Growth and saturation in random media

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ABSTRACT. We report here on the recent works [**3**] and [**4**]. There we consider a model of diffusion in random media with a two-way coupling (i.e. a model in which the randomness of the medium influences the diffusing particles, and where the diffusing particles change the medium). In this particular model, particles are injected at the origin with a time-dependent rate, and diffuse among random traps. Each trap has a finite (random) depth, so that when it has absorbed a finite (random) number of particles it is "saturated", and it no longer acts as a trap. Related models have been studied recently by Gravner and Quastel [**10**] and by Funaki [**9**] using hydrodinamic limit tools. We compute the asymptotic behaviour of the probability of survival of a particle born at some given time, both in the annealed and quenched cases, and show that three different situations occur depending on the injection rate. For weak injection, the typical survival strategy of the particle is as in Sznitman [**16**] and the asymptotic behaviour of this survival probability behaves as if there was no saturation effect. For medium injection rate, the picture is closer to that of Internal DLA, as given by Lawler, Bramson and Griffeath [**13**]. For large injection rates, the picture is less understood except in dimension one.

1. Introduction.

We present a model of growth, diffusion and trapping in a random environment. This model has three main features: a random environment, an injection pattern and a two way coupling between random walks and the random environment (i.e. the random environment acts on the particles by trapping, and the particles act on the environment by saturation of the traps).

We describe rapidly the three ingredients of the model.

- Firstly, the random environment is given by a collection of i.i.d. integer valued random variables $\eta(x)$ at each site x of the lattice $\mathbb{Z}^d$. $\eta(x)$ is the initial depth (or capacity) of the trap at site x, with the convention that the site x is not a trap if the depth $\eta(x)$ is zero.
- Secondly the injection pattern: at the origin of the lattice $\mathbb{Z}^d$ particles are injected (or born) at a time dependent rate. We will mainly study deterministic

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injection patterns but most of the results are true with Poissonian injections. We will indicate later where random injection does make a difference.

- Finally the interaction between the medium and the random walks: when born, the particles perform continuous time simple random walks on the cubic lattice, until they find a non-saturated trap (i.e. visited by less particles than its initial depth). When meeting such a trap the particle stops and stays forever in this trap. The depth of the trap is then decreased by one. When a trap is full or saturated, i.e. when it has been visited by as many particles as its initial depth, or equivalently when its depth has reached zero, it no longer acts as a trap and particles can walk on it.

Our initial motivation came from a simplified version of a problem of confinement of heavy nucleotides in nuclear waste management by high-performance clay barriers (see [**5**]). This context suggested the random injection at one point and the trapping and saturation mechanism. A more complete study would ask for a model with interaction between the particles, and the possibility for "desorption" (i.e. for the particles to leave the traps after a long time). But other various motivations can be proposed.

The model we examine is flexible. For instance, it encompasses two models recently studied. Namely, the Poissonian traps model of A.S. Sznitman (see [**14**] and references within) and the continuous time Internal Diffusion Limited Aggregation (IDLA) model (introduced by Diaconis and Fulton [**6**]) of Lawler, Bramson and Griffeath [**12, 13**]. The analogue of the Poissonian traps model in the discrete context of the cubic lattice corresponds in our model to the situation where saturation of the traps is omitted. For instance, this would be the case where the injection is limited to the injection of only one particle. Or equivalently, the case when the initial depth of the traps is infinite, whatever the injection rate is. IDLA deals with the case in which all sites are initially traps of depth 1 (no randomness of the medium), and the injection has a specific rate (i.e. the total number of particles born at time t is a Poisson process of constant intensity). The trapping and saturation mechanism is sometimes called "noise reduction" in the literature about growth models (see paragraph 4.1 of Krug and Spohn [**11**] in the context of the Eden model where the analogue of the depth of traps is non-random).

We have chosen to deal with the simplest possible description of the initial randomness of the medium (i.e. i.i.d. distribution of initial depths of traps) in order to use the very powerful machinery developed by A.S. Sznitman (see [**17, 15, 14**]). We had to adapt to the discrete context his latest version of the "enlargement of obstacles" method (see [**15, 14**]). We have limited ourself to the case of bounded depths. In fact what we really had in mind was the situation where $\eta(x)$ could take only the values 0 (the site x is then not a trap) and $m \in \mathbb{N}$. Situations where very deep traps would be present could produce very different behaviours.

After this description of the model we now state some of the natural questions about this model. There are at least four types of such questions (ordered from the simplest to the most difficult one). The first one is about the shape of the saturated set.

1. What is the shape of the set of saturated traps?

The second question is about survival probabilities.

2. What is the proportion of live particles at time t? What is their age distribution?

More precisely, what is the probability of survival of the k-th born particle until time t? This question can be asked first when k is fixed and the when both k and t go to infinity.
The third question is about the location of live particles.

3. What is the typical path of the k-th born particle if it is conditioned to live until time t?

And finally the last question is about the collective behaviour of the live particles.

4. What is the profile of the cloud of live particles at time t?

Gravner and Quastel [**10**] deal with the fourth question. In the context of IDLA (with zero-range dynamics), they prove among other things that when $d = 2$, under an hydrodynamic scaling limit, the profile of the cloud of live particles converges weakly in probability to the solution of the one phase Stefan problem with a source at the origin. An extension of this result to the model treated here is reported in [**3**].

The results of [**4**] concern the first two questions. The third one will be treated elsewhere. Before describing our results, we recall that two main lines of statements are possible: annealed and quenched. The first corresponds to an almost sure statement (theorem 2) and the second to a statement in average (theorem 3) with respect to the randomness of the medium. We will give results in both situations, though we believe that the most important are the quenched ones. The main result of this paper is the existence of three very different situations depending on the strength of the injection. We call naturally these the low, medium and high injection regimes. Let us call $N(t)$ the number of particles that have been born at time t. The high injection regime is reached when $N(t) \gg t^{d/2}$, both in the annealed and quenched situations. It differs from the other two injection regimes by the fact that most particles will survive (will not be trapped). In this high injection regime, we do not know too much about the answer to question 1 (the shape of the saturated region), except when the lattice has dimension 1 (this is reported elsewhere [**3**]).

Here we will concentrate on the two other regimes. We will see that in both of them, there is a growing saturated zone, spherical with a radius growing as $\left(\frac{1}{aw_d}N(t)\right)^{1/d}$, where a is the average depth of each obstacle and w_d the volume of a sphere of unit radius. We will also see that in both these regimes, the survival probability tends to zero, but at very different rates. The reason for this difference can be roughly explained as follows. When the injection rate is too low ($N(t) \ll \ln t$ in the quenched case and $N(t) \ll t^{d/(d+2)}$ in the annealed case), the saturated zone is too small to really matter for survival and the saturation effect is irrelevant, so that the result is essentially given by the survival probability in the Poissonian traps model without saturation. When the injection rate is in the medium range ($\ln t \ll N(t) \ll t^{d/2}$ in the quenched case and $t^{d/(d+2)} \ll N(t) \ll t^{d/2}$ in the annealed case), the saturated zone is large enough to modify the survival probability, which can now heuristically be computed as the probability that a Brownian motion does not cross some spherical moving boundary.

The main difficulty in the proof of the large deviation estimates of theorems 2 and 3, providing the quenched and annealed logarithmic asymptotics of the survival probability of a single particle, corresponds to the proof of the upper bounds. For the quenched and annealed medium regimes, we have not a good enough control of the probability that the saturated set of obstacles is not a ball. This means that the shape theorem of section 2 (theorem 1), answering question one and stating that with "high" probability the set of saturated traps corresponds to erasing obstacles within a ball of a certain radius, can not be used for the upper bounds, and therefore all possible shapes for the saturated set at some given time have to be considered. With the exception of part (ii) of the annealed theorem 3, an understanding about the asymptotic behaviour of the principal Dirichlet eigenvalue of the discrete Laplacian on large sets with random absorbing obstacles is needed. It is the case that the smallest possible value that one can obtain for this principal Dirichlet eigenvalue after erasing a high enough predetermined amount of obstacles, corresponds to erasing a ball. This is the content of part (ii) of theorem 4, which is proved by means of an adaptation of the latest version of the enlargement of obstacles technique of Sznitman [**14, 15**]. An analogue analysis is required for the proof of the upper bound of theorem 2 in the low regime. We would like to remark that the use of an adaptation of the latest version of the enlargement of obstacle, where different scales are introduced for the so called bad and density sets, has been crucial to obtain the upper bounds for injection rates close to the critical ones ($N(t) \sim \ln t$ in the quenched case and $N(t) \sim t^{d/(d+2)}$ in the annealed situation). As part of the proof of theorem 4 mentioned above, a discrete version of Faber-Krahn inequality was needed. It might be the case that this precise estimate is known and has already been proved, but we where unable to find the proper references.

The organisation of the paper is as follows. In section 2 we introduce the model together with the notation that will be subsequently used, and state the results. The detailed answers to question 1 are given in theorem 1. Theorems 2 and 3 deal with question 2, in the quenched and annealed situation respectively. In section 3 a sketch of the proof of the annealed logarithmic asymptotics of the survival probabilities (theorem 2) is given. The complete details of this proof, including the construction of the enlargement of obstacle method for the cubic lattice, and the version of Faber-Krahns inequality mentioned above, is given in [**4**]. In the final section of this paper, some other related results are discussed. This includes a shape theorem for the high injection regime ($N(t) \gg t^{d/2}$) for $d = 1$ and an almost sure hydrodynamic limit result for critical injections ($N(t) = Ct^{d/2}$), for some constant C). The proofs of these results will appear in [**3**].

2. Notation and Results.

In what follows we will define a stochastic process corresponding to the dynamics described in the introduction of random walks in a lattice with some absorbing sites or obstacles. Let m be some natural number and define $\mathcal{I} := \{n \in \mathbb{N} : 0 \le n \le m\}$.

The state space representing the obstacle configuration endowed with the natural topology will be denoted by $\Gamma := \mathcal{I}^{\mathbb{Z}^d}$. Let $\mathcal{B}$ be the corresponding Borel σ-algebra. Define for each finite set $F \in \mathbb{Z}$ the continuous projections π_F from Γ to $\mathcal{I}^F$. Given an element $\eta \in \Gamma$ we define its x-th coordinate by $\eta(x) := \pi_{\{x\}}\eta$. A site x such that $\eta(x) \geq 1$ represents a site with an obstacle present, while $\eta(x) = 0$ means that there is no obstacle. Furthermore, let P be the probability measure on $\Omega := D([0,\infty), \mathbb{Z}^d)^{\mathbb{N}}$ endowed with its Borel σ-algebra $\mathcal{D}$, under which the canonical coordinate process $\{Z_n : n \in \mathbb{N}\}$ represents independent simple random walks on $\mathbb{Z}^d$ of total jump rate 1, and such that $Z_n(0) = 0$. Let $\{T_n \in [0,\infty) : n \in \mathbb{N}\}$ be a sequence of strictly increasing times and define random walks $\{Y_n : n \in \mathbb{N}\}$ by $Y_n(t) := 0$ if $0 \leq t \leq T_n$ and $Y_n(t) := Z_n(t - T_n)$ if $t > T_n$. Let us define $N(t) := \sum_{n=1}^{\infty} \mathbf{1}_{[0,t]}(T_n)$, where $\mathbf{1}_B$ is the indicator function of $B \subset \mathbb{R}$, representing the total number of random walks that have been born at time t.

We now proceed to define a collection of probability measures $Q_{N,\eta}$ on the space $(\Omega, \mathcal{D})$, indexed by the set of right continuous increasing functions N from $[0,\infty)$ taking values on $\mathbb{N}$ and by the set of configurations $\eta \in \Gamma$, and which we will call the random saturation process. Under each measure $Q_{N,\eta}$, the canonical coordinate process $\{Z_n : n \in \mathbb{N}\}$ on Ω will represent the dynamics of interacting random walks. In the sequel it is understood that any infimum over an empty subset of $\mathbb{N}$ or $\mathbb{R}$ is infinity. For a given $n \in \mathbb{N}$, we define the stopping time

$$s_n^1 := \inf\{t \geq 0 : \eta(Y_n(t)) > 0\}$$

which is the first time that the random walk Y_n visits an obstacle. Now let

$$t_1 := \inf\{s_n^1 : n \in \mathbb{N}\}$$

This is the first time some obstacle has been visited. We now define,

$$Y_n^1(t) := \begin{cases} Y_n(t) & \text{if} \quad s_n^1 > t_1 \\ Y_n(t \wedge s_n^1) & \text{if} \quad s_n^1 = t_1 \end{cases} \tag{1}$$

Here we have stopped those random walks which hit a trap for the first time. Let $\mathbb{M}_1 := \{n \in \mathbb{N} : s_n^1 = t_1\}$ be the set of indices where the infimum in the definition of T_1 is attained. Similarly define $\mathbb{Z}_1 := \{x \in \mathbb{Z}^d : x = Y_n(s_n^1) \quad for \quad some \quad n \in \mathbb{M}_1\}$. It is easy to see that P-a.s. the set $\mathbb{M}_1$ has a unique element, which we will denote by n_1, and this is the index of the unique random walk which is stopped in (1). Therefore P-a.s. the set $\mathbb{Z}_1$ has a unique element x_1 such that $x_1 := Y_{n_1}(s_{n_1}^1)$. We now update the obstacle configuration by defining

$$\eta^1(x) := \begin{cases} \eta(x_1) - 1 & \text{if} \quad x = x_1 \\ \eta(x) & \text{if} \quad x \neq x_1 \end{cases}$$

In other words, we decrease by one the site which has the trap which has been visited first. Note again, that P-a.s. this site is unique. Now define recursively for $k \geq 2$, $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ the stopping times s_n^k and t_k, and the processes $\eta_t^k(x)$ and $Y_m^k(t)$, as follows

$$
\begin{aligned}
s_n^k &:= \inf\{t \geq 0 : \eta^{k-1}(Y_n^{k-1}(t)) > 0\} \\
t_k &:= \inf\{s_n^k : n \in \mathbb{N} \backslash (\mathbb{M}_1 \cup \cdots \cup \mathbb{M}_{k-1})\} \\
\mathbb{M}_k &:= \{n \in \mathbb{N} : s_n^k = t_k\} \\
\mathbb{Z}_k &:= \{x \in \mathbb{Z}^d : x = Y_n^k(s_n^k) \quad \text{for} \quad \text{some} \quad n \in \mathbb{M}_k\} \\
Y_n^k(t) &:= \begin{cases} Y_n^k(t) & \text{if} \quad s_n^k > t_k \\ Y_n^k(t \wedge s_n^k) & \text{if} \quad s_n^k = t_k \end{cases} \\
\eta^k(x) &:= \begin{cases} \eta^{k-1}(x) - 1 & \text{if} \quad x \in \mathbb{Z}_k \\ \eta^{k-1}(x) & \text{otherwise} \end{cases}
\end{aligned}
$$

Note that P-a.s. the sets $\mathbb{M}_k$ and $\mathbb{Z}_k$, for $k \geq 1$, have each a unique element. From the fact that the total number of random walks $\{Y_m : m \in \mathbb{N}\}$ in movement at a given time t is finite, it is not difficult to check that for each $n \in \mathbb{N}$, as $k \to \infty$ the sequence of processes Y_n^k converges P-a.s. on the Skorokhod topology of Ω. Let us call such a limit $X_n^{N,\eta}$. We then define $X^{N,\eta} := \{X_n^{N,\eta} : n \in \mathbb{N}\}$. Note that under the probability measure P, this process represents random walks which move freely until they visit the first site x which has received less than $\eta(x)$ visits, time at which they are freezed. To this process there corresponds a probability measure $Q_{N,\eta}$ under which the canonical coordinate process $Z := \{Z_n : n \in \mathbb{N}\}$ on Ω is distributed as $X^{N,\eta}$ under P. It is defined by $Q_{N,\eta}(A) := P(X^\eta \in A)$ for every set $A \subset \mathcal{D}$. In the sequel, we will say that Z under the probability measure $Q_{N,\eta}$ is a random saturation process on an obstacle configuration η and driven by an injection N. We will denote by $\tau_n := \inf\{t \geq 0 : Z_n(s) = Z_n(t) \ for \ s \geq t\}$, the time at which the random walk Z_n is frozen.

We will now endow the obstacle state space $(\Gamma, \mathcal{B})$ with a product probability measure μ defined by,

$$\mu(\eta(x) = \alpha) = p_\alpha$$

where $\sum_{\alpha \in \mathcal{I}} p_\alpha = 1$. Note that a random saturation process with injection $N(t) = [t]$ (where for $x \in \mathbb{R}$, $[x]$ represents the closest integer greater than or equal to x) and an obstacle configuration with law μ such that $\mathcal{I} = \{0, 1\}$ and $p_0 = 0, p_1 = 1$, corresponds closely to the IDLA model introduced by Lawler, Bramson and Griffeath [**13**] (see [**6**] for the original version of IDLA). In fact, the only difference is that in IDLA, the birth times $\{T_n : n \in \mathbb{N}\}$ are sums of exponentially distributed random variables. In contrast, a random saturation process with injection $N(t) = 1$ and obstacle configuration with a law given by μ is a random walk on a lattice where sites are absorbing independently of each other with some positive probability. This model was studied by Antal [**1, 2**] using an adaptation to the lattice of the first version of the enlargement of obstacle method developed by Sznitman's [**17**].

Now let,

$$\zeta(x,t) := \sum_{n\in\mathbb{N}} \mathbf{1}_{Z_n(t)}(x) \tag{2}$$

where for $A \subset \mathbb{Z}^d$, we define $\mathbf{1}_A : \mathbb{Z}^d \to \{0,1\}$ as the indicator function of the set A. $\zeta(x,t)$ represents the number of random walks at time t in site x. Then define,

$$S_t := \{x \in \mathbb{Z}^d : \zeta(x,t) \geq \eta(x) > 0\} \tag{3}$$

This set corresponds to the sites x of the cubic lattice $\mathbb{Z}^d$ which have an obstacle, and which been visited at least $\eta(x)$ times. We will call it the set of saturated obstacles at time t. In the sequel, for $x \in \mathbb{Z}^d$, we define the norm $||x|| := \sqrt{x_1^2 + \ldots + x_d^2}$, where for $1 \leq i \leq d$, x_i is the $i-th$ coordinate of x. An answer to question 1 of the introduction, asking about the shape of the set of saturated obstacles S_t is given by the following theorem.

THEOREM 1. *Consider a random saturation process on an obstacle configuration η and driven by an injection N. Let η be distributed according to some product measure μ and call $a := \mu(\eta(x))$ the average depth of obstacles at time* 0. *Define $B_r := \{x \in \mathbb{Z}^d : \eta(x) > 0 \quad \text{and} \quad ||x|| < r\}$. Assume that $1 \ll N(t) \ll \frac{t^{d/2}}{\ln t}$ and that $a > 0$. Then,*

(i) *For every $\epsilon > 0$, $Q_{N,\mu}$-a.s. there exists a $t_0 \geq 0$ such that*

$$B_{(1-\epsilon)\left(\frac{1}{aw_d}N(t)\right)^{1/d}} \subset S_t \subset B_{(1+\epsilon)\left(\frac{1}{aw_d}N(t)\right)^{1/d}}$$

whenever $t \geq t_0$.

(ii) *μ-a.s. the following is true: for every $\epsilon > 0$, $Q_{N,\eta}$-a.s. there exists a $t_0 \geq 0$ such that*

$$B_{(1-\epsilon)\left(\frac{1}{aw_d}N(t)\right)^{1/d}} \subset S_t \subset B_{(1+\epsilon)\left(\frac{1}{aw_d}N(t)\right)^{1/d}}$$

whenever $t \geq t_0$.

REMARK 1. *It is not difficult to see that if N is distributed according to some Poisson process R of time dependent rate $\lambda(t)$, the annealed and quenched versions with respect to R of parts (i) and (ii) of theorem 1, with N replaced by $\int_0^t \lambda(s)ds$ in the statement, are true.*

The proof of this theorem, based on arguments of Lawler, Bramson and Griffeath [**13**], will appear in [**4**]. Note that in the above shape theorem, at time t, the volume of the limiting sphere times the average depth of obstacles, equals the total number of particles $N(t)$ that have been born. This means that the proportion of frozen random walks converges to 1.

Let $k(t) : [0,\infty) \longrightarrow \mathbb{N}$ be an increasing function of time and let $g(t) := T_{k(t)}$ be the birth time of the random walk $Z_{k(t)}$. As explained in the Introduction, one of the natural questions (question 2) we want to answer is what is the asymptotic behaviour of the survival probability of the random walk $Z_{k(t)}$ with law given by $Q_{N,\eta}$, both when $k(t)$ is fixed as time goes to infinity, and when $k(t)$ goes to infinity together with time. To state the corresponding results let us introduce some notation. Let λ_d be the principal Dirichlet eigenvalue of the Laplacian operator divided $2d$ on the ball of unit radius of $\mathbb{R}^d$ and w_d its volume. Define $p := \mu(\eta(x) > 0)$, $a := \mu(\eta(x))$, $\bar{a} := \max\{n \in \mathcal{A}\}$ and denote by $p_c(d)$ the critical probability of site percolation on $\mathbb{Z}^d$. In the sequel we assume that $p > 0$. The first theorem is a quenched version of the asymptotics of the survival probability.

THEOREM 2. *Consider a random saturation process on an obstacle configuration η and driven by an injection N. Assume that $0 < N(t) \ll t^{d/2-\epsilon}$ for some $\epsilon \in (0,1)$, that $\limsup_{t\to\infty} k(t) > \bar{a}$ and that $t - g(t) \gg 1$. Then,*

(i) *Assume that $1 \ll N(t) \ll (t-g(t))^{d/2}$. If $\ln(t-g(t)) \ll N(t)$ or $p > p_c(d)$ then,*

$$\lim_{t\to\infty} \frac{1}{h_M(k,t)} \ln Q_{N,\eta}(\tau_{k(t)} > t) = -1 \qquad \mu - a.s. \tag{4}$$

where $h_M(k,t) := \lambda_d (a w_d)^{2/d} \int_{g(t)}^{t} \frac{ds}{N(s)^{2/d}}$.

(ii) *If $N(t) \ll \ln(t-g(t))$ and $p < p_c(d)$ then,*

$$\lim_{t\to\infty} \frac{1}{h_L(k,t)} \ln Q_{N,\eta}(\tau_{k(t)} > t) = -1 \qquad \mu - a.s.$$

where $h_L(k,t) = \lambda_d (w_d |\ln(1-p)|)^{2/d} \frac{t-g(t)}{(\ln(t-g(t)))^{2/d}}$.

REMARK 2. *The condition $N(t) \ll t^{-\epsilon+d/2}$ for some $\epsilon > 0$, is necessary to ensure the validity of the shape theorem 1. On the other hand, the less important condition $\liminf_{t\to\infty} k(t) \geq \bar{a}$ is included to rule out the possibility that the random walk born at time $g(t) = T_{k(t)}$ dies at the origin as soon as it is born. This might be the case when $k(t)$ is some constant smaller than $\bar{a}$, which is the maximum value of the obstacle capacity η at each site.*

Let us briefly discuss the meaning of the above result. For the sake of clarity, let us consider the case in which $k(t)$ is some constant grater than $\bar{a}$. When $p < p_c$, we know that μ-a.s. there exists a unique obstacle free cluster on the lattice $\mathbb{Z}^d$. The above theorem shows that for $N(t) \ll t^{-\epsilon+d/2}$, for some $\epsilon > 0$, there appear to be two different injection regimes when $p < p_c$. There is a regime which we will denote by *quenched low regime*, when $N(t) \ll \ln t$, given by part (ii). Here subscript L in h_L stands for *low*. The survival strategy for random walks in this regime consists essentially of travelling fast to a distance of order t to some region of the lattice free of obstacles and of radius of the order of $(\ln t)^{1/d}$. This is exactly the survival strategy of a Brownian motion on $\mathbb{R}^d$ with Poissonian obstacles (see Sznitman [**14**]) or of a simple random walk on the lattice with site obstacles distributed according to some product measure (see Antal [**2**]). There is a second injection regime for

$\ln t \ll N(t) \ll t^{-\epsilon+d/2}$, which we call *quenched medium regime*, given by part (i) of theorem 2. The subscript M in h_M stands for *medium*. Here random walks are provided with a better survival strategy than going far to find natural clearings, as in the low regime. Namely, by the shape theorem 1, the high enough injection produces a central clearing larger than those that can be found far away. Thus, the typical survival strategy of a particle is to stay in this central region. When $p > p_c$, so that μ-a.s. there is no infinite trap free cluster, theorem 2 states that for any injection rate satisfying the condition $N(t) \ll t^{-\epsilon+d/2}$, the decay of the survival probability is as in the medium regime. For the purpose of illustrating the above description let us consider the special situation in which the injection rate is of the form

$$N(t) = (\ln t)^{\alpha}$$

for $\alpha \geq 0$. In this case, the logarithm of the probability that a random walk born at some fixed time survives up to time t is going to decay like some function $h(t) = \frac{t}{(\ln t)^{\beta}}$, where β is a function of α. Figure 1 shows the dependence of β with respect to α. Note that when $\alpha > 1$, we are in the medium regime, and $h = h_M$. On the other hand, for $0 < \alpha < 1$ we are in the low regime, and depending on the value of the percolation parameter p, the decay function h takes the value h_M (for $p > p_c$) or h_L (for $p < p_c$).

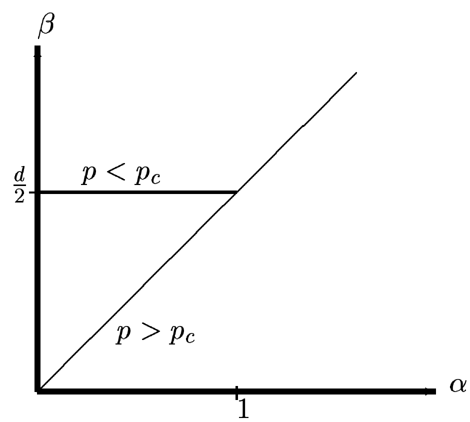

Figure 1

The second result giving a partial answer to question 2 of the introduction is the annealed version of theorem 2. As before, we are assuming that $p > 0$.

THEOREM 3. *Consider a Random Saturation Process in an obstacle configuration η and driven by an injection N. Assume that $g(t) < t$. Assume that $N(t) \ll t^{d/2-\epsilon}$ for some $\epsilon \in (0,1)$, that $\limsup_{t\to\infty} k(t) > \bar{a}$ and that $t - g(t) \gg 1$. Then,*

(i) *Assume that $1 \ll N(t) \ll (t-g(t))^{d/2}$. If $(t-g(t))^{d/(d+2)} \ll N(t)$ or $p = 1$, then*

$$\lim_{t\to\infty} \frac{1}{h_M(k,t)} \ln Q_{N,\mu}(\tau_{k(t)} > t) = -1$$

where $h_M(k,t) := \lambda_d (aw_d)^{2/d} \int_{g(t)}^t \frac{ds}{N(s)^{2/d}}$.

(ii) *If* $N(t) \ll (t-g(t))^{d/(d+2)}$ *and* $p<1$, *then*

$$\lim_{t\to\infty} \frac{1}{\tilde{h}_L(k,t)} \ln Q_{N,\mu}(\tau_{k(t)} > t) = -1$$

where $\tilde{h}_L(k,t) = (w_d|\ln(1-p)|)^{2/(d+2)} \left(\frac{d+2}{2}\right) \left(\frac{2\lambda_d}{d}\right)^{d/(d+2)} (t-g(t))^{d/(d+2)}$.

The main feature of the quenched theorem 2 is still in this annealed theorem: namely the presence of two injection regimes for $N(t)$ satisfying $N(t) \ll t^{-\epsilon+d/2}$ for some $\epsilon > 0$. The role of p_c is here played by $p = 1$. Furthermore, this time the transition between the two regimes occurs at the injection rate $N(t) \sim t^{d/(d+2)}$. What we call here *annealed low regime*, when $N(t) \ll t^{d/(d+2)}$, corresponds to studying the decay properties of the annealed survival probability of a Brownian motion in $\mathbb{R}^d$ with Poissonian obstacles [**7**] or a simple random walk on the lattice with each site being absorbing independently of the others [**8, 1**]. This means that the survival strategy of a particle in the low regime is to stay in a natural clearing of radius $t^{1/(d+2)}$ produced at the origin. What we call *annealed medium regime*, when $t^{d/(d+2)} \ll N(t) \ll t^{-\epsilon+d/2}$, gives the same decay as the quenched medium regime with particles taking advantage of the central clearing produced by saturation.

3. Main steps providing logarithmic asymptotics of survival probabilities.

In what follows we give an outline of the proof of theorem 2. The proof of the annealed version theorem 3, requires similar techniques, so we omit it. The detailed proofs of both theorems will appear in [**4**].

There are two main survival strategies for a random walk that determine the behaviour for long times, of the probability to survive up to time t. The first strategy is based on the observation that with a high probability, the set of saturated obstacles at time t is a ball of radius $\left(\frac{1}{aw_d}N(t)\right)^{1/d}$ intersected with the original set of obstacles. Such is the statement of part (i) of theorem 1. Then, to survive up to time t a particle tries to spend all the time in a ball of time dependent radius corresponding to such a theorem. We will call the obstacle free region produced in this way, the central clearing. The proof of this shape theorem 1 follows the lines of theorem 1 of [**13**] and requires a small modification of the methods there used.

The second survival strategy is to go very fast far away, to find a natural clearing of the obstacles of size of order $(\ln t)^{1/d}$, and to spend the rest of the time up to time t in this clearing. We will call such clearings, the natural clearings. This is precisely the survival strategy of a single random walk on the lattice with random obstacles on the sites having a Bernoulli product distribution [**2**].

There is going to be a competition between the two survival strategies described above. Depending on the injection rate and on the value of the percolation parameter p, one is going to dominate the other one, fact which is reflected on the different regimes appearing in the statement of theorem 2. In fact, part (i) of the quenched logarithmic asymptotics of theorem 2, corresponds to a random walk following the survival strategy of staying in the central clearing created by the saturated obstacles, while part (ii) corresponds to a random walk which survives by traveling fast and far away to some natural clearing of size of order $(\ln t)^{1/d}$.

To illustrate these two situations, in what follows we take a look at at the case $k(t) = 1$, corresponding to the behaviour of the first particle born. For high injection rates $N(t) \gg \ln t$, the central clearing produced by saturation (theorem 1) is going to dominate in size the natural clearings that can be found within a box of side t. Similarly, if the percolation parameter is higher than the critical value p_c, there is no infinite cluster of sites free of obstacles, and the posibility of traveling far away to find natural clearings is denied. In both cases, the central clearing determines the behaviour of the decay of the survival probability up to time t and the hypothesis of part (i) of theorem 2 are satisfied. More precisely, the logarithm of such probability decays like $\int_0^t \lambda(s)ds$, where $\lambda(s)$ is the principal Dirichlet eigenvalue of the discrete Laplacian operator divided by $2d$ on the set of sites free of obstacles at time s. For long times, such an eigenvalue decays as the principal Dirichlet eigenvalue of the continous Laplacian on a ball of radius $\left(\frac{1}{aw_d}N(s)\right)^{2/d}$. Such an argument via an application of the shape theorem 1 provides the lower bound on the asymptotics of part (i) of theorem 2. On the other hand at low injection rates $N(t) \ll \ln t$ and $p < p_c$, it is easy to see that the probability of survival of the our particle (the first born particle) is bounded below by the survival probability of a single simple random walk on a random environment of absorbing obstacles distributed according to μ. Now, by the logarithmic asymptotics proved by Antal [**1, 2**], this provides the lower bound of part (ii) of theorem 2.

The proof of the upper bounds of theorem 2 turn out to be more difficult. For part (i) (high injection or percolating obstacles case), the main problem is that there is no good control on the probability that the central clearing is a ball. In other words, the shape theorem 1, is useless. We therefore have to consider all possible shapes for the saturated set of obstacles at a given time and not only “balls”. It is the case that the smallest possible value that one can obtain for the principal Dirichlet eigenvalue of the discrete Laplacian on the obstacle free sites in a box after erasing a predetermined large enough amount of obstacles, corresponds to erasing a sphere. Part (i) of the theorem 4 is a statement of this fact. To state it, we need to introduce some notation. For a given configuration of obstacle depth $\eta \in \mathcal{I}^{\mathbb{Z}^d}$, we will denote by $\mathcal{N}_n(\eta)$ the set of configurations obtained from η after deleting n obstacles. Thus, for every $\varsigma \in \mathcal{N}_n(\eta)$ we have $\sum_{x\in\mathbb{Z}^d}(\eta(x)-\varsigma(x)) = n$. Now consider the space $\Upsilon := \{0,1\}^{\mathbb{Z}^d}$. This represents a space of site configurations on the lattice: sites in state 1 have an obstacle and are absorbing, and those in state 0 are empty and non-absorbing. Next, given $\xi \in \Upsilon$, call the subset of $\mathbb{Z}^d$ without obstacles $\mathcal{E}(\xi) := \{x \in \varepsilon\mathbb{Z}^d : \xi(x) = 0\}$. We now, given an open subset U of $\mathbb{R}^d$, denote by

$\lambda_\xi(U)$ the principal Dirichlet eigenvalue of the discrete Laplacian on $U \cap \mathbb{Z}^d$. We also define a mapping $\sigma : \mathcal{I}^{\mathbb{Z}^d} \to \Upsilon$ by $\sigma(\eta)(x) = 1$ if $\eta(x) \geq 1$ and $\sigma(\eta)(x) = 0$ if $\eta(x) = 0$. Finally, for given $\eta \in \mathcal{I}^{\mathbb{Z}^d}$, and open set $U \subset \mathbb{R}^d$ we adopt the convention $\lambda_\eta(U) := \lambda_{\sigma(\eta)}(U)$.

THEOREM 4. *On $\mathcal{I}^{\mathbb{Z}^d}$ consider a product measure μ such that $\mu(\eta(x) \geq 1) = p$, where $\eta \in \mathcal{I}^{\mathbb{Z}^d}$ and $0 < p < 1$. Let $f(t) : [0, \infty) \to [0, \infty)$ be an increasing function such that $f(t) \ll t$, w_d be the volume of a ball on $\mathbb{R}^d$ of unit radius and λ_d the principal Dirichlet eigenvalue of the Laplacian operator on this ball times $\frac{1}{2d}$. Then if $a := \mu(\eta)$, the following statements are true,*

i) *Suppose that $f(t) \gg (\ln t)^{1/d}$. Then,*

$$\lim_{t\to\infty} f(t)^2 \inf_{\varsigma \in \mathcal{N}_{a w_d f^d(t)}(\eta)} \lambda_\varsigma((-t,t)^d) = \lambda_d \qquad \mu - a.s.$$

ii) *Suppose that $f(t) \ll (\ln t)^{1/d}$. Then,*

$$\lim_{t\to\infty} (\ln t)^{2/d} \inf_{\varsigma \in \mathcal{N}_{a w_d f^d(t)}(\eta)} \lambda_\varsigma((-t,t)^d) = c(d,p) \qquad \mu - a.s.$$

where $c(d,p) := \lambda_d (w_d |\ln(1-p)|)^{2/d}$.

Now, if P is the probability that a simple random walk survives up to time t in a subset G of the lattice $\mathbb{Z}^d$, it is true that $P \leq c((\lambda t)^{d/2} + 1)e^{-\lambda t}$, where λ is the principal Dirichlet eigenvalue of the discrete Laplacian on the set G and c is a constant. A combination of this fact with part (i) of the above theorem enables us to prove the upper bound of the logarithmic asymptotics of part (i) of theorem 2.

On the other hand for part (ii) (low injection regime without percolation of obstacles) the difficulty in proving the upper bound lies in proving that the principal Dirichlet eigenvalue of the discrete laplacian on the obstacle free sites of a box does not change if we erase a low enough amount of obstacles. This is the content of part (ii) of theorem 4 stated above, which is enough to prove the upper bound of part (ii) of theorem 2.

Finally we would like to comment briefly on the proof of theorem 4. As mentioned, this is a key step in the proof of theorem 2. To prove it, we adapt Sznitman second version of his enlargement of obstacles for the continuous Laplacian on $\mathbb{R}^d$ minus translations of some compact non-polar set, to the discrete Laplacian on $\mathbb{Z}^d$ minus some sites. This version is then applied having in mind a lattice version of Faber-Krahn inequality.

4. Results concerning high and critical injection.

To begin we would like to mention recent results which give a partial answer to question 1 of the introduction concerning the shape of the saturated traps, and question 4 concerning the profile of live particles. The hydrodynamic scaling result of Quastel and Gravner [**10**] has been extended by Ben Arous, Quastel and Ramirez

[**3**] to include the random saturation process. To be more precise, let $\zeta(x,t)$ be the total number of random walks at site $x \in \mathbb{Z}^d$ and at time t in a random saturation process $Q_{N,\eta}$ on an obstacle configuration η and injection $N(t) = ct^{d/2}$. Now let $X \in \mathbb{R}^d$ and $T \in [0,\infty)$. Then,

THEOREM 5. *Let* $\epsilon > 0$, $X \in \mathbb{R}^d$ *and* $T \in [0,\infty)$ *and define* $\xi_\epsilon(X,T) = \zeta\left(\frac{X}{\epsilon}, \frac{t}{\epsilon}\right)$. *Then μ-a.s. it is true that as* $\epsilon \to 0$,

$$\xi_\epsilon(X,T) \to \rho(X,T)$$
$$\mathbf{1}_{\xi_\epsilon(X,T)\geq 0} \to \mathbf{1}_{s(X)\leq t}$$

weakly in $Q_{N,\eta}$-probability, where $\rho(X,T) \geq 0$ and $s(X,T)$ are solutions of the one-phase Stefan problem,

$$\frac{\partial\rho}{\partial T} = \Delta\rho + T^{\frac{d-2}{2}}\delta_0 \qquad s(X) < T$$
$$\rho = 0 \qquad s(X) \geq T$$
$$\nabla_0\rho\cdot\nabla s = -a \qquad s(X) = T$$

The proof of this theorem involves minor modifications to the method used to prove the hydrodynamic limit in Gravner and Quaste [**10**]. As a corollary of it, it is possible to obtain a certain form of a shape theorem for the set of saturated obstacles for injections $N(t) = Ct^{d/2}$, which correspond to the boundary between the medium and the high injection regime. More precisely, one can show that as μ-a.s. as t goes to infinity, the total variation norm distance between the set of obstacle free sites and the ball centered at the origin of radius $ct^{1/2}$ (for some apropriate constant c) converges to 0.

A second problem studied in [**3**] is the high regime ($N(t) \gg t^{d/2}$) of the random saturation model for $d = 1$. There, a shape theorem was proved. To state it, let us first given an obstacle configuration $\eta \in \Gamma$ define by $r_\eta(t)$ and $l_\eta(t)$ the positions at time t of the rightmost and leftmost random walks of the random saturation process on an obstacle configuration η,

$$r_\eta(t) := \sup_{1\leq n\leq N(t)} \{X_n^\eta(t)\}$$
$$l_\eta(t) := \inf_{1\leq n\leq N(t)} \{X_n^\eta(t)\}$$

Note that a site $x \in \mathbb{Z}$ such that $\eta(x) = 1$ is occupied at time t if and only if $l_\eta(t) \leq x \leq r_\eta(t)$. In this paper we prove the following theorem.

THEOREM 6. *Consider a random saturation process on an obstacle configuration η and driven by an injection $N(t)$. Assume that there is an $\epsilon > 0$ such that $N(t) \gg t^{\frac{1}{2}+\epsilon}$. Then, for every $\eta \in \Gamma$ such that* $\liminf_{t\to\infty} \frac{\sum_{x=0}^n \eta(x)}{n} > 0$ *it is true that*

$$\lim_{t\to\infty} \frac{r_\eta(t)}{\sup_{0\le y\le t}(t-y)\mathcal{I}\left(\frac{\ln\frac{N(y)}{t^{1/2}}}{t-y}\right)} = 1 \qquad P-a.s. \tag{5}$$

where $\mathcal{I} : [0,\infty) \to [0,\infty)$ *is the inverse of the function* $I : [0,\infty) \to [0,\infty)$ *defined by* $I(x) := x \sinh^{-1} x - \sqrt{1+x^2} + 1$

REMARK 3. *Under the assumption* $\liminf_{t\to\infty} \frac{\sum_{x=0}^{-n}\eta(x)}{n} > 0$, *equation (5) is satisfied with* $r_\eta(t)$ *replaced by* $-l_\eta(t)$.

We would like to comment on the expression governing the high regime asymptotics in 5 of the above theorem. The function $f(t) := \sup_{0\le y\le t}(t-y)\mathcal{I}\left(\frac{\ln\frac{N(y)}{t^{1/2}}}{t-y}\right)$, should be thought as a perturbation from the position of the rightmost random walk at time t, from a set of $N(t)$ simple random walks born at the origin all at time 0 (no obstacles present). The function $I(x)$ inverse of $\mathcal{I}$ is in fact the rate function of Cramer's theorem for continuous time simple random walks having a total jump rate equal to 1. The supremum in the expression for $f(t)$ is an effect coming from the fact that the random walks in the random saturation process are not born all at the same instant.

On the other hand, theorem 6 contains a result proved in [**10**] which answered afirmatively a conjecture of Lawler, Bramson and Griffeath when $d = 1$. Namely, that for Internal Diffusion Limited Aggregation (corresponding to the random saturation model with $N(t) = [t]$ and initially all sites being obstacles of depth one) with $d = 1$, if $S(t)$ is the set of occupied sites at time t, then μ-a.s. for every $\epsilon > 0$ there is a $t_0 > 0$ such that

$$[-(1-\epsilon)\sqrt{t\log t}, (1-\epsilon)\sqrt{t\log t}] \subset S(t) \subset [-(1+\epsilon)\sqrt{t\log t}, (1+\epsilon)\sqrt{t\log t}]$$

whenever $t \ge t_0$. In fact, when $N(t) = [t]$, the function $I(x)$ of theorem 6 (which is in fact the rate function for a continuous time simple random walk on the lattice) is close to x^2. Then, the inverse function $\mathcal{I}(x)$ is close to $\sqrt{x}$, which explains why the denominator inside the limit of equation (5) becomes $\sqrt{t\ln t}$.

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References

[1] P. Antal, *Enlargement of obstacles for the simple random walk*, Ann. Probab. **23**, 1061-1101 (1995).

[2] P. Antal, *Trapping problems for the simple random walk*, Thesis, ETH, Zürich (1994).

[3] G. Ben Arous, J. Quastel and A.F. Ramírez, in preparation.

[4] G. Ben Arous and A.F. Ramírez, "Asymptotic survival probabilities in the random saturation process", submitted to Ann. of Probab. (1999).

[5] G. Ben Arous and M. Renard, *Problème de l'argile dopé à la zéolithe*, Rapport Sience& Tec, (1997).
[6] P. Diaconis and W. Fulton, *A growth model, a game, an algebra, Lagrange inversion, and characteristic classes*, Rend. Sem. Mat. Univ. Pol. Torino **49**, 95-119 (1991).
[7] M. Donsker and S.R.S. Varadhan, *Asymptotics for the Wiener sausage*, Commun. Pure Appl. Math. **28**, 525-565 (1975).
[8] M. Donsker and S.R.S. Varadhan, *On the number of distinct sites visited by a random walk*, Commun. Pure Appl. Math. **32**, 721-747 (1979).
[9] T. Funaki, *Free boundary problem from stochastic lattice gas model*, preprint (1999).
[10] J. Gravner and J. Quastel, *Internal DLA and the Stefan problem*, preprint (1998).
[11] J. Krug and H. Spohn, *Kinetic roughening of growing surfaces*, Solids Far From Equilibrium, C. Godrèche ed., Cambridge Univ. Press, 479-582, (1991).
[12] G. Lawler, *Intersection of Random Walks*, Birkhäuser, Ann Arbor (1991).
[13] G. Lawler, M. Bramson and D. Griffeath, *Internal diffusion limited aggregation*, Ann. Probab. **20**, 2117-2140 (1992).
[14] A.S. Sznitman, *Brownian motion obstacles and random media*, Springer-Verlag (1998).
[15] A.S. Sznitman, *Capacity and principal eigenvalues: the method of enlargement of obstacles revisited*, The Annals of Probability **25**, No. 3, 1180-1209 (1997).
[16] A.S. Sznitman, *Fluctuations of principal eigenvalues and random scales*, Commun. Math. Phys. **189**, 337-363 (1997).
[17] A.S. Sznitman, *Lifschitz tail and Wiener sausage I*, J. Funct. Anal. **94**, 223-246 (1990).

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