\alpha_\ell\leq 1/2$.

Next, for each $\ell\in\{1,\cdots,L\}$, define a sequence $\{\overline{Z}_{\ell,j}\}_{j=1}^{\alpha_\ell}$, $\overline{Z}_{\ell,j}\in B_\ell$, satisfying the following properties:

a) $d(\overline{Z}_{\ell,j},\partial B_\ell)\geq\frac{1}{4}\sqrt{\frac{C}{N\delta'}}$

b) $|\overline{Z}_{\ell,j}-\overline{Z}_{\ell,i}|\geq\frac{1}{4\sqrt{NC}}$, $i\neq j$

(such a sequence always exists due to our construction of α_ℓ).

Finally, for $k \le \sum_{\ell=1}^{L} \alpha_\ell$, define $i_k = \max\left\{\ell' : \sum_{\ell=1}^{\ell'} \alpha_\ell < k\right\}$ and $\tilde{Z}_k = \overline{Z}_{i_k+1, k-\sum_{\ell=1}^{i_k}\alpha_\ell}$; Let $\mathcal{K} := \{\sum_{\ell=1}^{L}\alpha_\ell + 1, \ldots, N/2\}$, and let $\{\tilde{Z}_k\}_{k\in\mathcal{K}}$ be $K = \frac{N}{2} - \sum_{\ell=1}^{L}\alpha_\ell \le (\eta+\delta')N$ points equidistributed along the curve

$$\overline{D}_B = \left\{ z \in D^c : \ \mathrm{Im}\,(z) > \frac{1}{2}\,,\ d(z,D) = 1 \right\}. \tag{2.8}$$

Let $\overline{\nu} = \frac{1}{N}\sum_{k=1}^{N/2}(\delta_{\tilde{Z}_k} + \delta_{\tilde{Z}_k^*})$, then $\rho(\nu,\overline{\nu}) \le 2(\delta'+\eta)$. We remark that with this construction, for some $C(\eta,\delta')$,

$$|\tilde{Z}_j - \tilde{Z}_i| \ge \frac{C(\eta,\delta')}{N}, \quad i \ne j. \tag{2.9}$$

Fix now $\varepsilon > 0$ small enough. Returning to the proof of the lower bound, we have, by Lemma 2.3 (recall that N is assumed even, and reduce δ', η if needed).

$$\overline{Q}^N\Big(B(\nu,\delta)\Big) \ge C_N \int\cdots\int_{D_I^\varepsilon} \prod_{j=1}^{N/2} dx_j dy_j \exp\left(-N\sum_{j=1}^{N/2}(x_j^2+y_j^2)\right)$$
$$\cdot \prod_{j=1}^{N/2} \sqrt{N} y_j \,\mathrm{erfc}\,(y_j\sqrt{2N}) e^{-2y_j^2 N} \exp\left(\sum_{i\ne j}\log|z_i - z_j|\,|z_i - z_j^*|\right)$$

where $D_I^\varepsilon = \left\{(x_j,y_j) : |(x_j,y_j) - \tilde{Z}_j| \le \frac{\varepsilon}{N}\right\}$ and $z_j = (x_j,y_j)$.

Due to property a) and (2.8) we have that in D_I^ε, for $\varepsilon < \frac{1}{2}$,

$$\sqrt{N} y_j \,\mathrm{erfc}\,(y_j\sqrt{2N}) e^{-2y_j^2 N} \ge \frac{C_1}{N}$$

for some universal constant C_1. Furthermore, in D_I^ε

$$N\sum_{j=1}^{N/2}|z_j|^2 \le \left(N + \frac{1}{\sqrt{N}}\right)\sum_{j=1}^{N/2}|\tilde{Z}_j|^2 + 1.$$

Therefore, denoting by B_N a quantity (which may change from line to line) which satisfies $\frac{1}{N^2}\log B_N \longrightarrow 0$, one finds

$$\overline{Q}^N\Big(B(\nu,\delta)\Big) \ge C_N B_N \exp\left(-(N+2)\right)\sum_{j=1}^{N/2}|\tilde{Z}_j|^2$$
$$\cdot \int\cdots\int_{D_I^\varepsilon} \exp\left(\sum_{i\ne j}\log|z_i - z_j|\,|z_i - z_j^*|\right)\prod_{j=1}^{N/2} e^{-(x_j^2+y_j^2)/2} dx_j dy_j. \tag{2.10}$$

Next, note that due to (2.9), property a) and the definition of $\tilde{Z}_j$, we have

$$|z_i - z_j| \ge \left(1 - \frac{2\varepsilon}{C(\eta,\delta')}\right)|\tilde{Z}_i - \tilde{Z}_j|, \quad i \ne j$$

and

$$|z_i - z_j^*| \ge \left(1 - \frac{2\varepsilon}{C(\eta,\delta')}\right)|\tilde{Z}_i - \tilde{Z}_j^*|$$

Therefore,

$$\begin{aligned}
\overline{Q}_N\Big(B(\nu,\delta)\Big) &\geq B_N C_N \exp\left(-\left(N+\frac{1}{\sqrt{N}}\right)\sum_{j=1}^{N/2}|\tilde{Z}_j|^2\right)\\
&\cdot \exp\left(\sum_{i\neq j}\log|\tilde{Z}_i-\tilde{Z}_j|\,|\tilde{Z}_i-\tilde{Z}_j^*|\right)\\
&\cdot \exp\left(-N^2\log\left(1-\frac{2\varepsilon}{C(\eta,\delta')}\right)\right)\iint_{D_I^\varepsilon}\prod_{j=1}^{N/2}e^{-(x_j^2+y_j^2)/2}dx_j dy_j\\
&\geq B_N^{(\varepsilon)} C_N \exp\left(-\left(N+\frac{1}{\sqrt{N}}\right)\sum_{j=1}^{N/2}|\tilde{Z}_j|^2\right)\\
&\qquad \exp\left(\sum_{i\neq j}\log|\tilde{Z}_i-\tilde{Z}_j|\,|\tilde{Z}_i-\tilde{Z}_j|\right) \qquad (2.11)
\end{aligned}$$

where

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{1}{N^2}\log B_N^{(\varepsilon)}=0.$$

Next, for each ℓ, let

$$G_\ell^\varepsilon=\left\{\ell' :\ \max_{\substack{x\in B_\ell\\ y\in B_{\ell'}}}|x-y|>(1+\varepsilon)\min_{\substack{\tilde{Z}_k\in B_\ell\\ \tilde{Z}_{k'}\in B_{\ell'}}}|\tilde{Z}_k-\tilde{Z}_{k'}|\right\}$$

Note that

$$|G_\ell^\varepsilon|\leq\left(\frac{2}{\varepsilon}\right)^2.$$

Then, using the fact that for N large enough and for $x\in B_\ell, y\in B_{\ell'}$ with $\ell'\in G_\ell^\varepsilon$, one has that $\log|x-y|\,|x-y^*|\leq 0$,

$$\begin{aligned}
&\frac{1}{2}\iint_{D^F}\nu(dx)\nu(dy)\log|x-y| = \iint_D \nu(dx)\nu(dy)\log|x-y|\,|x-y^*|\\
&=\sum_{\substack{\ell\\ \ell'\notin G_\ell^\varepsilon}}\iint_{\substack{x\in B_\ell\\ y\in B_{\ell'}}}\nu(dx)\nu(dy)\log|x-y|\,|x-y^*|\\
&+\sum_{\substack{\ell\\ \ell'\in G_\ell^\varepsilon}}\iint_{\substack{x\in B_\ell\\ y\in B_{\ell'}}}\nu(dx)\nu(dy)\log|x-y|\,|x-y^*|\\
&\leq \sum_{\{i,j:\,\tilde{Z}_i\in B_\ell,\,\tilde{Z}_j\notin\cup_{\ell'\in G_\ell^\varepsilon}B_{\ell'}\}}\frac{\nu(B_\ell)\nu(B_{\ell'})}{\alpha_\ell\alpha_{\ell'}}\Big(\log|\tilde{Z}_i-\tilde{Z}_j|\,|\tilde{Z}_i-\tilde{Z}_j^*|\Big)+2\log(1+\epsilon).
\end{aligned}$$

Hence, for some constant $C_2 = C_2(\varepsilon, \delta', \eta)$,

$$\begin{aligned}
\frac{1}{2}\iint_{D^F} \nu(dx)\nu(dy)\log|x-y| &= \iint_{D} \nu(dx)\nu(dy)\log|x-y|\,|x-y^*| \\
&\le \sum_{i\neq j} N^{-2}\log|\tilde{Z}_i-\tilde{Z}_j|\,|\tilde{Z}_i-\tilde{Z}_j^*| \\
&+ \sum_{\{i,j:\,\tilde{Z}_i\in B_\ell,\,\tilde{Z}_j\notin\cup_{\ell'\in G_\ell^\varepsilon}B_{\ell'}\}} \left|N^{-2}-\frac{\nu(B_\ell)}{\alpha_\ell}\frac{\nu(B_{\ell'})}{\alpha_{\ell'}}\right|\left|\log|\tilde{Z}_i-\tilde{Z}_j|\,|\tilde{Z}_i-\tilde{Z}_j^*|\right| \\
&\quad + \frac{C_2}{N^2}N\left(\frac{2}{\varepsilon}\right)^2\log N \\
&- \sum_{i\in\mathcal{K},j\neq i} N^{-2}\log|\tilde{Z}_i-\tilde{Z}_j|\,|\tilde{Z}_i-\tilde{Z}_j^*| + 2\log(1+\epsilon). \qquad (2.12)
\end{aligned}$$

Note however that by construction,

$$\begin{aligned}
&- \sum_{i,j\in\mathcal{K},j\neq i} N^{-2}\log|\tilde{Z}_i-\tilde{Z}_j|\,|\tilde{Z}_i-\tilde{Z}_j^*| \\
&\le (\eta+\delta')N^{-1}\sum_{j=1}^{N}|\log(N/j)|X_+-X_-|\,| \\
&\le (\eta+\delta')\left(|\log|X_+-X_-|\,|+1\right). \qquad (2.13)
\end{aligned}$$

On the other hand,

$$-\sum_{i\in\mathcal{K},j\notin\mathcal{K}} N^{-2}\log|\tilde{Z}_i-\tilde{Z}_j|\,|\tilde{Z}_i-\tilde{Z}_j^*| \le (\eta+\delta')N^{-1}\sum_{j=1}^{N}\log 2\,. \qquad (2.14)$$

Substituting (2.12), (2.13) and (2.14) in (2.11), we obtain

$$\begin{aligned}
&\liminf_{N\to\infty}\frac{1}{N^2}\log\overline{Q}^N\Big(B(\nu,\delta)\Big) \\
&\ge \lim_{N\to\infty}\frac{1}{N^2}\log(C_N B_N^{(\varepsilon)}) - \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N/2}|\tilde{Z}_j|^2 \\
&\quad + \frac{1}{2}\iint_{D^F}\nu(dx)\nu(dy)\log|x-y| \\
&\quad - \lim_{N\to\infty}\frac{1}{N^2}\sum_{i\neq j}|1-\delta'|\Big|\log|\tilde{Z}_i-\tilde{Z}_j|\,|\tilde{Z}_i-\tilde{Z}_j^*|\Big| - \eta - \delta' - 2\log(1+\epsilon).
\end{aligned}$$

Taking first $\varepsilon\to 0$, then $\delta'\to 0$ and finally $\eta\to 0$, we obtain

$$\begin{aligned}
&\liminf_{N\to\infty}\frac{1}{N^2}\log\overline{Q}^N\Big(B(\nu,\delta)\Big) \\
&\ge \lim_{N\to\infty}\frac{1}{N^2}\log C_N - \frac{1}{2}\int|z|^2\nu(dz) + \frac{1}{2}\iint\log|x-y|\nu(dx)\nu(dy).
\end{aligned}$$

□

Proof of Theorem 1.1: The weak large deviations principle and the fact that $K = K_1$ (and thus that U minimizes $I(\cdot)$) follows from Lemmas 2.1, 2.4 and 2.5. The exponential tightness of $\overline{Q}^N$ can be proved exactly as property 4.1

in Ben Arous G., Guionnet A. (1997), and implies the full LDP for $\overline{Q}^N$. An alternative proof of the exponential tightness is obtained by observing that, due to lemma 2.3, with $A_{K,r} := \{\mu^{\underline{\lambda},\underline{X},\underline{Y}} : \mu^{\underline{\lambda},\underline{X},\underline{Y}}(B_r^c) > K\}$,

$$\begin{aligned}
&\overline{Q}^N(\hat{\mu}^N \in A_{K,r}) \\
&\leq g_N \max_{\ell=0,\ldots,[N/2]} \int \cdots \int \prod_{j=1}^{k} d\lambda_j \prod_{j=1}^{\ell} dX_j \prod_{j=1}^{\ell} dY_j \mathbf{1}_{\{\mu^{\underline{\lambda},\underline{X},\underline{Y}} \in A_K, \ell(\mu^{\underline{\lambda},\underline{X},\underline{Y}})=\ell\}} \\
&\qquad \cdot \exp\left(-\frac{N^2}{4}\int (|x|^2+|y|^2)\mu^{\underline{\lambda},\underline{X},\underline{Y}}(dx,dy)\right) \\
&\leq g_N \exp\left(-\frac{KN^2r^2}{4}\right),
\end{aligned}$$

where $N^{-2}\log g_N = O(1)$, B_r denotes the centered disc of radius r in the complex plane, and we have used the inequality $\log|x-y| \leq (|x|^2+|y|^2)/4+2$. The exponential tightness follows at once.

Point 2 in the statement of Theorem 1.1 is proved exactly as properties b.1, b.2 in Theorem 1.1 of Ben Arous G., Guionnet A. (1997). It thus remains to prove the uniqueness of U as a minimizer of $I(\cdot)$. To prove this, we begin by showing that if μ is a minimizer of $I(\cdot)$, then it must be of compact support. Indeed, fix R large (eventually, we take $R \to \infty$). With B_R denoting as above the centered disc of radius R in the complex plane, let $\theta := \mu(B_R^c)$, and write $\mu = (1-\theta)\mu^R + \theta\eta^R$, where μ^R is a probability measure supported on B_R and η^R is a probability measure supported on B_R^c. Define

$$\hat{\mu} := (1-\theta)\mu^R + \theta U.$$

Then,

$$\begin{aligned}
&2(I(\mu) - I(\hat{\mu})) \\
&= \theta\Big(\int_{\mathbb{C}} |x|^2\eta^R(dx) - \frac{1}{2}\Big) - 2\theta(1-\theta)\int_{\mathbb{C}}\int_{\mathbb{C}} \log|x-y|\mu^R(dx)(\eta^R-U)(dy) \\
&\quad -\theta^2\Big(\int_{\mathbb{C}}\int_{\mathbb{C}} \log|x-y|\eta^R(dx)\eta^R(dy) - (2K-\frac{1}{2})\Big). \qquad (2.15)
\end{aligned}$$

Note that for some constant C independent of R, for any $x \in B_R$,

$$\Big|\int_{\mathbb{C}} \log|x-y|U(dy)\Big| \leq C(1+\log R)\,,$$

while

$$\int_{\mathbb{C}} (\log|x-y|)_+\eta^R(dy) \leq \int_{\mathbb{C}} \log(3|y|)\eta^R(dy),$$

and, using the inequality $\log|x-y| \leq |\log|x|\,| + |\log|y|\,| + 2$,

$$\int_{\mathbb{C}}\int_{\mathbb{C}} \log|x-y|\eta^R(dx)\eta^R(dy) \leq 2(1+\int_{\mathbb{C}} \log|x|\eta^R(dx)).$$

Hence, substituting in (2.15), with C_1 denoting again a constant independent of R,

$$2(I(\mu) - I(\hat{\mu})) \geq \theta(\int_{\mathbb{C}} |x|^2 \eta^R(dx) - \frac{1}{2}) - 2\theta(1-\theta)(\int_{\mathbb{C}} \log(3|x|)\eta^R(dx) + C(1+\log R)) - C_1\theta^2(1 + \int_{\mathbb{C}} \log|x|\eta^R(dx)).$$

Since $\log R < R$ for R large enough and $\lim_{R\to\infty} \theta = 0$ but $\theta > 0$ for all R, one concludes that for some R large enough, $I(\mu) > I(\hat{\mu})$, leading to a contradiction.

Knowing that the minimizing μ is of compact support, we can now apply the potential theoretic arguments from Hille E. (1962). In fact, one follows verbatim the proof of Theorem 2.3b in Mhaskar H.N., Saff E.B. (1985). □

REMARK 2.6. An inspection of the proof reveals that a similar large deviations statement would be available either when other ensembles are considered such that a statement similar to Lemma 2.3 holds (possibly with different function replacing $|x|^2$) or when exponential weighting depending on the empirical measure of the eigenvalues is added. In the self-adjoint or in the unitary cases, this corresponds to well known families, we do not have however a concrete application in mind in the complex case and therefore have not pursued this direction here. We refer to Hiai F., Petz D. (1997a), Hiai F., D. Petz (1997b) for several large deviations statements for certain ensembles of matrices.

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