



ELSEVIER

# Large deviations in the Langevin dynamics of a random field Ising model

Gérard Ben Arous<sup>a</sup>, Michel Sortais<sup>b,\*</sup>,<sup>1</sup>

<sup>a</sup>*Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, USA*

<sup>b</sup>*Fakultät II der T.U.Berlin, Institut für Mathematik, MA 7-4, Strasse des 17ten Juni 136, D-10623 Berlin, Germany*

Received 29 May 2002; received in revised form 25 November 2002; accepted 27 November 2002

---

## Abstract

We consider a Langevin dynamics scheme for a  $d$ -dimensional Ising model with a disordered external magnetic field and establish that the *averaged* law of the empirical process obeys a large deviation principle (LDP), according to a good rate functional  $\mathcal{J}^a$  having a unique minimiser  $Q_\infty$ . The asymptotic dynamics  $Q_\infty$  may be viewed as the unique weak solution associated with an infinite-dimensional system of interacting diffusions, as well as the unique Gibbs measure corresponding to an interaction  $\Psi$  on infinite dimensional path space. We then show that the *quenched* law of the empirical process also obeys a LDP, according to a deterministic good rate functional  $\mathcal{J}^q$  satisfying:  $\mathcal{J}^q \geq \mathcal{J}^a$ , so that (for a typical realisation of the disordered external magnetic field) the quenched law of the empirical process converges exponentially fast to a Dirac mass concentrated at  $Q_\infty$ .

© 2003 Elsevier Science B.V. All rights reserved.

*Keywords:* Large deviations; Statistical mechanics; Disordered systems; Interacting diffusion processes

---

## 1. Introduction and statement of the main results

In several recent papers, Brézin and De Dominicis have considered the statics and the dynamics of an Ising model submitted to a Gaussian random field: in Brézin and De Dominicis (1998b) they proceed to a “Replica Theory” analysis of the statics of this random field Ising model (RFIM), whereas in Brézin and De Dominicis (1998a) they study the Langevin dynamics associated with the model. In the latter note, they

---

\* Corresponding author.

E-mail address: [sortais@math.tu-berlin.de](mailto:sortais@math.tu-berlin.de) (M. Sortais).

<sup>1</sup> Research supported by the Swiss National Science Foundation under contract no 21-54120.98.

examined the time correlator associated with these dynamics and made the following observation concerning the low temperature regime of low dimensional ( $d \leq 8$ ) RFIMs: the singularities appearing in the time correlator of the dynamics when letting the initial time  $t_0$  go to  $-\infty$  are precisely the same as those appearing in the Replica Theory of the statics in the  $n \rightarrow 0$  limit. They conclude to the occurrence of an aging phenomenon in the low temperature regime of such low dimensional RFIMs: the dynamical properties of such disordered spin systems depend upon a “waiting time”  $t_0$ .

In the present paper, our aim is to consider the very same Langevin dynamics framework for such Gaussian RFIM and establish some large deviation principles for the empirical process of the corresponding trajectories, both in the *averaged* regime (when performing an average over the realisations of the disordered external field) and in the *quenched* regime (i.e. for a fixed, typical realisation of the disorder variables). As a consequence of these large deviation results, we may also state the following strong law of large numbers: for a typical realisation of the disorder variables, the quenched law of the empirical process converges exponentially fast to a Dirac mass concentrated at some asymptotic dynamics  $Q_\infty$ , that may be explicitly described as the unique weak solution associated with some infinite-dimensional system of interacting diffusions. Such large deviations approach to the dynamics of disordered systems has already been applied in the context of the Sherrington–Kirkpatrick model (see Ben Arous and Guionnet, 1997; Ben Arous and Guionnet, 1998; Grunwald, 1996, 1998), and more recently the same kind of results have been derived in the context of a short range spin glass (Ben Arous and Soriais, 2002). Here the situation is certainly much simpler in the sense that we are dealing with a *site* disordered system, and not a bond disordered one; in such a simple situation we are able to characterise the asymptotic dynamics  $Q_\infty$  as the unique Gibbs measure corresponding to some finite range, translation invariant interaction on path space, and the averaged LDP obtained for the empirical process may then be seen as a consequence of the Gibbsian nature of the averaged regime.

Let us now be more precise about the context we are working in. We consider the  $d$ -dimensional lattice disordered system whose Hamiltonian  $H_A^k$  in the finite volume  $A = [-N, N]^d \cap \mathbb{Z}^d$  writes:

$$H_A^k(x) = -2 \left( \sum_{j \sim i} x^i x^j + \sum_{i \in A} k^i x^i \right), \quad x \in \{-1; 1\}^{\mathbb{Z}^d},$$

$k = (k^i)_{i \in \mathbb{Z}^d}$  being an i.i.d. family of centred Gaussian random variables with variance  $\sigma^2$  ( $j \sim i$  means that sites  $j$  and  $i$  are nearest neighbours in  $A$ ,  $A$  being equipped with its periodic boundary conditions).

In order to consider the Langevin dynamics corresponding to  $H_A^k$ , we replace the “hard” spin variables ( $x^i \in \{-1; 1\}$ ) by linear spin variables ( $x^i \in \mathbb{R}$ ), introduce the real polynomial function  $U$  given by

$$U(x) = Cx^4 - Dx^2$$

for some positive constants  $C$  and  $D$ , and consider the process

$$dx_t^i = dw_t^i - U'(x_t^i) dt, \quad i \in \mathbb{Z}^d,$$

as our reference process (here and in the sequel,  $(w^i)_{i \in \mathbb{Z}^d}$  denotes an i.i.d. family of standard brownian motions indexed by the lattice); observe that the equilibrium measure  $\alpha_{C,D}$  corresponding to this process is proportional to  $e^{-2U(x)} dx$ , so that

$$\alpha_{C,D} \xrightarrow[C,D \nearrow \infty]{} \frac{1}{2} (\delta_{-1} + \delta_1) \quad \text{when } D = 2C.$$

Fixing  $\beta > 0$  (inverse temperature parameter) and  $T > 0$  (terminal time of the experiment), we let  $\mu_0$  be some compactly supported probability measure on  $\mathbb{R}$ ; in the finite volume  $A$  equipped with periodic boundary conditions, the nearest neighbour interacting system of diffusions  $\mathcal{S}_A^k$  given by

$$dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \left( \sum_{j \sim i} x_t^j + k^i \right) dt \quad (i \in A, 0 \leq t \leq T)$$

with the “deep quench” initial condition:  $\text{law}(x|_{t=0}) = \mu_0^{\otimes A}$  then has an invariant reversible measure proportional to:  $\exp(-\beta H_A^k(x)) \alpha_{C,D}^{\otimes A}(dx)$ . Denoting by  $W_T$  the (Polish) space of all real valued continuous functions on the interval  $[0; T]$ , we let  $Q_A^k$  be the law of the system  $\mathcal{S}_A^k$ ;  $Q_A^k$  is a probability measure on  $W_T^A$ , which we shall write as

$$Q_A^k \in \mathcal{M}(W_T^A).$$

We define the empirical process  $\pi_x^{(A)}$  corresponding to a finite-dimensional vector of diffusions  $x \in W_T^A$  through the identity:

$$\pi_x^{(A)} = \frac{1}{|A|} \sum_{i \in A} \delta_{x^{(A), (i)}}$$

$\tau = x^{(A)} \in W_T^{\mathbb{Z}^d}$  being the infinite-dimensional vector of diffusions obtained from  $x$  by reproducing periodically on the lattice the information contained in the box  $A$  and  $\tau^{(i)} \in W_T^{\mathbb{Z}^d}$  being the new configuration obtained from  $\tau$  by shifting the origin of the lattice at site  $i$ :

$$(\tau^{(i)})_j = \tau_{j+i}, \quad \forall j \in \mathbb{Z}^d.$$

Now let  $\mathcal{M}_s(\Omega)$  be the (Polish) space of all spatially shift invariant probability measures on  $\Omega = W_T^{\mathbb{Z}^d}$ ;  $\pi_x^{(A)}$  is an  $\mathcal{M}_s(\Omega)$ -valued variable on  $W_T^A$ , and we shall denote by  $\Pi_A^k$  the law of  $\pi_x^{(A)}$  under  $dQ_A^k(x)$ , so that:

$$\Pi_A^k \in \mathcal{M}(\mathcal{M}_s(\Omega)).$$

Finally  $\Pi_A \in \mathcal{M}(\mathcal{M}_s(\Omega))$  is defined through the identity:

$$\Pi_A(\mathcal{A}) = \int d\gamma(k) \Pi_A^k(\mathcal{A})$$

holding for any Borel set  $\mathcal{A} \subset \mathcal{M}_s(\Omega)$  ( $\Pi_A$  is the averaged law of the empirical process).

Our main results are large deviation principles for the families  $(\Pi_A)_{A \subset \subset \mathbb{Z}^d}$  and  $(\Pi_A^k)_{A \subset \subset \mathbb{Z}^d}$  (for a typical realisation of  $k$ ), yielding a fast convergence of these families towards a Dirac mass  $\delta_{Q_\infty}$  as  $A \nearrow \mathbb{Z}^d$ ; the asymptotic dynamics  $Q_\infty$  may be described in several ways: as the unique weak solution corresponding to an infinite

dimensional system of interacting diffusions, as the mean of a family of disordered, infinite-dimensional dynamics and also as the unique Gibbs measure corresponding to a finite-range, translation-invariant interaction on path space  $\Omega$ . We refer the reader to Section 1.2 in (Dembo and Zeitouni, 1998, pp. 4,5) for precise definitions concerning large deviation principles, rate functions and good rate functions.

**Theorem 1.1.** (i) For any inverse temperature parameter  $\beta$  and any terminal time  $T$ , the family  $(\Pi_A)_{A \subset \mathbb{Z}^d}$  obeys a large deviation principle on  $\mathcal{M}_s(W_T^{\mathbb{Z}^d})$ , on the scale  $|A|$  and according to a good rate function

$$\mathcal{I}^a : \mathcal{M}_s(W_T^{\mathbb{Z}^d}) \rightarrow [0; +\infty]$$

having a unique minimiser  $Q_\infty$ .

Moreover,  $Q_\infty$  is the unique weak solution corresponding to the following infinite dimensional system of nearest neighbour interacting diffusions:

$$\begin{cases} dx_t^i = dv_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt \\ \text{law}(x|_{t=0}) = \mu_0^{\otimes \mathbb{Z}^d} \quad (i \in \mathbb{Z}^d, 0 \leq t \leq T) \end{cases}$$

where  $\{(v_t^i)_{0 \leq t \leq T}\}_{i \in \mathbb{Z}^d}$  is an i.i.d. family of time inhomogeneous Ornstein–Uhlenbeck processes satisfying:

$$v_0^i = 0 \quad \text{and} \quad dv_t^i = dw_t^i + \gamma_t v_t^i dt \quad \text{for} \quad \gamma_t = \frac{\sigma^2 \beta^2}{1 + \sigma^2 \beta^2 t}.$$

(ii) Furthermore, almost surely in the realisations of the disorder variables  $\mathbf{k}$ , the family  $(\Pi_A^{\mathbf{k}})_{A \subset \mathbb{Z}^d}$  also obeys a large deviation principle on  $\mathcal{M}_s(\Omega)$ , on the scale  $|A|$  and according to a good rate function  $\mathcal{I}^q$  satisfying:

$$\mathcal{I}^q \geq \mathcal{I}^a.$$

**Note.** Each of the processes  $(v_t^i)_{t \geq 0}$  may also be presented as

$$v_t^i = (1 + \sigma^2 \beta^2 t) \int_0^t \frac{dw_s^i}{1 + \sigma^2 \beta^2 s}$$

and is thus a centred Gaussian process having the covariance structure:

$$E[v_s^i v_t^i] = s(1 + \sigma^2 \beta^2 t), \quad \forall 0 \leq s \leq t.$$

So the only difference between the stochastic differential system characterising  $Q_\infty$  and the infinite volume Langevin dynamics of a standard Ising model lies in the nature of the “thermal noise” driving the system: in the case of a standard Ising model it is an i.i.d. family  $\{(w_t^i)_{0 \leq t \leq T}; i \in \mathbb{Z}^d\}$  of standard brownian motions, whereas here we have an i.i.d. collection of centred gaussian processes with a very simple time correlation structure. Actually, each of the processes  $(v_t^i)_{0 \leq t \leq T}$  may also be represented as

$$(z^i \cdot t + w_t^i)_{0 \leq t \leq T},$$

where  $(w_t^i)_{0 \leq t \leq T}$  is an i.i.d. family of standard Brownian Motions and  $(z^i)$  an i.i.d. family of centred Gaussian variables with variance  $\sigma^2 \beta^2$ , independent of  $(w_t^i)_{0 \leq t \leq T}$ . So as a consequence of the preceding theorem we may state that the disordered, infinite-volume Stochastic Differential System

$$(\mathcal{S}_\infty^k) \quad \begin{cases} dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \left( \sum_{j \sim i} x_t^j + k^i \right) dt, \\ \text{law}(\mathbf{x}|_{t=0}) = \mu_0^{\otimes \mathbb{Z}^d} \quad (i \in \mathbb{Z}^d, 0 \leq t \leq T) \end{cases}$$

has a unique weak solution  $P^k$ ,  $\mathbb{P}$ -a.s.  $(\mathbf{k})$ , and that

$$Q_\infty(\cdot) = \int P^k(\cdot) d\mathbb{P}(\mathbf{k}).$$

Furthermore,  $Q_\infty$  may also be presented as the unique Gibbs measure associated with a finite-range interaction  $\Psi$  on infinite-dimensional path space  $\Omega$  (equipped with an infinite tensor product of Wiener measures as reference measure); such presentation is proposed in Section 2 (see in particular Sections 2.2, 2.3 and Proposition 2.2 therein). The Gibbsian nature of  $Q_\infty$  plays an important role in our understanding of the level 3 large deviations occurring in the Langevin dynamics of the RFIM, hopefully this presentation of  $Q_\infty$  should also prove useful in the investigations related to its space and time decorrelation properties, at least in the high temperature regime. We may also state as an elementary remark that the limiting probability  $dQ_\infty(\mathbf{x})$  thus obtained does not correspond to a Markov field of interacting diffusions. This is no surprise; indeed, in the case of Sherrington–Kirkpatrick spin glass dynamics, Ben Arous and Guionnet had already isolated a limiting dynamics that was a *strongly* non-Markov one (see Ben Arous and Guionnet, 1998), thus confirming some of the predictions made by Sompolinsky and Zippelius in (Sompolinsky and Zippelius, 1983). In comparison, the limiting dynamics  $Q_\infty$  obtained in the context of the RFIM may not be considered as a strongly non-Markov one:  $Q_\infty$  is characterised through an explicit equation, and adding the variable  $(v_t^i)_{0 \leq t \leq T}$  at each site  $i \in \mathbb{Z}^d$  suffices to come back to a (time inhomogeneous) Markov field.

As a simple consequence of the preceding large deviations results, one may for example fix some bounded continuous functionals  $\varphi_i: W_T \rightarrow \mathbb{R}$  ( $1 \leq i \leq n$ ) and some bounded continuous  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  to state that for a typical realisation of the disorder variables  $\mathbf{k}$ , the distribution of

$$F \left( \frac{1}{|A|} \sum_{i \in A} \varphi_1(x^i), \dots, \frac{1}{|A|} \sum_{i \in A} \varphi_n(x^i) \right)$$

under  $dQ_A^k(\mathbf{x})$  converges exponentially fast to a Dirac mass concentrated at

$$F \left( \int \varphi_1 dQ_\infty, \dots, \int \varphi_n dQ_\infty \right)$$

when  $\Lambda \nearrow \mathbb{Z}^d$ , with an exponential speed of convergence that may be bounded from below uniformly in  $k$  by using the averaged LD rate functional  $\mathcal{I}^a$ .

Here is the plan we follow in order to establish these large deviation results: in Section 2 we perform a standard Gaussian computation in order to view the finite volume, averaged probability

$$Q_A = \int d\gamma(k) Q_A^k$$

as the law of a (new) system of interacting diffusions ( $\mathcal{S}_A$ ) that is spatially homogeneous. One may then remark that Shiga and Shimizu’s theorem (Shiga and Shimizu, 1980), asserting the existence and uniqueness of a strong solution for certain infinite dimensional systems of interacting diffusions, may be used here; we thus let  $Q_\infty$  denote the probability law corresponding to an infinite dimensional extension of ( $\mathcal{S}_A$ ) and show, using an integration by parts formula established by Cattiaux, Roelly and Zessin (Cattiaux et al., 1996), that  $Q_\infty$  may also be characterised as the unique Gibbs measure associated with some translation invariant interaction  $\Psi$  on  $\Omega$ .

Section 3 is devoted to a derivation of the LDP for  $(\Pi_A)_{A \subset \subset \mathbb{Z}^d}$ . The LD upper bound is established using the Gibbsian nature of the averaged regime and Varadhan’s method (cf. Olla, 1988); in the present context one has to proceed carefully since the continuous functionals entering in the definition of the interaction  $\Psi$  are not uniformly bounded on  $\Omega$ : fortunately one may establish that the confinement induced by the single site potential  $U$  is strong enough to compensate for the lack of compactness in the spin variables  $x^i$ . We then check the validity of the *Gibbsian variational principle* for the interaction  $\Psi$  and establish the LD lower bound following the method devised by Föllmer and Orey in the context of Gibbs measures on  $\{\pm 1\}^{\mathbb{Z}^d}$  (see Föllmer and Orey, 1988).

In Section 4 we give a quick derivation of the LDP for  $(\Pi_A^k)_{A \subset \subset \mathbb{Z}^d}$  following Comets (1989); the corresponding rate functional  $\mathcal{I}^q : \mathcal{M}_s(W_T^{\mathbb{Z}^d}) \rightarrow [0; +\infty]$  has a rather intricate expression, and studying the set of all minimisers corresponding to  $\mathcal{I}^q$  might seem hopeless at first. Nevertheless, according to some general considerations concerning Large Deviations in Disordered Systems or Random Media (see e.g. Lemma 2.2.8 in Chapter 2 of Zeitouni, 2003), the deterministic rate functionals  $\mathcal{I}^q, \mathcal{I}^a : \mathcal{M}_s(\Omega) \rightarrow [0; +\infty]$  associated with the Large Deviations of the empirical process in the quenched regime and in the annealed regime do certainly satisfy

$$\mathcal{I}^q \geq \mathcal{I}^a,$$

and we also know that the set of all minimisers of  $\mathcal{I}^a$  is reduced to a singleton  $\{Q_\infty\}$  (since this set also coincides with the set of all translation invariant Gibbs measures associated with  $\Psi$ , according to the Variational Principle).

Finally, in Section 5 we briefly show that one may change to a certain extent the initial and boundary conditions entering in the definition of  $Q_A^k$  and still derive such LD Principles for the empirical process.

## 2. Gibbsian nature of the averaged regime

### 2.1. Infinite-dimensional extension of the averaged regime probabilities

Before stating the main result of this section, we remind that  $\mathcal{S}_A^k$  denotes the following system of nearest neighbour interacting diffusions:

$$\begin{cases} dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \left( \sum_{j \sim i} x_t^j + k^i \right) dt, \\ \text{Law}(x|_{t=0}) = \mu_0^{\otimes A} \quad (i \in A, 0 \leq t \leq T) \end{cases}$$

( $k$  being some fixed realisation of the disordered external magnetic field,  $\mu_0$  some compactly supported probability measure on the real line, and  $A$  being equipped with its periodic boundary conditions).  $Q_A^k$  stands for the probability law corresponding to the system  $\mathcal{S}_A^k$ , and the probability  $Q_A$  corresponding to the averaged regime is then defined by

$$Q_A(\cdot) = \int d\gamma(k) Q_A^k(\cdot).$$

The following proposition shows that this new probability  $Q_A$  on Wiener space  $W_T^A$  may also be viewed as the law of a nearest neighbour interacting diffusions system  $\mathcal{S}_A$ .

**Proposition 2.1.**  $Q_A$  is the weak solution corresponding to the interacting diffusions system  $\mathcal{S}_A$  given by

$$\begin{cases} dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt + \frac{\sigma^2 \beta^2}{1 + \sigma^2 \beta^2 t} \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt \\ y_t^i = x_t^i - x_0^i + \int_0^t U'(x_s^i) ds, \quad z_t^i = \int_0^t x_s^i ds \\ \text{Law}(x|_{t=0}) = \mu_0^{\otimes A} \quad (i \in A, 0 \leq t \leq T). \end{cases}$$

**Proof.** We let  $P_T$  denote the probability law on Wiener space  $W_T$  corresponding to the process:

$$dx_t = dw_t - U'(x_t) dt$$

with initial condition:  $\text{law}(x|_{t=0}) = \mu_0$ .  $P_T$  is such that

$$x_t - x_0 + \int_0^t U'(x_v) dv \quad \text{is a standard Brownian motion under } dP_T(x).$$

Let also  $P_t$  denote the restriction of  $P_T$  to the  $\sigma$ -algebra  $\mathcal{F}_t$  corresponding to the time interval  $[0, t] \subset [0, T]$ , and  $Q_{A,t}$  denote similarly the restriction of  $Q_A$  to the  $\sigma$ -algebra

in  $W_T^A$  corresponding to this same time interval. According to Fubini and Girsanov’s theorems:

$$Q_A \ll P_T^{\otimes A},$$

and

$$M_t^A = \frac{dQ_{A,t}}{dP_t^{\otimes A}}$$

is a positive  $P_T^{\otimes A}$ -martingale with mean 1, such that

$$\begin{aligned} M_t^A(\mathbf{x}) &= \int d\gamma(\mathbf{k}) \left[ \exp \left\{ \beta \sum_{i \in A} \int_0^t \left( \sum_{j \sim i} x_u^j + k^i \right) dw_u^i \right. \right. \\ &\quad \left. \left. - \frac{\beta^2}{2} \sum_{i \in A} \int_0^t \left( \sum_{j \sim i} x_u^j + k^i \right)^2 du \right\} \right] \\ &= \exp \left( \beta \sum_{i \in A} \int_0^t \left( \sum_{j \sim i} x_u^j \right) dw_u^i - \frac{\beta^2}{2} \sum_{i \in A} \int_0^t \left( \sum_{j \sim i} x_u^j \right)^2 du \right) \\ &\quad \times \int d\gamma(\mathbf{k}) \left[ \exp \left\{ (\mathbf{k}_A; A_{\beta,t}(\mathbf{x})) - \frac{\beta^2 t}{2} (\mathbf{k}_A; \mathbf{k}_A) \right\} \right], \end{aligned}$$

$A_{\beta,t}(\mathbf{x})$  being the  $A$ -dimensional real vector defined by:

$$\begin{aligned} A_{\beta,t}^i(\mathbf{x}) &= \beta w_t^i(\mathbf{x}) - \beta^2 \int_0^t \left( \sum_{j \sim i} x_u^j \right) du \\ &= \beta \left( x_t^i - x_0^i + \int_0^t \left( U'(x_s^i) - \beta \sum_{j \sim i} x_s^j \right) ds \right). \end{aligned}$$

Averaging over the Gaussian vector  $\mathbf{k}$ , we then obtain

$$\begin{aligned} \log M_t^A(\mathbf{x}) &=_{\text{mart.}} \frac{\sigma^2}{2} \left( A_{\beta,t}(\mathbf{x}); \frac{1}{1 + \sigma^2 \beta^2 t} A_{\beta,t}(\mathbf{x}) \right) + \beta \sum_{i \in A} \int_0^t \left( \sum_{j \sim i} x_u^j \right) dw_u^i \\ &=_{\text{mart.}} \sigma^2 \int_0^t \left( \frac{A_{\beta,u}(\mathbf{x})}{1 + \sigma^2 \beta^2 u}; dA_{\beta,u}(\mathbf{x}) \right) + \beta \sum_{i \in A} \int_0^t \left( \sum_{j \sim i} x_u^j \right) dw_u^i \\ &=_{\text{mart.}} \beta \sigma^2 \sum_{i \in A} \int_0^t \frac{A_{\beta,u}^i(\mathbf{x})}{1 + \sigma^2 \beta^2 u} dw_u^i(\mathbf{x}) + \beta \sum_{i \in A} \int_0^t \left( \sum_{j \sim i} x_u^j \right) dw_u^i, \end{aligned}$$

the sign =<sub>mart.</sub> meaning here that the two  $P_T^{\otimes A}$ -semimartingales under consideration (on the left hand side and on the right hand side of the equality) have the same martingale part.

At this stage one should check that the positive martingale  $M_t^A(\mathbf{x})$  is a uniformly integrable one, so as to make sure that Girsanov’s theorem may be applied a second time. We prove a stronger fact in Section 3 (see Proposition 3.1):  $(M_t^A(\mathbf{x}))_{0 \leq t \leq T}$  is in fact bounded from above,  $P_T^{\otimes A}$ -a.s. in  $\mathbf{x}$ .

Girsanov’s theorem may thus be applied to the probability  $Q_A$  on path space  $W_T^A$  corresponding to the uniformly integrable martingale  $(M_t^A(\mathbf{x}))_{0 \leq t \leq T}$ , and the proposition is proved.  $\square$

**Note.** Adding the variables

$$y_t^i = x_t^i - x_0^i + \int_0^t U'(x_s^i) ds, \quad z_t^i = \int_0^t x_s^i ds$$

at each site  $i \in \Lambda$  and letting:  $\gamma_t = \sigma^2 \beta^2 / (1 + \sigma^2 \beta^2 t)$ , one may notice that the interacting diffusions system  $\mathcal{S}_\Lambda$  obtained in the averaged regime is a Markov one, since the differentials of the new variables  $y^i$  and  $z^i$  are

$$dy_t^i = dw_t^i + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt \quad \text{and} \quad dz_t^i = x_t^i dt.$$

Classical results of Ito calculus assert that  $\mathcal{S}_\Lambda$ , viewed as a Markov system of interacting three dimensional diffusions, has a unique strong solution for any inverse temperature parameter  $\beta$  and any terminal time  $T > 0$ , that may be constructed through Euler’s method (cf. Krylov, 1990).

Actually, the monotonicity properties of the drift term appearing in the differentials  $dx_t^i$ ,  $dy_t^i$  and  $dz_t^i$  are such that one may assert, following Shiga and Shimizu, a strong existence and uniqueness result for the corresponding infinite dimensional system  $\mathcal{S}_\infty$ , given by

$$\begin{cases} dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt, \\ dy_t^i = dw_t^i + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt, \\ dz_t^i = x_t^i dt, \\ \text{Law}(\mathbf{x}|_{t=0}) = \mu_0^{\otimes \mathbb{Z}^d}, \quad \mathbf{y}|_{t=0} = \mathbf{z}|_{t=0} = \mathbf{0} \quad (i \in \mathbb{Z}^d, 0 \leq t \leq T). \end{cases}$$

To be more precise, we next introduce some notations and give a strong existence and uniqueness theorem for infinite systems of  $n$ -dimensional diffusions that is convenient for our purpose.

**Definition 2.1.** Fix an integer  $n \geq 1$ . For each  $p \in \mathbb{Z}$  and  $X = (X_i)_{i \in \mathbb{Z}^d} \in (\mathbb{R}^n)^{\mathbb{Z}^d}$ , let

$$\|X\|_p^2 = \sum_{i \in \mathbb{Z}^d} \frac{1}{(1 + |i|)^{2p}} |X_i|^2 \quad (\text{with: } |i| = |i_1| + \dots + |i_d|)$$

and define the Hilbert space  $S_p(\mathbb{Z}^d; \mathbb{R}^n)$  as

$$S_p(\mathbb{Z}^d; \mathbb{R}^n) = \{X \in (\mathbb{R}^n)^{\mathbb{Z}^d} / \|X\|_p < +\infty\}.$$

The space of  $\mathbb{R}^n$ -valued rapidly decreasing sequences on  $\mathbb{Z}^d$  is then defined as

$$S(\mathbb{Z}^d; \mathbb{R}^n) = \bigcap_{p \in \mathbb{Z}} S_p(\mathbb{Z}^d; \mathbb{R}^n).$$

$S(\mathbb{Z}^d; \mathbb{R}^n)$  is a nuclear space for the sequence of norms  $\|\cdot\|_p$ , and its dual space is the space  $S'(\mathbb{Z}^d; \mathbb{R}^n)$  of all tempered  $\mathbb{R}^n$ -valued sequences on  $\mathbb{Z}^d$ :

$$S'(\mathbb{Z}^d; \mathbb{R}^n) = \bigcup_{p \in \mathbb{Z}} S_p(\mathbb{Z}^d; \mathbb{R}^n).$$

We also let:  $S(\mathbb{Z}^d) = S(\mathbb{Z}^d; \mathbb{R})$  and  $S_+(\mathbb{Z}^d) = S(\mathbb{Z}^d; \mathbb{R}_+)$ , and we recall that any  $S'(\mathbb{Z}^d; \mathbb{R}^n)$ -valued path  $X : [0, T] \rightarrow (\mathbb{R}^n)^{\mathbb{Z}^d}$  that is weakly continuous is also strongly continuous.

Following Shiga and Shimizu, we may now state a strong existence and uniqueness theorem for infinite systems of  $\mathbb{R}^n$ -valued diffusions whose coefficients satisfy some monotonicity condition.

**Theorem 2.1.** *Consider the infinite dimensional system of  $\mathbb{R}^n$ -valued diffusions (E) given by:*

$$(E): \quad dX_t^i = \alpha \cdot dw_t^i + f_i(t; X_t^i; X_t) dt, \quad i \in \mathbb{Z}^d, \quad t \geq 0$$

where:

- $X_t = (X_t^i)_{i \in \mathbb{Z}^d} \in (\mathbb{R}^n)^{\mathbb{Z}^d}$ .
- $\{(w_t^i)_{t \geq 0}; i \in \mathbb{Z}^d\}$  is an i.i.d. family of standard  $n$ -dimensional brownian motions.
- $\alpha$  is fixed in  $\mathbb{R}^{n \otimes n}$ .
- $f_i : \mathbb{R}_+ \times \mathbb{R}^n \times (\mathbb{R}^n)^{\mathbb{Z}^d} \rightarrow \mathbb{R}^n$  is a family of continuous mappings such that:
  - $f_i(t; 0; 0) = 0, \forall i \in \mathbb{Z}^d, \forall t \geq 0$
  - $\forall i \in \mathbb{Z}^d, \forall \Lambda \subset \subset \mathbb{Z}^d, f_i : \mathbb{R}_+ \times \mathbb{R}^n \times (\mathbb{R}^n)^{\mathbb{Z}^d} \rightarrow \mathbb{R}^n$  is locally Lipschitz in the variables  $(t; \xi; (X^i)_{i \in \Lambda})$
  - *Monotonicity Condition:*

$$\exists K, L > 0, \exists c = (c_i)_{i \in \mathbb{Z}^d} \in S_+(\mathbb{Z}^d), \forall i \in \mathbb{Z}^d, \forall t \geq 0, \forall \xi, \eta \in \mathbb{R}^n, \forall X, Y \in (\mathbb{R}^n)^{\mathbb{Z}^d},$$

$$((\xi - \eta); f_i(t; \xi; X) - f_i(t; \eta; Y)) \leq K |\xi - \eta|^2 + L |\xi - \eta| \cdot \sqrt{\sum_{j \in \mathbb{Z}^d} c_{i-j} |X^j - Y^j|^2}.$$

Then: for each  $X \in S'(\mathbb{Z}^d; \mathbb{R}^n)$ , (E) has a unique strong solution  $(X_t)_{t \geq 0}$  with initial condition  $X_0 = X$ , moreover  $(X_t)_{t \geq 0}$  has a.s.  $S'(\mathbb{Z}^d; \mathbb{R}^n)$ -valued paths that are continuous with respect to the strong topology in  $S'(\mathbb{Z}^d; \mathbb{R}^n)$ .

**Proof.** One merely needs to adapt the demonstration of Theorem 4.1 in (Shiga and Shimizu, 1980) (for a complete proof see Sordais (2001, Theorem 2.2.1 in Chapter 2)). □

**Corollary 2.1.** For any compactly supported probability  $\mu_0$  on the real line, the infinite system of three-dimensional interacting diffusions  $\mathcal{S}_\infty$  given by

$$\begin{cases} dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt \\ dy_t^i = dw_t^i + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt \\ dz_t^i = x_t^i dt \\ \text{Law}(x|_{t=0}) = \mu_0^{\otimes \mathbb{Z}^d}, \quad y|_{t=0} = z|_{t=0} = \mathbf{0} \quad (i \in \mathbb{Z}^d, 0 \leq t \leq T) \end{cases}$$

has a unique strong solution  $(X_t)_{t \geq 0}$ , and  $(X_t)_{t \geq 0}$  has a.s.  $S_d(\mathbb{Z}^d; \mathbb{R}^3)$ -valued, continuous paths.

**Proof.** Apply simply the preceding theorem to the situation where

$$n = 3, \quad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and  $f_i : \mathbb{R}_+ \times \mathbb{R}^3 \times (\mathbb{R}^3)^{\mathbb{Z}^d} \rightarrow \mathbb{R}^3$  is defined through the identity:

$$\forall t \geq 0, \forall \xi = (\xi^1, \xi^2, \xi^3) \in \mathbb{R}^3, \forall X = (x^i, y^i, z^i)_{i \in \mathbb{Z}^d} \in (\mathbb{R}^3)^{\mathbb{Z}^d},$$

$$f_i(t; \xi; X) = \begin{pmatrix} -U'(\xi_1) + \beta \sum_{j \sim i} x^j + \gamma_t \left( y^i - \beta \sum_{j \sim i} z^j \right) \\ \beta \sum_{j \sim i} x^j + \gamma_t \left( y^i - \beta \sum_{j \sim i} z^j \right) \\ x^i \end{pmatrix},$$

observe also that our “deep quench” initial condition  $\mu_0^{\otimes \mathbb{Z}^d}$  satisfies:  $\mu_0^{\otimes \mathbb{Z}^d}(S_d(\mathbb{Z}^d; \mathbb{R})) = 1$ . □

**Note.** (a)  $(X_t)_{0 \leq t \leq T} = (x_t^i, y_t^i, z_t^i)_{0 \leq t \leq T}$  being the unique strong solution corresponding to the system  $\mathcal{S}_\infty$ , we let  $Q_\infty$  denote the probability law of  $(x_t^i)_{0 \leq t \leq T}$ ;  $Q_\infty$  is a spatially shift invariant probability measure on infinite dimensional path space  $\Omega = W_T^{\mathbb{Z}^d}$ :  $Q_\infty \in \mathcal{M}_s(\Omega)$ .

In the next sections we show that  $Q_\infty$  may be viewed as a Gibbs measure on  $\Omega$  and give the corresponding interaction on path space explicitly.

(b) As an intermediate step in the proof of Theorem 2.1, one may show that  $Q_\infty$  satisfies:

$$\int dQ_\infty \left\{ \sup_{0 \leq t \leq T} \|x_t\|_d^2 \right\} < \infty.$$

Actually, still using Ito’s formula and an appropriate Gromwall inequality, one may also prove that the quantity

$$\sup_{0 \leq t \leq T} \left( \int dQ_\infty \|x_t\|_d^{2p} \right)$$

is finite for all  $p \geq 1$  (see the proof of Theorem 4.6 in Föllmer and Wakolbinger, 1986).

As an easy consequence, we may state that integrals such as

$$\int dQ_\infty(x) |x_T^i|^p \quad \text{or} \quad \int dQ_\infty(x) \left( \int_0^T |x_t^i|^p dt \right)$$

are bounded from above uniformly in  $i \in \mathbb{Z}^d$ , for all  $p \geq 1$ .

(c) Last but not least, the dynamics  $Q_\infty$  may be very conveniently compared to the dynamics of a standard Ising model by introducing the new variables:

$$v_t^i = y_t^i - \beta \sum_{j \sim i} z_t^j$$

at each site  $i \in \mathbb{Z}^d$ . Indeed, one obtains that  $Q_\infty$  may also be viewed as the probability law corresponding to the system

$$dx_t^i = dv_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt,$$

$$\text{Law}(x|_{t=0}) = \mu_0^{\otimes \mathbb{Z}^d},$$

$\{v_t^i; 0 \leq t \leq T\}_{i \in \mathbb{Z}^d}$  being an i.i.d. family of time inhomogeneous Ornstein–Uhlenbeck processes under  $Q_\infty$ , solving the following linear stochastic differential equation:

$$dv_t = dw_t + \gamma_t v_t dt,$$

with initial condition:  $v_0 = 0$ .

### 2.2. Gibbsian integration by parts formula on path space

Our aim in the present section is to present briefly some of the main results contained in (Cattiaux et al., 1996); to this end we first give general definitions for Gibbs measures based on the lattice  $\mathbb{Z}^d$ .

**Definition 2.2.** Let  $X$  be a Polish space (spin space) and let  $\mathcal{T}_d$  denote the set of all finite subsets  $A \subset \subset \mathbb{Z}^d$ .

Let  $\lambda = \otimes_{i \in \mathbb{Z}^d} \lambda_i$  be an infinite tensor product of  $\sigma$ -finite measures on  $X$  and  $\Gamma$  be a probability measure on  $X^{\mathbb{Z}^d}$ .

(1) A family  $\rho = (\rho_A)_{A \in \mathcal{F}_d}$  of  $\geq 0$ , measurable functionals on  $X^{\mathbb{Z}^d}$  is said to be a  $(\Gamma; \lambda)$ -modification when the following conditions are satisfied:

(i)  $\forall A \in \mathcal{F}_d, \int \rho_A(z \vee x_{A^c}) d\lambda_A(z) = 1, \Gamma_{A^c}$ -a.s.  $(x_{A^c})$

(ii)  $\forall \Lambda \subset A \in \mathcal{F}_d, \Gamma_{A^c}$ -a.s.  $(x_{A^c}), (\lambda_A \otimes \delta_{x_{A^c}})$ -a.s.  $(y)$ ,

$$\rho_A(y) = \rho_A(y) \cdot \int \rho_\Lambda(z \vee y_{\Lambda^c}) d\lambda_\Lambda(z)$$

(here and in the sequel,  $(z \vee y_{\Lambda^c})$  denotes the combination of configurations  $z \in X^\Lambda$  and  $y_{\Lambda^c} \in X^{\Lambda^c}$ ).

(2) An interaction  $\Psi = (\psi_A)_{A \in \mathcal{F}_d}$  is a family of measurable functionals on  $X^{\mathbb{Z}^d}$  such that:

(i)  $\forall A \in \mathcal{F}_d, \psi_A : X^{\mathbb{Z}^d} \rightarrow \mathbb{R}$  is  $\mathcal{F}_A$ -measurable

(ii)  $\forall A \in \mathcal{F}_d, \sum_{A' \cap A \neq \emptyset} |\psi_{A'}(\mathbf{x})| < \infty, \forall \mathbf{x} \in X^{\mathbb{Z}^d}$

( $\mathcal{F}_A$  being the  $\sigma$ -algebra generated by the projections  $p_i : X^{\mathbb{Z}^d} \rightarrow X, i \in \mathbb{Z}^d$ ).

One may then consider the Hamiltonian potential  $H^\Psi = (H_A^\Psi)_{A \subset \mathbb{Z}^d}$  corresponding to the interaction  $\Psi$ .  $H^\Psi$  is defined through the identities:

$$H_A^\Psi(\mathbf{x}) = \sum_{A' \cap A \neq \emptyset} \psi_{A'}(\mathbf{x})$$

holding for all  $A \in \mathcal{F}_d$  and all  $\mathbf{x} \in X^{\mathbb{Z}^d}$ .

In the situation where the partition function corresponding to the interaction  $\Psi$  is well defined, i.e.:

$$\forall A \in \mathcal{F}_d, \Gamma_{A^c}$$
-a.s.  $(x_{A^c}),$

$$\mathcal{Z}_A^\Psi(x_{A^c}) = \int d\lambda_A(z) \exp(-H_A^\Psi(z \vee x_{A^c})) < +\infty,$$

one may then easily check that

$$\rho_A(x_A \vee x_{A^c}) = \frac{\exp(-H_A^\Psi(x_A \vee x_{A^c}))}{\mathcal{Z}_A^\Psi(x_{A^c})}$$

defines a  $(\Gamma; \lambda)$ -modification on  $X^{\mathbb{Z}^d}$ .

(3) Given any  $(\Gamma; \lambda)$ -modification  $\rho$ , one says that the probability  $\Gamma$  is a Gibbs measure on  $X^{\mathbb{Z}^d}$  whenever

$$\forall A \in \mathcal{F}_d, \Gamma_{A^c}$$
-a.s.  $(x_{A^c}), \Gamma(dx_A | x_{A^c}) = \rho_A(x_A \vee x_{A^c}) d\lambda_A(x_A).$

**Note.** (a) When the modification  $\rho$  is constructed via an interaction potential  $\Psi$ , the corresponding Gibbs measures are called  $(\Psi; \lambda)$ -Gibbs measures.

(b) When one knows that the family of  $\geq 0$  functionals  $\rho = (\rho_A)_{A \in \mathcal{F}_d}$  on  $X^{\mathbb{Z}^d}$  is a  $(\Gamma; \lambda)$ -modification, to make sure that  $\Gamma$  is a  $(\rho; \lambda)$ -Gibbs measure one simply has to check that the identity

$$\Gamma(dx|\xi) = \rho_{\{i\}}(x \vee \xi) d\lambda_i(x)$$

holds  $\Gamma_{\{i\}^c}$ -a.s.  $(\xi)$ , for each site  $i \in \mathbb{Z}^d$ .

Several authors (Föllmer and Orey, 1988; Olla, 1988; Comets, 1989) have investigated the LD asymptotics of the empirical process  $\hat{\pi}_x^{(A)}$  in the situation where  $x$  is distributed according to some  $(\Psi; \lambda)$ -Gibbs measure  $\mathcal{G}$  on  $X^{\mathbb{Z}^d}$ . In each of these articles, the Gibbsian interaction  $\Psi$  is assumed to satisfy translation invariance, so that

$$\psi_A(x^{(i)}) = \psi_{i+A}(x) \quad \text{for all } A, i, x,$$

as well as the following summability property

$$\|\Psi\| = \sum_{A \ni 0} \sup_x |\psi_A(x)| < +\infty.$$

Assuming additionally that each of the functionals defining the interaction  $\Psi$  is continuous and that the reference measure  $\lambda$  is simply the infinite tensor product of a single site probability measure  $\lambda_0$ , one may then state that the law of the empirical process  $\hat{\pi}_x^{(A)}$  under  $d\mathcal{G}(x)$  obeys a LDP on  $\mathcal{M}_s(X^{\mathbb{Z}^d})$ , on the scale  $|A|$  and according to the good rate function  $\mathcal{I}^\Psi : \mathcal{M}_s(X^{\mathbb{Z}^d}) \rightarrow [0; +\infty]$  given by

$$\forall P \in \mathcal{M}_s(X^{\mathbb{Z}^d}), \quad \mathcal{I}^\Psi(P) = \mathcal{H}_\lambda(P) - \int_{X^{\mathbb{Z}^d}} V^\Psi(x) dP(x) - p^\Psi,$$

where

$$V^\Psi(x) = - \sum_{A \ni 0} \frac{1}{|A|} \psi_A(x)$$

and

$$p^\Psi = \inf_{Q \in \mathcal{M}_s(X^{\mathbb{Z}^d})} \left\{ \mathcal{H}_\lambda(P) - \int_{X^{\mathbb{Z}^d}} V^\Psi(x) dP(x) \right\},$$

$\mathcal{H}_\lambda(P) = \mathcal{H}(P|\lambda_0^{\otimes \mathbb{Z}^d})$  being the specific entropy relative to  $\lambda = \lambda_0^{\otimes \mathbb{Z}^d}$ .

One should also add that the very same LD results are valid when considering the law of the empirical process  $\hat{\pi}_x^{(A)}$  under  $d\mathcal{G}(x|\xi_{A^c})$ , the probability  $d\mathcal{G}(x|\xi_{A^c})$  being a conditional version of  $d\mathcal{G}(x)$  knowing that  $x_{A^c} = \xi_{A^c}$ , and  $\xi \in X^{\mathbb{Z}^d}$  being any fixed boundary condition.

Before stating the Integration by Parts Formula that was developed by Cattiaux, Roelly and Zessin in the context of Gibbs measures on path space  $\Omega = W_T^{\mathbb{Z}^d}$ , we would like to motivate such developments and show how the infinite dimensional dynamics  $Q_\infty$  (defined after Corollary 2.1) may also be presented as a Gibbs measure on  $\Omega$ . It is actually a rather straightforward task to find a satisfactory expression for the interaction  $\Psi^a$  corresponding to  $Q_\infty$  when  $\Omega$  is equipped with the reference measure  $R_T^{\otimes \mathbb{Z}^d}$ ,  $R_T$

denoting the Wiener measure on  $W_T$  having initial condition  $\mu_0$  (cf. Corollary 2.1). Indeed, setting periodic boundary conditions on the finite cubic box  $\Lambda = [-N, N]^d \cap \mathbb{Z}^d$ , one may notice that the Radon–Nykodím derivative of the averaged regime probability  $Q_\Lambda$  with respect to the reference probability measure  $R_T^{\otimes \Lambda}$  writes

$$\frac{dQ_\Lambda}{dR_T^{\otimes \Lambda}}(\mathbf{x}_\Lambda) = \exp \left\{ \sum_{i \in \Lambda} \int_0^T \left( -U'(x_t^i) + \beta \sum_{j \sim i} x_t^j + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) \right) dx_t^i - \frac{1}{2} \sum_{i \in \Lambda} \int_0^T \left( -U'(x_t^i) + \beta \sum_{j \sim i} x_t^j + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) \right)^2 dt \right\},$$

$j \sim i$  meaning here that sites  $i, j \in \Lambda$  are nearest neighbours when  $\Lambda$  is considered with its periodic boundary conditions, and  $\gamma$  still denoting the function given by:  $\gamma_t = \sigma^2 \beta^2 / (1 + \sigma^2 \beta^2 t)$ .

Introducing the functionals  $F_i, G_i, H_i : \mathbb{R}_+ \times \Omega^3 \rightarrow \mathbb{R}$  given by

$$F_i(t; \mathbf{x}, \mathbf{y}, \mathbf{z}) = G_i(t; \mathbf{x}, \mathbf{y}, \mathbf{z}) + H_i(t; \mathbf{x}, \mathbf{y}, \mathbf{z})$$

and

$$G_i(t; \mathbf{x}, \mathbf{y}, \mathbf{z}) = -U(x^i) + \gamma_t x^i y^i, \quad H_i(t; \mathbf{x}, \mathbf{y}, \mathbf{z}) = \beta x^i \sum_{j \sim i} (x^j - \gamma_t z^j) - \beta \gamma_t z^i \sum_{j \sim i} x^j$$

we then have

$$\frac{dQ_\Lambda}{dR_T^{\otimes \Lambda}}(\mathbf{x}_\Lambda) = \exp\{-K_T^A(\mathbf{x}_\Lambda)\},$$

where (under  $dR_T^{\otimes \Lambda}(\mathbf{x}_\Lambda)$ ):

$$\begin{aligned} -K_T^A(\mathbf{x}_\Lambda) &= \sum_{i \in \Lambda} \int_0^T \left( \frac{\partial F_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) \right) dx_t^i - \frac{1}{2} \sum_{i \in \Lambda} \int_0^T \left( \frac{\partial F_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) \right)^2 dt \\ &= \sum_{i \in \Lambda} \int_0^T \left( \frac{\partial G_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) \right) dx_t^i + \sum_{i \in \Lambda} \int_0^T \left( \frac{\partial H_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) \right) dx_t^i \\ &\quad - \frac{1}{2} \sum_{i \in \Lambda} \int_0^T \left( \frac{\partial G_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) \right)^2 dt \\ &\quad - \sum_{i \in \Lambda} \int_0^T \left( \frac{\partial G_i}{\partial x^i} \cdot \frac{\partial H_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) \right) dt \\ &\quad - \frac{1}{2} \sum_{i \in \Lambda} \int_0^T \left( \frac{\partial H_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) \right)^2 dt. \end{aligned}$$

Using Ito’s formula, we then obtain

$$\begin{aligned}
 -K_T^A(\mathbf{x}_A) &= \sum_{i \in A} [G_i(T; \mathbf{x}_T, \mathbf{y}_T, \mathbf{z}_T) - G_i(0; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)] \\
 &\quad - \frac{1}{2} \int_0^T \left( \frac{\partial G_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) \right)^2 dt \\
 &\quad - \int_0^T \frac{\partial G_i}{\partial y^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) (dx_t^i + U'(x_t^i) dt) - \int_0^T \frac{\partial G_i}{\partial t}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) dt \\
 &\quad - \frac{1}{2} \int_0^T \left( \frac{\partial^2 G_i}{\partial (x^i)^2} + 2 \frac{\partial^2 G_i}{\partial x^i \partial y^i} \right) (t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) dt \Big] \\
 &\quad + \sum_{i \in A} [H_i(T; \mathbf{x}_T, \mathbf{y}_T, \mathbf{z}_T) - H_i(0; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)] \\
 &\quad - \frac{1}{2} \int_0^T \frac{\partial H_i}{\partial z^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) x_t^i dt \\
 &\quad + \beta \int_0^T \gamma_t \left( x_t^i \sum_{k \sim i} x_t^k + x_t^i \sum_{l \sim j} x_t^l \right) dt - \beta \int_0^T \gamma_t^2 \left( z_t^i x_t^j + z_t^j x_t^i \right) dt \Big] \\
 &\quad - \frac{1}{2} \sum_{i \in A} \int_0^T \left( -U'(x_t^i) + \beta \sum_{j \sim i} x_t^j + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) \right)^2 dt.
 \end{aligned}$$

In this last expression for the Girsanov exponent  $-K_T^A(\mathbf{x}_A)$ , the first sum obviously corresponds to a sum of self interactions, the second sum corresponds to a sum of nearest neighbours two spins interactions and the third sum corresponds to a sum of self interactions, nearest neighbours two spins interactions and three spins  $\{x^i, x^j, x^k\}$  interactions, where  $i \sim j \sim k$  in  $\Lambda_{\text{per}}$ . and  $i \neq k$ , so that finally

$$K_T^A(\mathbf{x}_A) = \sum_{\Gamma \subset A} \Psi_\Gamma^a(\mathbf{x}),$$

$\Psi^a$  being the interaction given by

$$\begin{aligned}
 \Psi_i^a(\mathbf{x}) &= U(x_T^i) - U(x_0^i) + \frac{1}{2} (\gamma_T (x_T^i)^2 - 2x_T^i y_T^i) - x_0^i{}^2 \\
 &\quad - \frac{1}{2} \int_0^T (U''(x_t^i) - U'(x_t^i)^2) dt + \frac{1}{2} \int_0^T (1 + 2(x_t^i - y_t^i)U'(x_t^i)) \gamma_t dt \\
 &\quad + \frac{1}{2} \int_0^T (\gamma_t (x_t^i - y_t^i))^2 dt, \\
 \Psi_{i,j}^a(\mathbf{x}) &= -\beta (x_T^i x_T^j - x_0^i x_0^j) + \beta \gamma_T (z_T^j x_T^i + z_T^i x_T^j) \\
 &\quad - \beta \int_0^T (U'(x_t^i) x_t^j + U'(x_t^j) x_t^i) dt + \frac{\beta^2}{2} \int_0^T (x_t^i{}^2 + x_t^j{}^2) dt
 \end{aligned}$$

$$\begin{aligned}
 & + \beta \int_0^T (U'(x_t^i)z_t^j + U'(x_t^j)z_t^i + x_t^i y_t^j + x_t^j y_t^i - 2x_t^i x_t^j) \gamma_t \, dt \\
 & - \beta^2 \int_0^T (x_t^i z_t^i + x_t^j z_t^j) \gamma_t \, dt - \beta \int_0^T (z_t^i (y_t^j - x_t^j) + z_t^j (y_t^i - x_t^i)) \gamma_t^2 \, dt \\
 & + \frac{\beta^2}{2} \int_0^T (z_t^i{}^2 + z_t^j{}^2) \gamma_t^2 \, dt \quad \text{if : } j \sim i, \\
 \psi_{i,j,k}^a(\mathbf{x}) & = \beta^2 \left( \int_0^T x_t^i x_t^k \, dt - \int_0^T (x_t^j z_t^k + x_t^k z_t^j) \gamma_t \, dt + \int_0^T z_t^j z_t^k \gamma_t^2 \, dt \right) \\
 & \text{if : } j \sim i \sim k, \quad j \neq k,
 \end{aligned}$$

$$\psi_A^a(\mathbf{x}) = 0 \quad \text{else.}$$

**Note.** Remembering that:  $y_t^i = x_t^i - x_0^i + \int_0^t U'(x_s^i) \, ds$ , we also have

$$\int_0^T \gamma_t (x_t^i - y_t^i) U'(x_t^i) \, dt = - \int_0^T \gamma_t \tilde{y}_t^i \, d\tilde{y}_t^i \quad \text{for : } \tilde{y}_t^i = y_t^i - x_t^i,$$

so that

$$\begin{aligned}
 & \int_0^T \gamma_t (x_t^i - y_t^i) U'(x_t^i) \, dt + \frac{1}{2} \int_0^T (\gamma_t (x_t^i - y_t^i))^2 \, dt \\
 & = - \frac{1}{2} (\gamma_T (\tilde{y}_T^i)^2 - \gamma_0 (\tilde{y}_0^i)^2) \\
 & = - \frac{\gamma_T}{2} \left( \int_0^T U'(x_t^i) \, dt \right)^2 + \frac{1}{2} (x_0^i)^2,
 \end{aligned}$$

which yields the following alternative expression for  $\psi_i$ :

$$\begin{aligned}
 \psi_i^a(\mathbf{x}) & = U(x_T^i) - U(x_0^i) + \frac{1}{2} (\gamma_T (x_T^i{}^2 - 2x_T^i y_T^i) - x_0^i{}^2) \\
 & - \frac{1}{2} \int_0^T (U''(x_t^i) - U'(x_t^i)^2) \, dt + \frac{1}{2} \ln(1 + \sigma^2 \beta^2 T) \\
 & - \frac{\gamma_T}{2} \left( \int_0^T U'(x_t^i) \, dt \right)^2 + \frac{1}{2} (x_0^i)^2.
 \end{aligned}$$

Let us next present the elements of Malliavin calculus enabling us to give a proper statement of an integration by parts formula for Gibbs measures on  $\Omega$ .

**Definition 2.3.** •  $W^{1,2}(C[0, T])$  denotes the Sobolev space of all functionals  $F : C[0, T] \rightarrow \mathbb{R}$  such that:  $F$  is square integrable with respect to  $R_T$  ( $F \in L^2(C[0, T])$ ) and there exists a family

$$\{(D_t F(\omega))_{0 \leq t \leq T}; \omega \in C[0, T]\}$$

in  $L^2([0, T]; C[0, T])$  such that

$$\forall g \in L^2([0, T]), \quad D_g F(\omega) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( F \left( \omega + \varepsilon \int_0^\cdot g_s ds \right) - F(\omega) \right)$$

exists as a strong limit in  $L^2(C[0, T])$  and equals:  $\int_0^T g_t(D_t F)(\omega) dt$ .

- $W^{1,\infty}(C[0, T])$  is the subspace of  $W^{1,2}(C[0, T])$  consisting of all those  $F$  for which the family  $\{(D_t F(\omega))_{0 \leq t \leq T}; \omega \in C[0, T]\}$  takes its values in  $L^\infty(C[0, T]; L^2([0, T]))$ .
- We also let  $\mathcal{W}(C[0, T]) \subset W^{1,\infty}(C[0, T])$  denote the set of all regular functionals  $F : C[0, T] \rightarrow \mathbb{R}$  such that:

$$F(\omega) = f(\omega_{t_0}, \omega_{t_1}, \dots, \omega_{t_n})$$

for some finite sequence  $0 \leq t_0 < t_1 < \dots < t_n \leq T$  and some  $\mathcal{C}^\infty$ , compactly supported  $f$ .

Let us recall at this stage that the very first integration by parts formula of Malliavin calculus (see Nualart, 1998, Chapter 1) states that, relatively to the standard Wiener measure on  $C[0, T]$ :

$$\begin{aligned} \forall F \in \mathcal{W}(C[0, T]), \quad \forall g \in L^2([0, T]), \\ \mathbb{E} \left( \int_0^T g_t D_t F dt \right) = \mathbb{E}(D_g F) = \mathbb{E} \left( F \int_0^T g_t dw_t \right). \end{aligned}$$

We next define Sobolev spaces corresponding to the infinite product  $\Omega$ .

- For any  $i \in \mathbb{Z}^d$  and  $g = g^i \in L^2([0, T])$ , for  $\omega \in \Omega_T$ , let  $\omega + \varepsilon \int_0^\cdot g_s^i ds$  denote the element  $\omega'$  of  $\Omega$  such that

$$\omega'_j = \omega_j \quad \text{for } j \neq i \quad \text{and} \quad \omega'_i = \omega_i + \varepsilon \int_0^\cdot g_s^i ds.$$

The Sobolev space  $W^{1,2}(\Omega)$  then consists of all functionals  $F : \Omega \rightarrow \mathbb{R}$  that are square integrable with respect to the reference measure  $R_T^{\otimes \mathbb{Z}^d}$  and such that there exists a family  $\{(D_t^i F(\omega))_{i \in \mathbb{Z}^d, 0 \leq t \leq T}; \omega \in W_T\}$  in  $(L^2([0, T]; W_T))^{\mathbb{Z}^d}$  satisfying:

$$\begin{aligned} \forall i \in \mathbb{Z}^d, \quad \forall g_i \in L^2([0, T]), \\ \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( F \left( \omega + \varepsilon \int_0^\cdot g_s^i ds \right) - F(\omega) \right) = D_{g_i}^i F(\omega) = \int_0^T g_t^i (D_t^i F)(\omega) dt, \end{aligned}$$

the preceding limit being taken in the strong sense in  $L^2(\Omega)$ .

- $W^{1,\infty}(\Omega)$  is the subspace of  $W^{1,2}(\Omega)$  consisting of all those  $F$  for which the family  $\{(D_t^i F(\omega))_{i \in \mathbb{Z}^d, 0 \leq t \leq T}; \omega \in C[0, T]\}$  takes its values in  $(L^2(\Omega; L^\infty([0, T])))^{\mathbb{Z}^d}$ .
- $\mathcal{W}_{\text{loc}}(\Omega)$  is the set of all regular functionals  $F : \Omega \rightarrow \mathbb{R}$  satisfying:

$$F(\omega) = f(\omega_{t_0}^\Gamma, \omega_{t_1}^\Gamma, \dots, \omega_{t_n}^\Gamma)$$

for some finite subset  $\Gamma \subset \mathbb{Z}^d$  and some finite sequence  $0 \leq t_0 < t_1 < \dots < t_n \leq T$ ,  $f$  being  $\mathcal{C}^\infty$  and compactly supported.

- $W_{\text{loc}}^{1,\infty}(\Omega)$  (resp.  $W_{\text{loc}}^{1,2}(\Omega)$ ) consists of all functionals  $F = F(\omega) \in W^{1,\infty}(\Omega)$  (resp.  $W^{1,2}(\Omega)$ ) depending on  $\omega$  only through a finite number of coordinates:  $F(\omega) = F(\omega_\Gamma)$  for some  $\Gamma \subset \subset \mathbb{Z}^d$ .
- Finally, let  $\mathcal{E}_{[0,T]}$  denote the set of all elementary  $L^2$  functions  $g : [0, T] \rightarrow \mathbb{R}$ :

$$\forall g \in L^2([0, T]),$$

$$(g \in \mathcal{E}_{[0,T]}) \Leftrightarrow (\exists 0 = t_0 < t_1 < \dots < t_n = T, g \text{ is constant}$$

$$\text{in each interval } [t_{i-1}, t_i[).$$

We may now state an integration by parts formula holding for all Gibbs measures  $Q$  on infinite dimensional path space  $\Omega$  corresponding to some (reasonable) interaction  $\Phi$ .

**Theorem 2.2.** *Let  $H^\Phi$  denote the Hamiltonian potential corresponding to some interaction  $\Phi$  on  $(\Omega; R_T^{\otimes \mathbb{Z}^d})$  (c.f. Definition 3.1) and suppose that for each  $i \in \mathbb{Z}^d$  and for all  $\eta \in W_T^{(\mathbb{Z}^d \setminus \{i\})}$ ,  $H_i^\Phi(\omega; \eta)$  is a  $W^{1,2}$  functional of  $\omega \in C[0, T]$  and  $(D_t^i H_i^\Phi(\omega; \eta))_{0 \leq t \leq T}$  has a measurable version.*

Let  $Q$  be a probability measure on  $\Omega$  satisfying:

$$\forall t \in [0, T], \forall i \in \mathbb{Z}^d, \quad \mathbb{E}_Q(|\omega_t^i|) < \infty \quad \text{and} \quad \mathbb{E}_Q \left( \int_0^T |D_t^i H_i^\Phi| dt \right) < \infty.$$

If  $Q$  is a Gibbs measure with respect to  $\Phi$  and  $R_T^{\otimes \mathbb{Z}^d}$ , then

$$\forall F \in W_{\text{loc}}^{1,\infty}(\Omega_T), \forall i \in \mathbb{Z}^d, \forall g_i \in \mathcal{E}_{[0,T]},$$

$$\mathbb{E}_Q \left( F \int_0^T g_t^i d\omega_t^i \right) = \mathbb{E}_Q(D_{g_t^i}^i F) - \mathbb{E}_Q(F D_{g_t^i}^i H_i^\Phi).$$

**Proof.** See (Cattiaux et al., 1996, Theorem 2.11, first part).  $\square$

### 2.3. Characterisation of $Q_\infty$ as a Gibbs measure

Although the infinite-dimensional system of interacting diffusions  $\mathcal{L}_\infty$  given by

$$\left\{ \begin{array}{l} dx_t^i = d\omega_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt \\ dy_t^i = d\omega_t^i + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt \\ dz_t^i = x_t^i dt \\ \text{Law}(\mathbf{x}|_{t=0}) = \mu_0^{\otimes \mathbb{Z}^d}, \quad \mathbf{y}|_{t=0} = \mathbf{z}|_{t=0} = \mathbf{0} \quad (i \in \mathbb{Z}^d, 0 \leq t \leq T) \end{array} \right.$$

is not a gradient one, we may still use the Gibbsian integration by parts formula as an essential tool to prove the following.

**Proposition 2.2.** *Let  $Q$  be a probability measure on  $\Omega$ . If  $Q$  is Gibbsian with respect to the interaction  $\Psi^a$  and to the reference measure  $R_T^{\otimes \mathbb{Z}^d}$ , then  $Q$  is the  $\mathbf{x}$ -marginal of a weak solution corresponding to the system  $\mathcal{S}_\infty$ , consequently:  $Q = Q_\infty$  (since  $\mathcal{S}_\infty$  has a unique strong solution).*

**Proof.** We shall simply check that the second half of the proof of Theorem 3.7 in (Cattiaux et al., 1996) is also valid in our context.

First of all, let us remark that our Gibbs measure  $Q$  is such that

$$\forall i \in \mathbb{Z}^d, \quad Q \ll R_T \otimes Q_{\mathbb{Z}^d \setminus \{i\}},$$

hence there is a unique adapted process  $(\beta_t^i)_{0 \leq t \leq T}$  for which  $\{x_t^i - x_0^i - \int_0^t \beta_s^i ds\}_{0 \leq t \leq T}$  is a standard brownian motion under  $dQ(x)$ .

Secondly,  $Q$  satisfies

$$\forall t \in [0, T], \quad \forall i \in \mathbb{Z}^d, \quad \mathbb{E}_Q(|\omega_t^i|) < \infty$$

and

$$\forall i \in \mathbb{Z}^d, \quad \mathbb{E}_Q \left( \int_0^T |D_t^i H_i| dt \right) < \infty$$

(cf. Lemma 3.2 in the next section).

Therefore we may use the Gibbsian integration by parts formula to show that

$$Q\text{-a.s.}(\mathbf{x}), \quad \forall i \in \mathbb{Z}^d, \quad \beta_t^i(\mathbf{x}) = -\frac{\partial F_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t),$$

the functional  $F_i$  being defined by

$$F_i(t; \mathbf{x}, \mathbf{y}, \mathbf{z}) = U(x^i) - \beta x^i \sum_{j \sim i} x^j - \gamma_t x^i \left( y^i - \beta \sum_{j \sim i} z^j \right).$$

Indeed,  $\{(\beta_t^i)_{0 \leq t \leq T}; i \in \mathbb{Z}^d\}$  is uniquely characterised (up to  $Q$  indistinguishability) by the fact that

$$\forall 0 \leq s < t \leq T, \quad \forall i \in \mathbb{Z}^d, \quad \mathbb{E}_Q \left[ A_s \left( x_t^i - x_s^i - \int_s^t \beta_r^i dr \right) \right] = 0$$

$A_s : \Omega_T \rightarrow \mathbb{R}$  being any  $\mathcal{W}_{loc}$ ,  $\mathcal{F}_s$ -measurable functional on  $\Omega_T$ .

We may also write the preceding equation as

$$\mathbb{E}_Q \left[ A_s \int_0^T 1_{]s, t]} dx_r^i \right] = \mathbb{E}_Q \left[ A_s \int_s^t \beta_r^i dr \right]$$

and after integrating by parts we obtain

$$\mathbb{E}_Q \left[ A_s \left( \int_s^t \beta_r^i dr + D_{1]s, t]}^i H_i^{\Psi^a} \right) \right] = 0 \tag{*}$$

since  $D_{1]s, t]}^i A_s \equiv 0$ .

One then simply has to compute  $D_{1|s,t}^i H_i^{\Psi^a}$  in order to make sure that (\*) holds true when setting:

$$\beta_t^i(\mathbf{x}) = -\frac{\partial F_i}{\partial x^i}(t; \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t).$$

Taking into account the fact that

$$\begin{aligned} H_i^{\Psi^a}(\mathbf{x}) &= F_i(T; \mathbf{x}_T, \mathbf{y}_T, \mathbf{z}_T) - F_i(0; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) \\ &\quad - \int_0^T \frac{\partial F_i}{\partial y_i} dy_r^i - \int_0^T \frac{\partial F_i}{\partial z_i} x_r^i dr - \sum_{j \sim i} \int_0^T \frac{\partial F_i}{\partial z_j} x_r^j dr - \int_0^T \frac{\partial F_i}{\partial r} dr \\ &\quad - \frac{1}{2} \int_0^T \left( \frac{\partial^2 F_i}{\partial x_i^2} + 2 \frac{\partial^2 F_i}{\partial x_i \partial y_i} \right) dr \\ &\quad + \frac{1}{2} \int_0^T \left( \frac{\partial F_i}{\partial x_i} \right)^2 dr + \sum_{j \sim i} \int_0^T \left( \frac{\partial F_i}{\partial x_j} \cdot \frac{\partial F_j}{\partial x_j} \right) dr \\ &\quad - \frac{1}{2} \sum_{j \sim i} \int_0^T \left( \frac{\partial F_i}{\partial x_j} \right)^2 dr \\ &= \int_0^T \frac{\partial F_i}{\partial x_i} dx_r^i + \sum_{j \sim i} \int_0^T \frac{\partial F_i}{\partial x_j} dx_r^j - \frac{1}{2} \int_0^T \left( \frac{\partial F_i}{\partial x_i} \right)^2 dr \\ &\quad - \frac{1}{2} \sum_{j \sim i} \int_0^T \left( \frac{\partial F_i}{\partial x_j} \right)^2 dr, \end{aligned}$$

we may then proceed to the (rather lengthy) Malliavin differentiation of each term in the preceding sum. To this end, we define

$$\forall r \in [0, T], \quad \varphi_r = \int_0^r 1_{|s,t|}(v) dv = ((r \wedge t) \vee s - s), \quad \phi_r = \int_0^r \varphi_v dv$$

and

$$\begin{aligned} \forall \varepsilon > 0, \quad x_r^{i,\varepsilon} &= x_r^i + \varepsilon \varphi_r \\ y_r^{i,\varepsilon} &= x_r^{i,\varepsilon} - x_0^{i,\varepsilon} + \int_0^r U'(x_v^{i,\varepsilon}) dv \\ z_r^{i,\varepsilon} &= \int_0^r x_v^{i,\varepsilon} dv, \end{aligned}$$

we then observe that

$$\begin{aligned} \frac{1}{\varepsilon} (x_r^{i,\varepsilon} - x_r^i) &\xrightarrow{\varepsilon \searrow 0} \varphi_r \\ \frac{1}{\varepsilon} (y_r^{i,\varepsilon} - y_r^i) &\xrightarrow{\varepsilon \searrow 0} \varphi_r + \int_0^r U''(x_v^i) \varphi_v dv \end{aligned}$$

$$\frac{1}{\varepsilon} (z_r^{i,\varepsilon} - z_r^i) \xrightarrow{\varepsilon \searrow 0} \phi_r,$$

these convergences taking place in  $L^2_{R_t}$ .

One thus obtains for example

$$D^i_{1[s,t]} \left\{ \int_0^T \frac{\partial F_i}{\partial x_i} dx_r^i \right\} = \int_s^t \frac{\partial F_i}{\partial x_i} dr + \int_s^T \left( \frac{\partial^2 F_i}{\partial x_i^2} \cdot \phi_r + \frac{\partial^2 F_i}{\partial x_i \partial y_i} \right. \\ \left. \times \left( \phi_r + \int_0^r U''(x_v^i) \phi_v dv \right) + \frac{\partial^2 F_i}{\partial x_i \partial z_i} \cdot \phi_r \right) (dM_r^i + \beta_r^i dr),$$

where  $(M_r^i)$  is a standard brownian motion under  $dQ$ , and

$$D^i_{1[s,t]} \left\{ -\frac{1}{2} \int_0^T \left( \frac{\partial F_i}{\partial x_i} \right)^2 dr \right\} \\ = - \int_s^T \left( \frac{\partial F_i}{\partial x_i} \frac{\partial^2 F_i}{\partial x_i^2} \cdot \phi_r + \frac{\partial F_i}{\partial x_i} \frac{\partial^2 F_i}{\partial x_i \partial y_i} \right. \\ \left. \times \left( \phi_r + \int_0^r U''(x_v^i) \phi_v dv \right) + \frac{\partial F_i}{\partial x_i} \frac{\partial^2 F_i}{\partial x_i \partial z_i} \cdot \phi_r \right) dr,$$

so that all in all the process  $\{(\beta_r^i)_{0 \leq t \leq T}; i \in \mathbb{Z}^d\}$  satisfies Eq. (\*) if and only if

$$\mathbb{E}_Q \left[ A_s \left\{ \int_s^t \left( \beta_r^i + \frac{\partial F_i}{\partial x_i}(r, \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r) \right) dr \right\} \right] \\ = \mathbb{E}_Q \left[ A_s \left\{ - \int_s^T \left( \frac{\partial^2 F_i}{\partial x_i^2} \cdot \phi_r + \frac{\partial^2 F_i}{\partial x_i \partial y_i} \cdot \left( \phi_r + \int_0^r U''(x_v^i) \phi_v dv \right) \right. \right. \right. \\ \left. \left. + \frac{\partial^2 F_i}{\partial x_i \partial z_i} \cdot \phi_r \right) \beta_r^i dr - \sum_{j \sim i} \int_s^T \left( \frac{\partial^2 F_i}{\partial x_j \partial x_i} \cdot \phi_r + \frac{\partial^2 F_i}{\partial x_j \partial y_i} \right. \right. \\ \left. \left. \times \left( \phi_r + \int_0^r U''(x_v^i) \phi_v dv \right) + \frac{\partial^2 F_i}{\partial x_j \partial z_i} \cdot \phi_r \right) \beta_r^j dr \right. \right. \\ \left. + \int_s^T \left( \left( \frac{\partial F_i}{\partial x_i} \frac{\partial^2 F_i}{\partial x_i^2} \right) \cdot \phi_r + \left( \frac{\partial F_i}{\partial x_i} \frac{\partial^2 F_i}{\partial x_i \partial y_i} \right) \cdot \left( \phi_r + \int_0^r U''(x_v^i) \phi_v dv \right) \right. \right. \\ \left. \left. + \left( \frac{\partial F_i}{\partial x_i} \frac{\partial^2 F_i}{\partial x_i \partial z_i} \right) \cdot \phi_r \right) dr + \sum_{j \sim i} \int_s^T \left( \left( \frac{\partial F_i}{\partial x_j} \frac{\partial^2 F_i}{\partial x_j \partial x_i} \right) \cdot \phi_r \right. \right. \\ \left. \left. + \left( \frac{\partial F_i}{\partial x_j} \frac{\partial^2 F_i}{\partial x_j \partial y_i} \right) \cdot \left( \phi_r + \int_0^r U''(x_v^i) \phi_v dv \right) \right. \right. \\ \left. \left. + \left( \frac{\partial F_i}{\partial x_j} \frac{\partial^2 F_i}{\partial x_j \partial z_i} \right) \cdot \phi_r \right) dr \right\} \right].$$

This last equality is certainly satisfied for any of our test functionals  $A_s$  if

$$\beta_r^i(\mathbf{x}) = -\frac{\partial F_i}{\partial x^i}(r; \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r),$$

and the proposition is proved.  $\square$

Since we are in a situation where the spin space  $W_T$  is non-compact and where the translation invariant interaction  $\Psi^a$  is not a summable one, it is not obvious at first sight that there exists indeed a Gibbs measure  $Q \in \mathcal{M}(\Omega)$  corresponding to  $\Psi^a$  and to the reference measure  $R_T^{\otimes \mathbb{Z}^d}$ ; however, this fact may be seen to follow from two results contained in Chapter 4 of (Georgii, 1988), namely Corollary (4.13, p. 63) together with Theorem (4.17, p. 67). So the set consisting of all  $(\Psi^a; R_T^{\otimes \mathbb{Z}^d})$ -Gibbs measures is non-empty, and according to the preceding proposition this set is actually reduced to  $\{Q_\infty\}$ .

### 3. Averaged large deviations of the empirical process

Although  $Q_\infty$  is the unique Gibbs measure associated with the interaction  $\Psi^a$  and reference measure  $R_T^{\otimes \mathbb{Z}^d}$ , some verifications are needed in order to state a LDP when considering the law of the empirical process  $\hat{\pi}_x^{(A)}$  under  $dQ_A(\mathbf{x})$ , since the interaction we are dealing with is not a summable one. Nevertheless, proceeding just as in Comets (1989) we may observe, still using periodic boundary conditions for  $A$ , that the following identity holds true for each Borel set  $\mathcal{A} \subset \mathcal{M}_s(\Omega)$ :

$$\begin{aligned} \Pi_A(\mathcal{A}) &= \int_{W_T^{\otimes A}} dR_T^{\otimes A}(\mathbf{x}) \mathbf{1}_{\{\hat{\pi}_x^{(A)} \in \mathcal{A}\}} e^{-\sum_{A' \subset A} \psi_{A'}^a(\mathbf{x})} \\ &= \int_{W_T^{\otimes A}} dR_T^{\otimes A}(\mathbf{x}) \mathbf{1}_{\{\hat{\pi}_x^{(A)} \in \mathcal{A}\}} e^{-\sum_{i \in A} \left( \sum_{A' \ni i} \frac{\psi_{A'}^a(\mathbf{x})}{|A'|} \right)} \\ &= \int_{\mathcal{A}} d\Pi_A^0(\pi) e^{|A| \int V^a d\pi}, \end{aligned}$$

$\Pi_A^0$  denoting the law of the empirical process  $\hat{\pi}_x^{(A)}$  under  $dR_T^{\otimes A}(\mathbf{x})$ , and  $V^a: \Omega \rightarrow \mathbb{R}$  being naturally defined by

$$\forall \mathbf{x} \in \Omega, \quad V^a(\mathbf{x}) = -\sum_{A \ni 0} \frac{\psi_A^a(\mathbf{x})}{|A|}.$$

Letting  $L$  be some positive real number for which  $\mu_0([-L; L])=1$  and  $\mathcal{M}_{s,L}(\Omega)$  denote the subset of  $\mathcal{M}_s(\Omega)$  consisting of all  $Q$  such that

$$-L \leq x_0^i \leq L, \quad Q\text{-a.s.}(\mathbf{x}) \quad (\forall i \in \mathbb{Z}^d),$$

we first establish the large deviation upper bound for  $\{\Pi_A\}$  by making use of Varadhan’s method (see Varadhan, 1966, Section 3 or Dembo and Zeitouni, 1998, Section 4.3): this amounts essentially to check that the functional

$$\mathcal{V}: \pi \mapsto \int V^a d\pi$$

may be defined on the whole space  $\mathcal{M}_{s,L}$  as an upper semicontinuous functional taking values in the interval  $[-\infty; M]$  for some  $M < +\infty$  depending only on the choices of  $\beta, \sigma, T$  and  $L$ .

$\mathcal{H}$  still denoting the specific entropy relative to the reference measure  $R_T^{\otimes \mathbb{Z}^d}$ , we then check that the good rate functional

$$\mathcal{I}^a = \mathcal{H} - \mathcal{V}$$

vanishes only at  $Q_\infty$ . To be more precise, we establish that our nonsummable interaction  $\Psi^a$  also satisfies the variational principle of standard Gibbsian theory (see Georgii, 1988, Theorem 15.39, p. 325), and this is enough to make sure that  $\mathcal{I}^a$  vanishes exclusively at  $Q_\infty$  since we know that the set of all  $(\Psi^a; R_T^{\otimes \mathbb{Z}^d})$ -Gibbs measures reduces to  $\{Q_\infty\}$ .

We finally prove the large deviations lower bound for  $\{\Pi_\Lambda\}$  following the method of Föllmer and Orey (see Föllmer and Orey, 1988, Section 3) and using the fact that for any open set  $\mathcal{O} \subset \mathcal{M}_{s,L}(\Omega)$ :

$$\inf_{\pi \in \mathcal{O}} \{\mathcal{H}(\pi) - \mathcal{V}(\pi)\} = \inf_{\pi \in \mathcal{O} \cap \mathcal{M}_\mathcal{V}} \{\mathcal{H}(\pi) - \mathcal{V}(\pi)\},$$

$\mathcal{M}_\mathcal{V}$  denoting the set of all points  $\pi \in \mathcal{M}_{s,L}(\Omega)$  at which the functional  $\mathcal{V} : \pi \mapsto \int V^a d\pi$  is lower semicontinuous.

### 3.1. Large deviations upper bound

Let us begin with the following

**Proposition 3.1.**  *$\mathcal{V} : \pi \mapsto \int V^a d\pi$  may be defined on  $\mathcal{M}_{s,L}(\Omega)$  as an upper semicontinuous functional taking values in the interval  $[-\infty; M]$ , for some  $M > 0$  depending only on the choices of  $\beta, \sigma, T$  and  $L$ .*

**Proof.** A quick glance at the expression for the interaction  $\Psi^a$  given earlier enables one to make sure that

$$\forall \mathbf{x} \in W_T^{\mathbb{Z}^d},$$

$$V^a(\mathbf{x}) = P_1(\gamma_T, \{x_0^i, x_T^i, y_0^i, y_T^i, z_0^i, z_T^i\}_{i \in \Delta}) + \int_0^T P_2(\gamma_t, \{x_t^i, y_t^i, z_t^i\}_{i \in \Delta}) dt,$$

where

$$\Delta = \{i \in \mathbb{Z}^d \mid (i = O) \text{ or } (i \sim O) \text{ or } (i \sim j \text{ for some } j \sim O)\},$$

$P_1$  and  $P_2$  being polynomial functions.

To make sure that

$$\int d\pi(\mathbf{x}) \left( \int_0^T P_2(\gamma_t, \{x_t^i, y_t^i, z_t^i\}_{i \in \Delta}) dt \right)$$

is bounded from above uniformly when  $\pi$  varies in  $\mathcal{M}_s(\Omega)$ , we shall first deal with the terms in  $(1) = \int_0^T P_2(\gamma_t, \{x_t^i, y_t^i, z_t^i\}_{i \in \Delta}) dt$  coming from the “single site interaction

potential”  $\psi_0^a$ ; they sum to

$$(1)' = \frac{\beta}{2} \int_0^T (U''(x_t^O) - U'(x_t^O)^2) dt - \frac{\beta}{2} \int_0^T (1 + 2(x_t^O - y_t^O)U'(x_t^O))\gamma_t dt - \frac{\beta}{2} \int_0^T (\gamma_t(x_t^O - y_t^O))^2 dt,$$

which is also equal to

$$\frac{\beta}{2} \int_0^T (U''(x_t^O) - U'(x_t^O)^2) dt - \frac{\beta}{2} \ln(1 + \sigma^2 \beta^2 T) + \frac{\beta}{2} \gamma_T \left( \int_0^T U'(x_t^O) dt \right)^2 - \frac{\beta}{2} (x_0^O)^2$$

according to the remark preceding Definition 2.3.

Next, according to Jensen’s inequality:

$$\frac{1}{T^2} \left( \int_0^T U'(x_t^O) dt \right)^2 \leq \frac{1}{T} \int_0^T U'(x_t^O)^2 dt,$$

so that finally

$$(1)' \leq K + \frac{\beta}{2} \int_0^T U''(x_t^O) dt - \frac{\beta}{2} \delta_T \int_0^T U'(x_t^O)^2 dt$$

for  $\delta_T = 1 - T\gamma_T = 1/(1 + \sigma^2 \beta^2 T)$  (here and in the sequel of the proof,  $K, K_1, K_2, \dots$  are some positive constants depending merely on the choices of  $\beta, \sigma, T$  and  $\mu_0$ ).

The terms remaining in the integral:  $\int d\pi(\mathbf{x})(\int_0^T P_2 dt)$  (coming from the “two sites and three sites interaction potentials”) may be dealt with by using Young inequalities and the translation invariance of  $\pi$ . For example: the integral term

$$\int d\pi(\mathbf{x}) \left\{ \frac{\beta^2}{2} \int_0^T U'(x_t^i)x_t^O dt \right\}$$

(coming from the interaction  $\psi_{O,i}^a$  for some  $i \sim O$ ) is not greater than

$$\frac{\beta^2}{2} \int d\pi(\mathbf{x}) \left\{ \frac{3}{5} \int_0^T |U'(x_t^O)|^{5/3} dt + \frac{2}{5} \int_0^T |x_t^O|^{5/2} dt \right\}$$

and the integral term

$$\int d\pi(\mathbf{x}) \left\{ \frac{\beta^5}{3} \int_0^T \gamma_t x_t^i z_t^j dt \right\}$$

(coming from the interaction  $\psi_{O,i,j}^a$  for some  $j \sim i \sim O$ ) is not greater than

$$\frac{\beta^5}{3} \int d\pi(\mathbf{x}) \left\{ \frac{1}{2} \int_0^T \gamma_t^2 (x_t^O)^2 dt + \frac{1}{2} \int_0^T (z_t^O)^2 dt \right\},$$

which may be bounded from above by

$$\frac{\beta^5}{3} \int d\pi(\mathbf{x}) \left\{ \frac{1}{2} \int_0^T \gamma_t^2 (x_t^O)^2 dt + \frac{T^2}{2} \int_0^T (x_t^O)^2 dt \right\},$$

remembering that:  $z_t^O = \int_0^t x_s^O ds$  and using Jensen’s inequality.

All in all, one finds out that for any translation invariant probability measure  $\pi \in \mathcal{M}_s(\Omega)$ , the integral:

$$\int d\pi(\mathbf{x}) \left( \int_0^T P_2(\gamma_t, \{x_t^i, y_t^i, z_t^i\}_{i \in \Delta}) dt \right)$$

is bounded from above by

$$\int d\pi(\mathbf{x}) \int_0^T \left( -\frac{\beta}{2} \delta_T U'(x_t^O)^2 + (K_1 |x_t^O|^5 + K_2) \right) dt$$

for some positive constants  $K_1$  and  $K_2$ .

We are now in a position to complete the proof of the proposition by taking into account the term  $P_1$  arising in the expression for  $V$  given above.

We have

$$P_1 = -\beta U(x_T^O) + \beta U(x_0^O) - \frac{\beta^3}{2} (\gamma_T (x_T^O)^2 - 2\gamma_T x_T^O y_T^O - x_0^O) + \beta^2 \sum_{i \sim O} (x_T^i x_T^O - x_0^i x_0^O) - \beta^2 \sum_{i \sim O} \gamma_T (z_T^i x_T^O + z_T^O x_T^i),$$

and we may here again use Jensen’s inequality, Young inequalities and the translation invariance of  $\pi$ . For example, the term

$$\int d\pi(\mathbf{x}) \{ \beta^2 x_T^i x_T^O \}$$

(coming from a nearest neighbour interaction with the origin) is bounded from above by

$$\beta^2 \int d\pi(\mathbf{x}) (x_T^O)^2$$

and the term

$$\int d\pi(\mathbf{x}) \{ -\beta^2 z_T^i x_T^O \}$$

is not greater than

$$\frac{\beta^2}{2} \int d\pi(\mathbf{x}) (x_T^O)^2 + \frac{\beta^2 T}{2} \int d\pi(\mathbf{x}) \left\{ \int_0^T (x_t^O)^2 dt \right\}.$$

Handling similarly the terms remaining in  $\int d\pi(\mathbf{x})P_1$ , one may then assert that

$$\begin{aligned} \int d\pi(\mathbf{x})V^a(\mathbf{x}) &= \int d\pi(\mathbf{x}) \left\{ P_1 + \int_0^T P_2 dt \right\} \\ &\leq \int d\pi(\mathbf{x}) \left\{ \int_0^T \left( -\frac{\beta}{2} \delta_T U'(x_t^O)^2 + (K_3|x_t^O|^5 + K_4) \right) dt \right\} \\ &\quad + \int d\pi(\mathbf{x}) \{ -\beta U(x_T^O) + K_5|x_T^O|^3 + K_6 \}. \end{aligned}$$

Hence, letting

$$M = T \cdot \sup_{x \in \mathbb{R}} \left\{ -\frac{\beta}{2} \delta_T U'(x)^2 + (K_3|x|^5 + K_4) \right\} + \sup_{x \in \mathbb{R}} \{ -\beta U(x) + K_5|x|^3 + K_6 \},$$

we finally have

$$\mathcal{V}(\pi) = \int V(\mathbf{x}) d\pi(\mathbf{x}) \leq M, \quad \forall \pi \in \mathcal{M}_{s,L}(\Omega).$$

Let us finally establish the upper semicontinuity of  $\mathcal{V} : \pi \mapsto \int V^a(\mathbf{x}) d\pi(\mathbf{x})$  on  $\mathcal{M}_{s,L}(\Omega)$ .

According to the two preceding points, the continuous functional  $V^a : \Omega \rightarrow \mathbb{R}$  may be decomposed as

$$V^a = V^+ - V^-,$$

with  $V^+ = V^a \cdot \mathbb{1}_{\{V \geq 0\}}$ ,  $V^- = V^a \cdot \mathbb{1}_{\{V < 0\}}$  and  $V^+$  is taking its values in  $[0; M]$  while  $V^-(\Omega) = [0; +\infty[$ .

Fixing  $\pi \in \mathcal{M}_{s,L}(\Omega)$  and a sequence  $(\pi_n)_{n \geq 1}$  on  $\mathcal{M}_{s,L}(\Omega)$  converging weakly to  $\pi$ , we certainly have

$$\int_{\Omega} V^+ d\pi_n \xrightarrow{n \rightarrow \infty} \int_{\Omega} V^+ d\pi$$

so that we only need to check that

$$\liminf_{n \rightarrow \infty} \int_{\Omega} V^- d\pi_n \geq \int_{\Omega} V^- d\pi$$

in order to establish that  $\mathcal{V}$  is upper semicontinuous.

Consider first the case where  $V^- \notin L^1(\pi)$ , so that

$$\int_{\Omega} V^- d\pi = +\infty.$$

Using Fatou’s lemma and Portmanteau’s theorem we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega} V^- d\pi_n &= \liminf_{n \rightarrow \infty} \int_0^{+\infty} \pi_n \{ V^- > y \} dy \\ &\geq \int_0^{+\infty} \left( \liminf_{n \rightarrow \infty} \pi_n \{ V^- > y \} \right) dy \\ &= \int_0^{+\infty} (\pi \{ V^- > y \}) dy = +\infty \end{aligned}$$

so that

$$\liminf_{n \rightarrow \infty} \int_{\Omega} V^- \, d\pi_n \geq \int_{\Omega} V^- \, d\pi.$$

In the case where  $\int_{\Omega} V^- \, d\pi = a < +\infty$ , one may consider for each  $A > 0$  a bounded continuous functional  $V_A^- : \Omega \rightarrow \mathbb{R}_+$  such that  $V_A^-$  takes its values in  $[0; A]$  and  $V_A^-(\mathbf{x})$  coincides with  $V^-(\mathbf{x})$  whenever  $V^-(\mathbf{x}) \leq A$ . One may then fix  $\varepsilon > 0$  and choose  $A$  large enough so that

$$\int_{\Omega} (V^- - V_A^-) \, d\pi < \varepsilon.$$

Setting:  $\Omega_A = (V^-)^{-1}([0; A])$ , we then have

$$\int_{\Omega} V^- \, d\pi_n = \int_{\Omega} V_A^- \, d\pi_n + \int_{\Omega_A^c} (V^- - V_A^-) \, d\pi_n,$$

the first term in the preceding sum converges to  $\int_{\Omega} V_A^- \, d\pi$ , which is greater than  $(a - \varepsilon)$ , while the second term is  $\geq 0$  for each  $n$ .

Hence

$$\forall \varepsilon > 0, \quad \liminf_{n \rightarrow \infty} \int_{\Omega} V^- \, d\pi_n > a - \varepsilon$$

so that

$$\liminf_{n \rightarrow \infty} \int_{\Omega} V^- \, d\pi_n \geq \int_{\Omega} V^- \, d\pi,$$

and  $\mathcal{V}$  is indeed upper semicontinuous.  $\square$

As a consequence of Proposition 3.1, we may now state a large deviations upper bound for the family  $\{\Pi_A\}_{A \subset \subset \mathbb{Z}^d}$ .

**Corollary 3.1.**  $\{\Pi_A\}_{A \subset \subset \mathbb{Z}^d}$  satisfies a large deviation upper bound on  $\mathcal{M}_{s,L}(\Omega)$  according to the good rate function  $\mathcal{I}^a = \mathcal{H} - \mathcal{V}$ , i.e.:

$$\limsup_{A \nearrow \mathbb{Z}^d} \frac{1}{|A|} \log \Pi_A(\mathcal{C}) \leq - \inf_{\pi \in \mathcal{C}} \{\mathcal{I}(\pi)\}$$

for each closed subset  $\mathcal{C} \subset \mathcal{M}_{s,L}(\Omega)$ .

**Proof.** Lemma 4.3.6 in (Dembo and Zeitouni, 1998, Section 4.3) may be applied here when considering the functional  $\phi$  defined on  $\mathcal{M}_{s,L}(\Omega)$  through the identities

$$\phi(\pi) = \begin{cases} -\infty & \text{if } \pi \notin \mathcal{C}, \\ \mathcal{V}(\pi) & \text{if } \pi \in \mathcal{C}. \end{cases} \quad \square$$

Before turning to the proof of the Gibbsian variational principle in the case of our nonsummable interaction  $\Psi^a$ , we find it appropriate to make here the following (elementary) remarks

(1) An easy consequence of Proposition 3.1 is that

$$Q_\Lambda(\mathcal{A}) \leq e^{M|\Lambda|} R_T^{\otimes \Lambda}(\mathcal{A})$$

for each finite cubic box  $\Lambda$  and each Borel set  $\mathcal{A} \subset W_T^A$ .

(2) According to the remarks made at the end of Section 2.1, we may also state that  $Q_\infty$  is such that

$$\int_\Omega dQ_\infty(\mathbf{x}) \left( \int_0^T |x_t^O|^6 dt \right) < \infty \quad \text{and} \quad \int_\Omega dQ_\infty(\mathbf{x}) |x_T^O|^4 < \infty.$$

(3) Still using Proposition 3.1 and the estimations established during its proof, one may also check that any  $\eta \in \mathcal{M}_{s,L}(\Omega)$  satisfying

$$\mathcal{V}(\eta) = \int_\Omega V^a(\mathbf{x}) d\eta(\mathbf{x}) > -\infty$$

cannot have “thick tails” in the sense that the variables

$$\int_0^T |x_t^O|^6 dt \quad \text{and} \quad |x_T^O|^4$$

are both integrable under  $d\eta(\mathbf{x})$ .

### 3.2. Gibbsian variational principle for $\Psi^a$

Recall that the specific entropy  $\mathcal{H}(\eta)$  defined with respect to the infinite dimensional Wiener measure  $R_T^{\otimes \mathbb{Z}^d}$  is such that for each translation invariant  $\eta \in \mathcal{M}_s(\Omega)$ :

$$\mathcal{H}(\eta) = \lim_{A \nearrow \mathbb{Z}^d} \frac{1}{|A|} H(\eta_A | R_T^{\otimes A}) = \sup_{A \subset \subset \mathbb{Z}^d} \frac{1}{|A|} H(\eta_A | R_T^{\otimes A}),$$

$H(\cdot | R_T^{\otimes A})$  denoting the usual relative entropy with respect to  $R_T^{\otimes A}$ .

We now prove the variational principle for  $\Psi^a$  as a consequence of the following lemmata:

**Lemma 3.1.** *One has for each  $\eta \in \mathcal{M}_{s,L}(\Omega)$ :*

$$\mathcal{H}(\eta) - \mathcal{V}(\eta) \geq 0.$$

**Proof.** We may assume that  $V^a \in L_\eta^1$ , since there is nothing to prove otherwise.

In this case, the  $L^1$  version of the multidimensional ergodic theorem (stated as Corollary 14.A5 in Georgii (1988, p. 304)) enables us to view  $\mathcal{V}(\eta)$  as a limit in the following way:

$$\mathcal{V}(\eta) = \int V^a d\eta = \lim_A \frac{1}{|A|} \sum_{i \in A} \int V^a(\mathbf{x}^{(i)}) d\eta(\mathbf{x}).$$

On the other hand,  $\eta$  has “thin tails” in the sense of remark (3) above, and this enables one to see that the preceding limit coincides with

$$\lim_A \frac{1}{|A|} \sum_{i \in A} \int V^a(\mathbf{x}_A^{(A),(i)}) d\eta_A(\mathbf{x}_A).$$

Indeed, the difference between the two preceding limits may be viewed as

$$-\beta \lim_A \frac{1}{|A|} \sum_{A' \cap A \neq \emptyset, A' \cap A^c \neq \emptyset} \int d\eta(\mathbf{x}_{A^c}) \left\{ \int d\eta(\mathbf{x}_A) (\psi_{A'}(\mathbf{x}_A^{(A)}) - \psi_{A'}(\mathbf{x}_A \vee \mathbf{x}_{A^c})) \right\},$$

the number  $n_A$  of nonzero terms in the preceding sum (over  $A'$  such that:  $A' \cap A \neq \emptyset, A' \cap A^c \neq \emptyset$ ) satisfies

$$\frac{n_A}{|A|} \rightarrow_{A \nearrow \mathbb{Z}^d} 0,$$

since  $\Psi^a$  has finite range, and these nonzero terms have absolute values that are uniformly bounded from above by some positive constant  $K$  (depending only on  $\eta$ , cf. remark (3)).

Hence

$$\mathcal{V}(\eta) - \mathcal{H}(\eta) = \lim_A \frac{1}{|A|} \int d\eta_A(\mathbf{x}_A) \left\{ \sum_{i \in A} V^a(\mathbf{x}^{(A),(i)}) - \ln \left( \frac{d\eta_A}{dR_T^{\otimes A}}(\mathbf{x}_A) \right) \right\}$$

and Jensen’s inequality applied to  $\ln$  yields for each  $A$ :

$$\begin{aligned} & \int d\eta_A \left\{ \sum_{i \in A} V^a(\mathbf{x}^{(A),(i)}) - \ln \left( \frac{d\eta_A}{dR_T^{\otimes A}}(\mathbf{x}_A) \right) \right\} \\ & \leq \ln \left\{ \int d\eta_A \frac{\exp(\sum_{i \in A} V(\mathbf{x}^{(A),(i)}))}{(d\eta_A/dR_T^{\otimes A}(\mathbf{x}_A))} \right\} = 0 \end{aligned}$$

(because the computation of the pressure corresponding to  $\Psi^a$  becomes trivial when  $A$  is equipped with its periodic boundary conditions).  $\square$

**Lemma 3.2.**  $Q_\infty$  is such that

$$\mathcal{H}(Q_\infty) = \int V^a dQ_\infty.$$

**Proof.** To prove this equality we shall merely use the fact that  $Q_\infty$  is a Gibbs state corresponding to  $\Psi^a$ , so that for each  $A \subset \subset \mathbb{Z}^d$ :

$$dQ_\infty(\mathbf{x}_A) = \int dQ_\infty(\mathbf{x}_{A^c}) \frac{\exp \left\{ -\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_\Gamma^a(\mathbf{x}_A \vee \mathbf{x}_{A^c}) \right\}}{\mathcal{Z}_A^{\Psi^a}(\mathbf{x}_{A^c})},$$

where:  $\mathcal{Z}_A^{\Psi^a}(\mathbf{x}_{A^c}) = \int dR_T^{\otimes A}(\mathbf{x}_A) e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_\Gamma^a(\mathbf{x}_A \vee \mathbf{x}_{A^c})}$ .

Using again the  $L^1$  multidimensional ergodic theorem and taking into account remark (2), we also have

$$\begin{aligned} \int_{\Omega} V^a dQ_{\infty} &= \lim_A \frac{1}{|A|} \sum_{i \in A} \int_{\Omega} dQ_{\infty}(\mathbf{x}) V^a(\mathbf{x}^{(i)}) \\ &= \lim_A \frac{1}{|A|} \sum_{i \in A} \int dQ_{\infty}(\mathbf{x}_{A^c}) \left\{ \int dR_T^{\otimes A}(\mathbf{x}_A) \frac{e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A \vee \mathbf{x}_{A^c})}}{\mathcal{Z}_A^{\Psi^a}(\mathbf{x}_{A^c})} \right. \\ &\quad \left. \times \left( -\beta \sum_{\Gamma \ni i} \frac{\psi_{\Gamma}^a(\mathbf{x}_A \vee \mathbf{x}_{A^c})}{|\Gamma|} \right) \right\} \\ &= \lim_A \frac{1}{|A|} \int dQ_{\infty}(\mathbf{x}_{A^c}) \left\{ \int dR_T^{\otimes A}(\mathbf{x}_A) \frac{e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A \vee \mathbf{x}_{A^c})}}{\mathcal{Z}_A^{\Psi^a}(\mathbf{x}_{A^c})} \right. \\ &\quad \left. \times \left( -\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A \vee \mathbf{x}_{A^c}) \right) \right\}. \end{aligned}$$

On the other hand:

$$\begin{aligned} \mathcal{H}(Q_{\infty}) &= \lim_A \frac{1}{|A|} \int dQ_{\infty}(\mathbf{x}_{A^c}) \left\{ \int dR_T^{\otimes A}(\mathbf{x}_A) \left( \frac{e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A \vee \mathbf{x}_{A^c})}}{\mathcal{Z}_A^{\Psi^a}(\mathbf{x}_{A^c})} \right) \right. \\ &\quad \left. \times \ln \left( \frac{e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A \vee \mathbf{x}_{A^c})}}{\mathcal{Z}_A^{\Psi^a}(\mathbf{x}_{A^c})} \right) \right\} \end{aligned}$$

hence

$$\mathcal{H}(Q_{\infty}) - \int_{\Omega} V^a dQ_{\infty} = \lim_A \frac{1}{|A|} \int dQ_{\infty}(\mathbf{x}_{A^c}) - \ln(\mathcal{Z}_A^{\Psi^a}(\mathbf{x}_{A^c})).$$

Jensen's inequality applied to  $-\ln$  and to the probability measure  $e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A^{(A)})} dR_T^{\otimes A}(\mathbf{x}_A)$  now yields:

$$\begin{aligned} &\frac{1}{|A|} \int dQ_{\infty}(\mathbf{x}_{A^c}) - \ln(\mathcal{Z}_A^{\Psi^a}(\mathbf{x}_{A^c})) \\ &= \frac{1}{|A|} \int dQ_{\infty}(\mathbf{x}_{A^c}) - \ln \left\{ \int dR_T^{\otimes A}(\mathbf{x}_A) e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A \vee \mathbf{x}_{A^c})} \right\} \\ &= \frac{1}{|A|} \int dQ_{\infty}(\mathbf{x}_{A^c}) - \ln \left\{ \int (e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A^{(A)})} dR_T^{\otimes A}(\mathbf{x}_A)) \right. \\ &\quad \left. \times e^{-\beta(\sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A \vee \mathbf{x}_{A^c}) - \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A^{(A)})} \right\} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{|A|} \int dQ_\infty(\mathbf{x}_{A^c}) \left\{ \int (e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_\Gamma^a(\mathbf{x}_A^{(A)})} dR_T^{\otimes A}(\mathbf{x}_A)) \right. \\ &\quad \left. \times \left( \beta \sum_{\Gamma \cap A \neq \emptyset, \Gamma \cap A^c \neq \emptyset} (\psi_\Gamma^a(\mathbf{x}_A \vee \mathbf{x}_{A^c}) - \psi_\Gamma^a(\mathbf{x}_A^{(A)})) \right) \right\}, \end{aligned}$$

and the preceding term may be easily seen to converge to 0, using again the fact that the number  $n_A$  of nonzero terms in the sum ranging over  $\Gamma$  such that:  $\Gamma \cap A \neq \emptyset, \Gamma \cap A^c \neq \emptyset$  satisfies

$$\frac{n_A}{|A|} \xrightarrow{A \nearrow \mathbb{Z}^d} 0$$

and Proposition 3.1.

By Lemma 3.1, we also know that

$$\mathcal{H}(Q_\infty) \geq \int V^a dQ_\infty,$$

and the proof is now complete.  $\square$

**Lemma 3.3.** Any  $\eta \in \mathcal{M}_{s,L}(\Omega)$  satisfying

$$\mathcal{H}(\eta) - \mathcal{V}(\eta) = 0$$

is also such that

$$\lim_A \frac{1}{|A|} \int \ln \left( \frac{d\eta_A}{dQ_{\infty,A}}(\mathbf{x}_A) \right) d\eta_A(\mathbf{x}_A) = 0$$

(here and in the sequel,  $Q_{\infty,A}$  simply denotes the  $A$ -marginal of  $Q_\infty$ ).

**Proof.** Consider  $\eta \in \mathcal{M}_{s,L}(\Omega)$  satisfying:  $\mathcal{H}(\eta) < \infty$  and  $\int_\Omega V^a d\eta = \mathcal{H}(\eta)$ .

One has for each  $A \subset \subset \mathbb{Z}^d$ :

$$\begin{aligned} &\frac{1}{|A|} \int \ln \left( \frac{d\eta_A}{dQ_{\infty,A}}(\mathbf{x}_A) \right) d\eta_A(\mathbf{x}_A) \\ &= \frac{1}{|A|} \int \ln \left( \frac{d\eta_A}{dR_T^{\otimes A}}(\mathbf{x}_A) \right) d\eta_A(\mathbf{x}_A) - \frac{1}{|A|} \int \ln \left( \frac{dQ_{\infty,A}}{dR_T^{\otimes A}}(\mathbf{x}_A) \right) d\eta_A(\mathbf{x}_A) \\ &= \frac{1}{|A|} \int \ln \left( \frac{d\eta_A}{dR_T^{\otimes A}}(\mathbf{x}_A) \right) d\eta_A(\mathbf{x}_A) - \frac{1}{|A|} \int d\eta_A(\mathbf{x}_A) \\ &\quad \times \ln \left\{ \int dQ_{\infty,A^c}(\mathbf{x}_{A^c}) \frac{e^{-\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_\Gamma^a(\mathbf{x}_A \vee \mathbf{x}_{A^c})}}{\mathcal{Z}_A^{\psi^a}(\mathbf{x}_{A^c})} \right\} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{|A|} \int \ln \left( \frac{d\eta_A}{dR_T^{\otimes A}}(\mathbf{x}_A) \right) d\eta_A(\mathbf{x}_A) - \frac{1}{|A|} \int d\eta_A(\mathbf{x}_A) \int dQ_{\infty, A^c}(\mathbf{x}_{A^c}) \\ &\quad \times \left\{ \left( -\beta \sum_{\Gamma \cap A \neq \emptyset} \psi_{\Gamma}^a(\mathbf{x}_A \vee \mathbf{x}_{A^c}) \right) - \ln(\mathcal{L}_A^{\Psi^a}(\mathbf{x}_{A^c})) \right\}, \end{aligned}$$

in the right hand side of the preceding inequality, the first term converges to  $\mathcal{H}(\eta)$  while the second term has a limit that may easily be seen to coincide with  $\mathcal{V}(\eta)$ , using once again the multidimensional ergodic theorem (since  $V^a \in L^1_{\eta}$ ) and then remarks (2), (3).  $\square$

**Proposition 3.2** (Variational principle for  $\Psi$ ). *For  $\eta \in \mathcal{M}_{s,L}(\Omega)$ , the following are equivalent:*

- (a)  $\eta$  is a Gibbs measure corresponding to  $\Psi^a$  (i.e.:  $\eta = Q_{\infty}$ )
- (b)  $\mathcal{H}(\eta) < \infty$  and  $\int_{\Omega} V^a d\eta = \mathcal{H}(\eta)$
- (c)  $\lim_A 1/|A| \int \ln(d\eta_A/dQ_{\infty,A}(\mathbf{X}_A)) d\eta_A(\mathbf{X}_A) = 0$ .

**Proof.** (a)  $\Rightarrow$  (b) is the content of Lemma 2.4.2, (b)  $\Rightarrow$  (c) is the content of Lemma 2.4.3, and (c)  $\Rightarrow$  (a) is a fact holding in great generality for translation invariant Gibbs measures (see Georgii, 1988, Theorem 15.37, p. 323).  $\square$

### 3.3. Large deviations lower bound

Following Föllmer and Orey’s approach to level 3 large deviations in a Gibbsian setup (see Föllmer and Orey, 1988), we decompose the proof of the large deviations lower bound for  $\{\Pi_A\}_{A \subset \subset \mathbb{Z}^d}$  into two steps:

*Step 1:* we first prove that any  $\pi \in \mathcal{M}_{s,L}(\Omega)$  such that:  $\mathcal{I}^a(\pi) < +\infty$  may be approximated by means of a sequence  $\{\pi_n\}_{n \geq 1}$  of ergodic probability measures on  $\Omega$  satisfying

- (a)  $\pi_n \in \mathcal{M}_{s,L}(\Omega)$ ,  $\pi_n \xrightarrow{n \rightarrow \infty} \pi$ ,
- (b)  $\mathcal{H}(\pi_n) \xrightarrow{n \rightarrow \infty} \mathcal{H}(\pi)$ , and
- (c)  $\mathcal{V}(\pi_n) \xrightarrow{n \rightarrow \infty} \mathcal{V}(\pi)$ .

*Step 2:*  $\pi \in \mathcal{M}_{s,L}(\Omega)$  being any ergodic probability measure such that  $\mathcal{I}^a(\pi) < +\infty$ , the inequality

$$\liminf_{A \nearrow \mathbb{Z}^d} \frac{1}{|A|} \log \Pi_A(\mathcal{G}) \geq -\mathcal{I}^a(\pi)$$

holds for all open neighbourhoods  $\mathcal{G}$  of  $\pi$ .

**Proof of Step 1.** Just as in (Föllmer and Orey, 1988), we may consider the sequence of volumes  $A_n = ]-n; n]^d \cap \mathbb{Z}^d$ , let  $\tilde{\pi}_n$  be the probability measure coinciding with  $\pi$

on each of the  $\sigma$ -fields

$$\Sigma_{n,\mathbf{k}} = \sigma\{x_u^i; 0 \leq u \leq T, i \in (A_n + 2n\mathbf{k})\}, \quad \mathbf{k} \in \mathbb{Z}^d$$

and making these  $\sigma$ -fields independent, and then set

$$\pi_n = \frac{1}{|A_n|} \sum_{i \in A_n} \tilde{\pi}_n o \theta_i^{-1},$$

where  $\theta_i$  simply denotes a shift on  $\Omega = W_T^{\mathbb{Z}^d}$ . Each  $\pi_n \in \mathcal{M}_{s,L}(\Omega)$  defines an ergodic probability measure on  $\Omega$ , and  $(\pi_n)_{n \geq 1}$  converges weakly to  $\pi$ .

Moreover

$$\liminf_{n \rightarrow \infty} \mathcal{H}(\pi_n) \geq \mathcal{H}(\pi),$$

since  $\mathcal{H}$  is lower semicontinuous. In order to prove the complementary inequality

$$\mathcal{H}(\pi) \geq \limsup_{n \rightarrow \infty} \mathcal{H}(\pi_n),$$

we may simply observe that for each  $n \geq 1$ :

$$\mathcal{H}(\pi_n) = \frac{1}{|A_n|} H(\pi_n|_{\Sigma_{n,o}}; R_T^{\otimes A_n})$$

since  $\pi_n$  defines a  $A_n$ -periodical probability on  $\Omega$ .

We then have

$$\mathcal{H}(\pi_n) \leq \frac{1}{|A_n|^2} \sum_{i \in A_n} H((\tilde{\pi}_n o \theta_i^{-1})|_{\Sigma_{n,o}}; R_T^{\otimes A_n})$$

since  $H(\cdot; R_T^{\otimes A_n})$  is convex. Moreover, for each  $i \in A_n$ , the measure  $(\tilde{\pi}_n o \theta_i^{-1})|_{\Sigma_{n,o}}$  may be viewed as a tensor product based on  $(2^d)$  block marginals taken from  $\pi|_{\Sigma_{n,o}}$  (see Fig. 1); recalling that relative entropy is sub-additive in the sense that

$$\sum_{l=1}^m H(\pi|_{\Sigma_{A_l}}; R_T^{\otimes A_l}) \leq H(\pi|_{\Sigma_{n,o}}; R_T^{\otimes A_n}),$$

whenever  $A_1, A_2, \dots, A_m$  is a partition of  $A_n$  (with corresponding sub- $\sigma$ -fields  $\Sigma_{A_1}, \Sigma_{A_2}, \dots, \Sigma_{A_m} \subset \Sigma_{n,o}$ ), one obtains

$$H((\tilde{\pi}_n o \theta_i^{-1})|_{\Sigma_{n,o}}; R_T^{\otimes A_n}) \leq H(\pi|_{\Sigma_{n,o}}; R_T^{\otimes A_n}), \quad \forall i \in A_n.$$

All in all, one finds out that

$$\mathcal{H}(\pi_n) \leq \frac{1}{|A_n|} H(\pi|_{\Sigma_{n,o}}; R_T^{\otimes A_n}) \leq \mathcal{H}(\pi).$$

A fortiori

$$\limsup_{n \rightarrow \infty} \mathcal{H}(\pi_n) \leq \mathcal{H}(\pi),$$

so that  $(\mathcal{H}(\pi_n))_{n \geq 1}$  converges to  $\mathcal{H}(\pi)$  as  $n \rightarrow \infty$ .

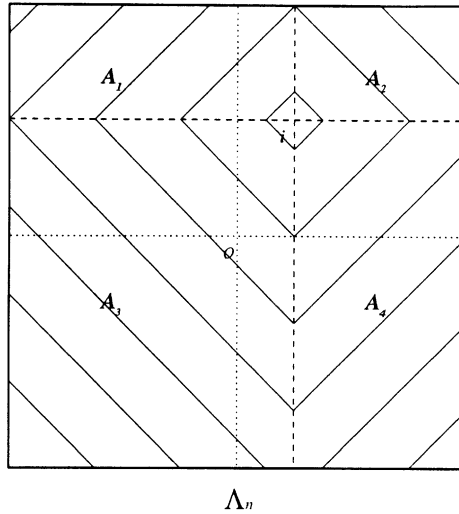


Fig. 1. In  $d = 2$ , each measure  $(\tilde{\pi}_n \circ \theta_i^{-1})|_{\Sigma_{n,O}}$  may be viewed as a tensor product where four distinct block marginals of  $\pi|_{\Sigma_{n,O}}$  are being used.

Finally, the convergence

$$\mathcal{V}(\pi_n) \xrightarrow{n \rightarrow \infty} \mathcal{V}(\pi)$$

also takes place simply because  $\mathcal{V}(\pi)$  is finite, so that  $\pi$  integrates the variables  $\{\int_0^T |x_t^O|^6 dt\}$  and  $\{|x_T^O|^4\}$ , and because most of the terms  $\int_{\Omega} (V \circ \theta_i) d\tilde{\pi}_n$  appearing in the decomposition

$$\mathcal{V}(\pi_n) = \int_{\Omega} V^a d\pi_n = \frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} \int_{\Omega} (V^a \circ \theta_i) d\tilde{\pi}_n$$

coincide with  $\int_{\Omega} (V^a \circ \theta_i) d\pi$ . To be more precise, there are at most  $3 \times |\partial \Lambda_n|$  sites  $i \in \Lambda_n$  for which

$$\int_{\Omega} (V^a \circ \theta_i) d\tilde{\pi}_n \neq \int_{\Omega} (V^a \circ \theta_i) d\pi,$$

these differences are averaged out in the large volume limit. This finishes the proof of Step 1.  $\square$

**Proof of Step 2.** As a consequence of the  $L^1$  version of the multidimensional ergodic theorem (cf. Georgii, 1988) we know that the convergence

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \pi\{\mathbf{x} \in \Omega \mid \hat{\pi}_x^{(\Lambda)} \in \mathcal{O}\} = 1$$

holds true for any open neighbourhood  $\mathcal{O}$  of  $\pi$ , and that the following  $L^1$  convergences

$$\frac{1}{|A|} \sum_{i \in A} V^a(\mathbf{x}^{(A),(i)}) \xrightarrow[A \nearrow \mathbb{Z}^d]{L^1_\pi} \int V^a d\pi, \quad \frac{1}{|A|} \log \frac{d\pi_A}{dR_T^{\otimes A}}(\mathbf{x}) \xrightarrow[A \nearrow \mathbb{Z}^d]{L^1_\pi} \mathcal{H}(\pi; R_T^{\otimes \mathbb{Z}^d})$$

also take place.

Let us now consider an elementary open neighbourhood  $\mathcal{O} \subset \mathcal{G}$  of  $\pi$ , of the form

$$\mathcal{O} = \bigcap_{k=1}^n \left\{ \eta \in \mathcal{M}_{s,L}(\Omega) \left| \left| \int f_k d\eta - \int f_k d\pi \right| < \varepsilon \right. \right\}$$

for some  $\varepsilon > 0$  and some bounded continuous functions  $f_1, f_2, \dots, f_n : \Omega \rightarrow \mathbb{R}$ , each  $f_k$  being measurable with respect to a fixed  $\sigma$ -algebra

$$\Sigma_{A_0} = \sigma\{x_u^i; 0 \leq u \leq T, i \in A_0\},$$

for some finite  $A_0 \subset \subset \mathbb{Z}^d$ .

Observe that for each  $A \subset \subset \mathbb{Z}^d$ , the event

$$E_A = \{\mathbf{x} \in \Omega \mid \hat{\pi}_x^{(A)} \in \mathcal{O}\}$$

decomposes into

$$E_A = \bigcap_{k=1}^n \left\{ \mathbf{x} \in \Omega \left| \left| \frac{1}{|A|} \sum_{i \in A} (f_k \circ \theta_i) - \int f_k d\pi \right| < \varepsilon \right. \right\}.$$

Setting

$$\overset{\circ}{A} = \{i \in A \mid (i + A_0) \subset A\},$$

we may next observe that for  $A$  sufficiently large the event

$$E'_A = \bigcap_{k=1}^n \left\{ \mathbf{x} \in \Omega \left| \left| \frac{1}{|\overset{\circ}{A}|} \sum_{i \in \overset{\circ}{A}} (f_k \circ \theta_i) - \int f_k d\pi \right| < \frac{\varepsilon}{2} \right. \right\}$$

lies in the  $\sigma$ -algebra  $\Sigma_A$  and satisfies:  $E'_A \subset E_A$ .

Taking into account the fact that  $\pi_A \ll Q_A$ , with

$$\pi_A\text{-a.s.}(\mathbf{x}), \quad \frac{d\pi_A}{dQ_A}(\mathbf{x}) = \frac{d\pi_A}{dR_T^{\otimes A}}(\mathbf{x}) \cdot \frac{dR_T^{\otimes A}}{dQ_A}(\mathbf{x}) = \frac{d\pi_A}{dR_T^{\otimes A}}(\mathbf{x}) \exp\left(\sum_{i \in A} V^a(\mathbf{x}^{(A),(i)})\right),$$

we then have

$$\begin{aligned} \Pi_A(\mathcal{G}) &= Q_A\{\mathbf{x} \mid \hat{\pi}_x^{(A)} \in \mathcal{G}\} \\ &\geq Q_A\{\mathbf{x} \mid \hat{\pi}_x^{(A)} \in \mathcal{O}\} \\ &\geq Q_A(E'_A) \end{aligned}$$

$$\begin{aligned} &\geq Q_A \left( E'_A \cap \left\{ \frac{d\pi_A}{dR_T^{\otimes A}}(\mathbf{x}) > 0; \frac{1}{|A|} \log \left( \frac{d\pi_A}{dR_T^{\otimes A}}(\mathbf{x}) \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{|A|} \sum_{i \in A} V^a(\mathbf{x}^{(A),(i)}) \leq \mathcal{I}^a(\pi) + \delta \right\} \right), \end{aligned}$$

$\delta$  being some positive number. Hence

$$\begin{aligned} \Pi_A(\mathcal{G}) &\geq \exp(-|A|(\mathcal{I}^a(\pi) + \delta))\pi_A \left( E'_A \cap \left\{ \frac{d\pi_A}{dR_T^{\otimes A}}(\mathbf{x}) > 0; \frac{1}{|A|} \log \left( \frac{d\pi_A}{dR_T^{\otimes A}}(\mathbf{x}) \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{|A|} \sum_{i \in A} V^a(\mathbf{x}^{(A),(i)}) \leq \mathcal{I}^a(\pi) + \delta \right\} \right) \end{aligned}$$

and taking into account the convergences obtained earlier via the  $L^1$  multidimensional ergodic theorem finishes the proof of Step 2.  $\square$

We may now state and prove a large deviations lower bound for the empirical process  $\hat{\pi}_x^{(A)}$  considered in the averaged regime.

**Proposition 3.3.** *The family  $(\Pi_A)_{A \subset \mathbb{Z}^d}$  of probability measures corresponding to the averaged regime of the empirical process also satisfies a large deviations lower bound on  $\mathcal{M}_{s,L}(\Omega)$ , in the sense that for any Borel set  $\mathcal{B} \subset \mathcal{M}_{s,L}(\Omega)$ :*

$$\liminf_{A \nearrow \mathbb{Z}^d} \frac{1}{|A|} \log \Pi_A(\mathcal{B}) \geq - \inf_{\pi \in \overset{\circ}{\mathcal{B}}} \mathcal{I}^a(\pi),$$

$\overset{\circ}{\mathcal{B}}$  denoting the interior of  $\mathcal{B}$  (with respect to the topology of weak convergence).

**Proof.** The proof is a straightforward consequence of Steps 1 and 2: assuming that

$$\inf_{\pi \in \overset{\circ}{\mathcal{B}}} \mathcal{I}^a(\pi) = A < +\infty,$$

one may choose an ergodic  $\pi \in \overset{\circ}{\mathcal{B}}$  satisfying:  $\mathcal{I}^a(\pi) < A + \varepsilon$ , for some fixed positive  $\varepsilon$ , and then consider an open neighbourhood  $\mathcal{O}$  of  $\pi$  such that  $\mathcal{O} \subset \overset{\circ}{\mathcal{B}}$ .

According to Step 2:

$$\liminf_{A \nearrow} \frac{1}{|A|} \log \Pi_A(\mathcal{O}) \geq -(A + \varepsilon),$$

and letting  $\varepsilon \searrow 0$  finishes the proof of the proposition.  $\square$

The large deviation principle we have just proved may naturally be viewed as an LDP taking place in  $\mathcal{M}_s(\Omega)$ , once we extend  $\mathcal{I}^a$  by setting

$$\mathcal{I}^a(Q) = +\infty,$$

whenever  $Q \in (\mathcal{M}_s(\Omega) \setminus \mathcal{M}_{s,L}(\Omega))$ . Moreover, as may be seen from the proofs of Proposition 3.1 and Lemmata 3.1–3.3, one may relax the compact support condition on  $\mu_0$  by requiring simply that

$$\int_{\mathbb{R}} x^6 d\mu_0(x) < +\infty$$

and still derive such LD bounds. One may generalise further the hypotheses made on the initial conditions, e.g. by requiring these initial conditions to be Gibbsian with respect to some (reasonable) interaction potential defined on  $\mathbb{R}^{\mathbb{Z}^d}$ ; such generalisations are carried out briefly in Section 5. But let us first come to a study of the quenched LD asymptotics of the empirical process  $\hat{\pi}_x^{(A)}$ .

#### 4. Quenched large deviation estimates

We now consider a fixed (typical) realisation  $\mathbf{k}_0$  of the external field variables, and let  $\mathcal{R}_A^{\mathbf{k}_0}$  denote the joint law of  $\mathbf{x}$  and  $\mathbf{k}$  when  $\mathbf{x}$  is distributed according to  $R_T^{\otimes A}$  while  $\mathbf{k}$  is fixed at the value  $\mathbf{k}_0$  (or rather, at the projection of  $\mathbf{k}_0$  onto  $\mathbb{R}^A$ ). According to Theorem III.1 in Comets (1989), we know that: a.s. in the realisation of the disorder variables  $\mathbf{k}_0$ , the law of the “joint empirical process”<sup>2</sup>  $\hat{\pi}_{\mathbf{x},\mathbf{k}}^{(A)}$  considered under  $d\mathcal{R}_A^{\mathbf{k}_0}(\mathbf{x}; \mathbf{k})$  obeys a LDP on  $\mathcal{M}_s((W_T \times \mathbb{R})^{\mathbb{Z}^d})$ , on the scale  $|A|$  and according to the deterministic rate functional  $\mathcal{H}^q$  given by

$$\mathcal{H}^q(\mu) = \begin{cases} \mathcal{H}_{R_T^{\otimes \mathbb{Z}^d}}(\mu_{\mathbf{x}}) & \text{if } \mu \text{ has a second marginal } \mu_{\mathbf{k}} \text{ coinciding} \\ & \text{with } \mathcal{N}(0; \sigma^2)^{\otimes \mathbb{Z}^d}, \\ +\infty & \text{else.} \end{cases}$$

Denoting by  $\mathcal{Q}_A^{\mathbf{k}_0}$  the joint law of  $\mathbf{x}$  and  $\mathbf{k}$  when  $\mathbf{x}$  is distributed according to  $dQ_A^{\mathbf{k}_0}$  and  $\mathbf{k}$  is fixed at the value  $\mathbf{k}_0$ , we also know that

$$\frac{d\mathcal{Q}_A^{\mathbf{k}_0}}{d\mathcal{R}_A^{\mathbf{k}_0}} = \exp \left\{ \sum_{i \in A} \int_0^T \left( -U'(x_t^i) + \beta \sum_{j \sim i} x_t^j + \beta k^i \right) dx_t^i - \frac{1}{2} \sum_{i \in A} \int_0^T \left( -U'(x_t^i) + \beta \sum_{j \sim i} x_t^j + \beta k^i \right)^2 dt \right\}.$$

Considering  $(\mathbf{x}; \mathbf{k})$  as a spin configuration where each spin variable  $(x^i; k^i)$  lies in  $(W_T \times \mathbb{R})$ , one may then express the preceding Radon–Nykodim derivative as

$$\frac{d\mathcal{Q}_A^{\mathbf{k}_0}}{d\mathcal{R}_A^{\mathbf{k}_0}}(\mathbf{x}; \mathbf{k}) = \exp \left\{ \sum_{i \in A} \mathcal{F}^i(\mathbf{x}; \mathbf{k}) \right\},$$

<sup>2</sup> The joint empirical process  $\hat{\pi}_{\mathbf{x},\mathbf{k}}^{(A)}$  is defined as the empirical process corresponding to the configuration  $\mathbf{z} = (z^i)_{i \in A} = ((x^i; k^i))_{i \in A}$ .

the functional  $\mathcal{F}^i : (W_T \times \mathbb{R})^{(\mathbb{Z}^d)} \rightarrow \mathbb{R}$  being of the form:

$$\mathcal{F}^i = - \sum_{\Gamma \ni i} \frac{\psi_\Gamma^q}{|\Gamma|}$$

for some finite range, translation invariant Gibbsian interaction

$$\Psi^q = (\psi_\Gamma^q)_{\Gamma \subset \subset \mathbb{Z}^d}$$

on  $(W_T \times \mathbb{R})^{(\mathbb{Z}^d)}$ . The “quenched” interaction  $\Psi^q$  is precisely given by

$$\psi_{\{i\}}^q(\mathbf{x}; \mathbf{k}) = [U(x_t^i) - \beta k^i x_t^i]_0^T + \frac{1}{2} \int_0^T \{U''(x_t^i) + (\beta k^i - U'(x_t^i))^2 + 2\beta^2 d(x_t^i)^2\} dt,$$

$$\psi_{\{i,j\}}^q(\mathbf{x}; \mathbf{k}) = -\beta [x_t^i x_t^j]_0^T + \beta \int_0^T \{(-U'(x_t^i) + \beta k^i) x_t^j + (-U'(x_t^j) + \beta k^j) x_t^i\} dt$$

$$\text{if } |j - i| = 1,$$

$$\psi_{\{i,j\}}^q(\mathbf{x}; \mathbf{k}) = 2\beta \int_0^T x_t^i x_t^j dt \quad \text{if } |j - i| = \sqrt{2},$$

$$\psi_{\{i,j\}}^q(\mathbf{x}; \mathbf{k}) = \beta \int_0^T x_t^i x_t^j dt \quad \text{if } |j - i| = 2,$$

$$\psi_\Gamma^q \equiv 0 \quad \text{for any other } \Gamma.$$

Just as in Comets (1989), we may now use the fact that

$$\frac{d\mathcal{Q}_A^{k_0}}{d\mathcal{R}_A^{k_0}}(\mathbf{x}; \mathbf{k}) = \exp \left\{ |A| \int \mathcal{F}^0 d\hat{\pi}_{\mathbf{x}, \mathbf{k}}^{(A)} \right\},$$

and applying the Laplace–Varadhan method in this context leads us to conjecture that the joint empirical process  $\hat{\pi}_{\mathbf{x}, \mathbf{k}}^{(A)}$  should also satisfy an LDP when  $((x^i, k^i))_{i \in A}$  is distributed according to  $d\mathcal{Q}_A^{k_0}(\mathbf{x}; \mathbf{k})$ . The proposition is more precisely the following:

**Proposition 4.1.** *Almost surely in the realisation of  $\mathbf{k}_0$ , the law of the “joint empirical process”  $\hat{\pi}_{\mathbf{x}, \mathbf{k}}^{(A)}$  under  $d\mathcal{Q}_A^{k_0}(\mathbf{x}; \mathbf{k})$  satisfies a LDP on the scale  $|A|$  and according to the good rate function  $\mathcal{I}^q : \mathcal{M}_s((W_T \times \mathbb{R})^{(\mathbb{Z}^d)}) \rightarrow [0; +\infty]$  given by*

$$\mathcal{I}^q(\mu) = \begin{cases} \mathcal{H}^q(\mu) - \int \mathcal{F}^0 d\mu & \text{if } \mathcal{H}^q(\mu) < +\infty, \\ +\infty & \text{else.} \end{cases}$$

Here again, some verifications are needed since the functional  $\mathcal{F}^0$  lacks a property of uniform boundedness; one should precisely check that the quantity

$$A_\delta = \limsup_{A \nearrow \mathbb{Z}^d} \frac{1}{|A|} \ln \int_{(W_T \times \mathbb{R})^A} e^{\delta \cdot |A| \langle \mathcal{F}^0; \hat{\pi}^{(A)} \rangle} d\mathcal{R}_A^{k_0}$$

is finite for some  $\delta > 1$  in order to prove the LD upper bound. In the present case one may actually prove the following stronger statement

**Lemma 4.1.** *a.s.  $(\mathbf{k}_0)$ , the quantity  $A_\delta$  is finite for all  $\delta > 1$ .*

**Proof.** Let us fix  $\delta > 1$  and notice first that the exponential of  $\delta \cdot |A| \langle \mathcal{F}_0; \hat{\pi}^{(A)} \rangle$  may be decomposed as

$$\begin{aligned} & \exp \left\{ \delta \sum_{i \in A} \int_0^T \left( -U'(x_t^i) + \beta \sum_{j \sim i} x_t^j + \beta k^i \right) dx_t^i \right. \\ & \quad \left. - \delta^2 \sum_{i \in A} \int_0^T \left( -U'(x_t^i) + \beta \sum_{j \sim i} x_t^j + \beta k^i \right)^2 dt \right\} \\ & \quad \times \exp \left\{ \frac{2\delta^2 - \delta}{2} \sum_{i \in A} \int_0^T \left( -U'(x_t^i) + \beta \sum_{j \sim i} x_t^j + \beta k^i \right)^2 dt \right\}. \end{aligned}$$

Using the Cauchy–Swartz inequality together with the martingale property of the square of the first term thus leads to the following inequality:

$$A_\delta \leq \limsup_{A \nearrow \mathbb{Z}^d} \frac{1}{2|A|} \ln \int e^{(2\delta^2 - \delta) \sum_{i \in A} \int_0^T (-U'(x_t^i) + \beta \sum_{j \sim i} x_t^j + \beta k^i)^2 dt} d\mathcal{P}_A^{\mathbf{k}_0}.$$

Taking into account the translation invariance of the  $\mathbf{x}$ -marginal of  $\mathcal{P}_A^{\mathbf{k}_0}$ , we then have

$$\begin{aligned} A_\delta & \leq \limsup_{A \nearrow \mathbb{Z}^d} \frac{1}{2|A|} \ln \int e^{(4\delta^2 - 2\delta) \sum_{i \in A} \int_0^T (2U'(x_t^i)^2 + 4d\beta^2(x_t^i)^2) dt} d\mathcal{P}_A^{\mathbf{k}_0} \\ & \quad + \limsup_{A \nearrow \mathbb{Z}^d} \frac{1}{2|A|} \cdot \beta^2(4\delta^2 - 2\delta) \sum_{i \in A} (\mathbf{k}_0^i)^2. \end{aligned}$$

Now the first of these two terms is clearly finite, while the second coincides *a.s.* with  $\sigma^2 \beta^2 (2\delta^2 - \delta)$ , due to the strong law of large numbers.  $\square$

The LD lower bound may then be proved similarly as in Section 3.3, i.e. by considering an arbitrary open neighbourhood  $\mathcal{N}$  of  $\mu \in \mathcal{M}_s((W_T \times \mathbb{R})^{\mathbb{Z}^d})$ , where  $\mu$  may be chosen as an ergodic p.m. such that  $\mathcal{J}^q(\mu) < +\infty$ , and establishing the inequality

$$\liminf_{A \nearrow \mathbb{Z}^d} \frac{1}{|A|} \ln \mathcal{Q}_A^{\mathbf{k}_0} \{ \hat{\pi}^{(A)} \in \mathcal{N} \} \geq -\mathcal{J}^q(\mu)$$

via the  $L^1$  ergodic theorem.

One thus obtains a quenched LDP for the joint empirical process  $\hat{\pi}_{(\mathbf{x}, \mathbf{k})}^{(A)}$  (Proposition 4.1), which may naturally be contracted to a quenched LDP for the empirical process  $\hat{\pi}_{\mathbf{x}}^{(A)}$ , yielding part (ii) of Theorem 1.1.

**Proposition 4.2.** *Almost surely in the realisation of  $\mathbf{k}_0$ , the law of the empirical process  $\hat{\pi}_{\mathbf{x}}^{(A)}$  under  $d\mathcal{P}_A^{\mathbf{k}_0}$  obeys a large deviations principle on  $\mathcal{M}_s(\Omega)$ , on the scale  $|A|$*

and according to the (deterministic) good rate functional  $\mathcal{I}^q : \mathcal{M}_s(\Omega) \rightarrow [0; +\infty]$  given by:

$$\mathcal{I}^q(\eta) = \inf_{1^{\text{st marg.}}(\mu)=\eta} (\mathcal{I}^q(\mu)).$$

The preceding expression for  $\mathcal{I}^q$  does not enable one to see immediately that  $Q_\infty$  is also the unique minimiser associated with  $\mathcal{I}^q$ ; this fact follows however from the general inequality

$$\mathcal{I}^q \geq \mathcal{I}^a \geq 0,$$

a proof of which may be found in (Zeitouni, 2003, Lemma 2.2.8).

### 5. General initial and boundary conditions

Our aim in the present section is to show that one may also consider the quenched dynamics

$$(P_A^k) \quad \begin{cases} dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt + \beta k^i dt \\ \text{law}(x|_{t=0}) = \mu_A \quad (i \in A, 0 \leq t \leq T) \end{cases}$$

obtained when using a (nonproduct) probability measure  $\mu_A \in \mathcal{M}(\mathbb{R}^A)$  as initial condition and some fixed boundary condition  $(\xi_t^j)_{0 \leq t \leq T, j \in A^c}$ , and still derive averaged and quenched LDPs for the empirical process.

Our general strategy consists in viewing the finite dimensional, averaged dynamics  $(P_A)$  as the  $A$ -dimensional projection corresponding to an infinite volume Gibbs measure  $Q_\infty$ ; as we shall see this strategy may still be carried out when the initial condition  $\mu_A$  is simply the  $A$ -dimensional projection of a (reasonable) translation invariant Gibbs measure  $\mu$  corresponding to some deterministic interaction  $\mathcal{P} = (p_A)_{A \subset \subset \mathbb{Z}^d}$  on  $\mathbb{R}^{\mathbb{Z}^d}$ .

As a first step towards this goal, let us remark that we may replace the compactly supported probability measure  $u_0 \in \mathcal{M}(\mathbb{R}^A)$  (corresponding to the initial condition  $u_0^{\otimes A}$  used in the preceding section) by a probability  $m_0$  supported on the whole real line and having thin enough tails; one should precisely require that:

$$\int_{\mathbb{R}} x^6 dm_0(x) < +\infty$$

since, according to the remarks made after Corollary 1.2.1 (cf. proof of Theorem 4.6 in Föllmer and Wakolbinger, 1986), this fact guarantees the finiteness of the integrals

$$I_1 = \int dQ_\infty(x) |x_T^i|^6 \quad \text{and} \quad I_2 = \int dQ_\infty(x) \left( \int_0^T |x_t^i|^6 dt \right),$$

$Q_\infty$  being the  $x$ -marginal corresponding to the system

$$dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt$$

$$dy_t^i = dw_t^i + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt$$

$$dz_t^i = x_t^i dt$$

$$\text{Law}(\mathbf{x}|_{t=0}) = m_0^{\otimes \mathbb{Z}^d}, \quad \mathbf{y}|_{t=0} = \mathbf{z}|_{t=0} = \mathbf{0} \quad (i \in \mathbb{Z}^d, 0 \leq t \leq T)$$

(such system certainly has a unique strong solution since  $m_0^{\otimes \mathbb{Z}^d}$  is supported by  $\mathcal{S}'(\mathbb{Z}^d)$ ).

In view of the computations carried out in the preceding section, the finiteness of  $I_1$  and  $I_2$  suffices to establish that  $Q_\infty$  is the unique translation invariant Gibbs measure corresponding to the interaction  $\Psi^a$  (on infinite dimensional path space  $\Omega = W_T^{\otimes \mathbb{Z}^d}$ ) and to the reference measure  $\mathcal{R} = R_T^{\otimes \mathbb{Z}^d}$ ,  $R_T$  now denoting the law of a one dimensional Brownian motion having initial condition  $m_0$ . We may thus choose e.g.

$$dm_0(x) = \frac{e^{-2U(x)}}{\int e^{-2U(y)} dy} dx$$

as the reference probability corresponding to a “deep quench” initial condition.

At this stage we should make an important remark stated as Proposition 2.5, (ii), in (Cattiaux et al., 1996).

**Lemma 5.1.** For each  $\mathbf{y} \in \mathcal{S}'(\mathbb{Z}^d)$ , denote by  $\mathcal{R}^y$  (resp.  $Q_\infty^y$ ) the probability  $\mathcal{R}$  (resp.  $Q_\infty$ ) conditioned to start at  $\mathbf{y}$ :

$$\mathcal{R}^y(\omega \in A) = \mathcal{R}(\omega \in A \mid \omega(0) = \mathbf{y}).$$

For  $Q_\infty$ -almost all  $\mathbf{y}$ ,  $Q_\infty^y$  defines a Gibbs measure on  $\Omega$  with respect to the interaction  $\Psi^a$  and to the reference measure  $\mathcal{R}^y$ .

Let  $\mu$  be a (reasonable) Gibbs measure associated to an interaction  $\mathcal{P}$  on  $\mathbb{R}^{\mathbb{Z}^d}$  and to the reference measure  $\mathcal{R}$ ; according to the preceding lemma, the infinite volume dynamics

$$(\mathcal{S}_\infty^\mu) \begin{cases} dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt \\ dy_t^i = dw_t^i + \beta \sum_{j \sim i} x_t^j dt + \gamma_t \left( y_t^i - \beta \sum_{j \sim i} z_t^j \right) dt \\ dz_t^i = x_t^i dt \\ \text{Law}(\mathbf{x}|_{t=0}) = \mu, \mathbf{y}|_{t=0} = \mathbf{z}|_{t=0} = \mathbf{0} \quad (i \in \mathbb{Z}^d, 0 \leq t \leq T) \end{cases}$$

may now be viewed as a Gibbsian average of Gibbs measures, and we next give a necessary and sufficient condition devised by Cattiaux, Roelly and Zessin for such an average to define a Gibbs measure on path space  $\Omega$ :

**Lemma 5.2.** Assume that the Gibbs measure  $\mu$  on  $\mathbb{R}^{\mathbb{Z}^d}$  is supported by the space  $\mathcal{S}'(\mathbb{Z}^d)$  of all tempered sequences on  $\mathbb{Z}^d$ ; let  $(Q^y; y \in \mathcal{S}'(\mathbb{Z}^d))$  be a measurable family of  $(\Psi; \mathcal{R}^y)$  Gibbs measures, and:

$$Q = \int Q^y d\mu(y).$$

Assume further that the expectation  $E_Q[e^{H_A^\Psi}]$  is finite for each  $A \subset\subset \mathbb{Z}^d$ , and set

$$N_A^\Psi(y) = \log E_Q[e^{H_A^\Psi} | \omega(0) = y] = \log E_{Q^y}[e^{H_A^\Psi}].$$

In such conditions,  $Q$  is a Gibbs measure corresponding to the reference measure  $\mathcal{R}$  and to the Boltzmann weights  $\rho$  given by:

$$\rho_A = (Z_A^\Psi(\omega_{A^c}))^{-1} \exp\{- (H_A^\Psi(\omega) - N_A^\Psi(\omega(0)) + H_A^\mathcal{P}(\omega(0)))\}$$

if and only if the variables  $\omega_A(0)$  and  $\omega_{A^c}$  are independent under the probability measure  $S_A$  given by:

$$dS_A(\omega) = (Z_A^\mathcal{P}(\omega_{A^c}(0)))^{-1} \exp\{- (H_A^\Psi(\omega) - N_A^\Psi(\omega(0)) + H_A^\mathcal{P}(\omega(0)))\} dQ(\omega).$$

**Proof.** See (Cattiaux et al., 1996, Proposition 2.6).  $\square$

In the case of interest to us, where  $Q$  is the  $x$ -marginal corresponding to the infinite volume dynamics  $(\mathcal{S}_\infty^\mu)$ , we have

$$N_A^\Psi(y) \equiv 0$$

and  $Q$  is a Gibbs measure corresponding to the interaction  $(\psi_A + p_A \circ pr.|_{l=0})_{A \subset\subset \mathbb{Z}^d}$  and with the reference measure  $R_T^{\otimes \mathbb{Z}^d}$ .

Of course, one should also require that the Gibbs measures corresponding to the interaction  $\mathcal{P} = (p_A)_{A \subset\subset \mathbb{Z}^d}$  enjoy the level 3 large deviation property; this will certainly be the case if  $\mathcal{P}$  defines a translation invariant, summable interaction on  $\mathbb{R}^{\mathbb{Z}^d}$  (so that  $\|\Psi\| = \sum_{A \ni 0} \sup_{x \in \mathbb{R}^{\mathbb{Z}^d}} |\psi_A(x)|$  is finite), and it will also be the case in the natural situation where  $\mathcal{P}$  is the nearest neighbour interaction corresponding to a standard Ising model:

$$p_{\{i,j\}}(x) = \beta x^i x^j \quad \text{whenever } i \sim j, \quad p_A \equiv 0 \text{ else.}$$

The preceding observations may now be gathered into

**Theorem 5.1.** Let  $\mathcal{P}$  be a translation invariant, summable interaction on  $\mathbb{R}^{\mathbb{Z}^d}$ , and  $\mu$  be a translation invariant Gibbs measure on  $\mathbb{R}^{\mathbb{Z}^d}$  corresponding to  $\mathcal{P}$  and to a reference probability  $m_0^{\otimes \mathbb{Z}^d}$ . Assume that

$$\int x^6 dm_0(x) < +\infty, \quad \int (x^0)^6 d\mu(x) < +\infty,$$

and denote by  $Q_\infty^\mu$  the  $\mathbf{x}$ -marginal corresponding to the infinite dimensional system  $(\mathcal{S}_\infty^\mu)$ . For each finite volume  $\Lambda$ , consider the law  $P_\Lambda^k$  of the quenched dynamics

$$\begin{cases} dx_t^i = dw_t^i - U'(x_t^i) dt + \beta \sum_{j \sim i} x_t^j dt + \beta k^i dt \\ \text{law}(\mathbf{x}|_{t=0}) = \mu_\Lambda(\cdot | \xi_{\Lambda^c}(0)) \quad (i \in \Lambda, 0 \leq t \leq T) \end{cases}$$

with boundary condition  $\xi = (\xi_t^j; j \in \mathbb{Z}^d, 0 \leq t \leq T)$ ,<sup>3</sup> and denote by  $P_\Lambda$  the corresponding averaged dynamics:

$$P_\Lambda = \int d\gamma(\mathbf{k}) P_\Lambda^k.$$

Then: (i)  $Q_\infty^\mu$ -a.s.  $(\xi)$ , the law of the empirical process  $\hat{\pi}_x^{(\Lambda)}$  under  $dP_\Lambda(\mathbf{x})$  obeys a large deviation principle on the scale  $|\Lambda|$  and according to the good rate function  $\mathcal{I}^a : \mathcal{M}_s(\Omega) \rightarrow [0; +\infty]$  given by

$$\forall P \in \mathcal{M}_s(\Omega), \quad \mathcal{I}^a(P) = \mathcal{H}(P; R_T^{\otimes \mathbb{Z}^d}) + \mathcal{H}(\text{pr}_0(P); \mu) - \int V^p dP,$$

where

$$V^p(\omega) = - \sum_{A \ni 0} \frac{\{\psi_A + p_A \circ \text{pr}_{|_{t=0}}\}(\mathbf{x})}{|A|},$$

and where  $\text{pr}_0(P)$  denotes the projection of  $P$  at time  $t = 0$ .

Moreover, if  $\mu$  is the only Gibbs measure corresponding to  $\mathcal{P}$  and  $m_0^{\otimes \mathbb{Z}^d}$  then  $Q_\infty^\mu$  is the unique minimiser corresponding to the good rate function  $\mathcal{I}^a$ .

(ii) Furthermore, for a fixed, typical realisation  $\mathbf{k}$  of the disordered external field, the law of the empirical process  $\hat{\pi}_x^{(\Lambda)}$  under  $dP_\Lambda^k(\mathbf{x})$  also obeys a large deviation principle on  $\mathcal{M}_s(\Omega)$ , on the scale  $|\Lambda|$  and according to a good rate function  $\mathcal{I}^q$  satisfying:

$$\mathcal{I}^q(P) \geq \mathcal{I}^a(P), \quad \forall P \in \mathcal{M}_s(\Omega).$$

### Acknowledgements

The second author gratefully acknowledges financial support from the Swiss National Science Foundation.

### References

Ben Arous, G., Guionnet, A., 1997. Symmetric Langevin spin glass dynamics. *Ann. Probab.* 25, pp. 1367–1422.  
 Ben Arous, G., Guionnet, A., 1998. Langevin dynamics for Sherrington–Kirkpatrick spin glasses. In: Bovier, A., Picco, P. (Eds.), *Mathematical Aspects of Spin Glasses and Neural Networks*. Birkhäuser, Basel, London, pp. 355–382.

<sup>3</sup> For  $i \in \partial\Lambda$ ,  $\xi$  appears in the drift term associated with  $dx_t^i$ .

- Ben Arous, G., Sordani, M., 2002. Large Deviations in the Langevin dynamics of a short range spin glass, Preprint.
- Brézin, E., De Dominicis, C., 1998a. Dynamics versus replicas in the Random Field Ising Model, preprint available on the xxx server under: cond-mat/9807113 8 July 1998.
- Brézin, E., De Dominicis, C., 1998b. New phenomena in the Random Field Ising Model, preprint available on the xxx server under: cond-mat/9804266 v2 21 August 1998.
- Cattiaux, P., Roelly, S., Zessin, H., 1996. Une approche gibbsienne des diffusions browniennes infini-dimensionnelles. *Probab. Theor. Relat. Fields* 104, 147–179.
- Comets, F., 1989. Large deviation estimates for a conditional probability distribution. Applications to random interacting Gibbs measures. *Probab. Theor. Relat. Fields* 80, 407–432.
- Dembo, A., Zeitouni, O., 1998. *Large Deviations Techniques and Applications*, 2nd Edition. Springer Verlag, Berlin.
- Föllmer, H., Orey, S., 1988. Large deviations for the empirical field of a Gibbs measure. *Ann. Probab.* 16, 961–977.
- Föllmer, H., Wakolbinger, A., 1986. Time reversal of infinite dimensional diffusion processes. *Stochast. Proc. Appl.* 22, 59–78.
- Georgii, H.-O., 1988. *Gibbs Measures and Phase Transitions*. de Gruyter, Berlin.
- Grunwald, M., 1996. Sanov results for Glauber spin glass dynamics. *Probab. Theor. Relat. Fields* 106, 187–232.
- Grunwald, M., 1998. Sherrington–Kirkpatrick spin glass dynamics: the discrete setting. In: Bovier, A., Picco, P. (Eds.), *Mathematical Aspects of Spin Glasses and Neural Networks*. Birkhäuser, Basel, pp. 355–382.
- Krylov, N.V., 1990. A simple proof of the existence of a solution of Itô’s equation with monotone coefficients. *Theor. Probab. Appl.* 35 (3), 583–587.
- Nualart, D., 1998. Analysis on Wiener space and anticipating stochastic calculus. In: Bernard, P. (Ed.), *Lectures on Probability and Statistics, St-Flour 1995*. Springer Verlag, Berlin, pp. 123–227.
- Olla, S., 1988. Large deviations for Gibbs random fields. *Probab. Theor. Relat. Fields* 77, 343–357.
- Shiga, T., Shimizu, A., 1980. Infinite dimensional stochastic differential equations and their applications. *J. Math. Kyoto Univ.* 20 (3), 395–416.
- Sompolinsky, H., Zippelius, A., 1983. Relaxational dynamics of the Edwards-Anderson model and the mean-field theory of spin glasses. *Phys. Rev. B* 25 (11), 6860–6875.
- Sordani, M., 2001. Large deviations in the Langevin dynamics of lattice disordered systems, Ph.D. Thesis.
- Varadhan, S.R.S., 1966. Asymptotic probabilities and differential equations. *Comm. Pure Appl. Math.* 19, 261–286.
- Zeitouni, O., 2003. *Lecture Notes on RWRE, Saint-Flour 2001 Lecture Notes*, Springer Verlag, Berlin, to appear.