Spatio-temporal large deviations principle for coupled circle maps

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Abstract

We consider the $(d+1)-$dimensional dynamical system constituted by weakly coupled expanding circle maps on $\mathbb{Z}^d$ together with the spatial shifts. This viewpoint allows us to use Thermodynamic Formalism, and to describe the asymptotic behavior of the system in this setup. We obtain a Volume Lemma, which describes the exponential behavior of the size under Lebesgue measure of dynamical balls around any orbit, then a Large Deviations Principle for the empirical measure associated to this dynamical system. The proofs are direct: we do not use the coding constructed by Jiang in [12] for such systems.

1 Introduction

Coupled map lattices have been introduced in 1983 by Kuhiniko Kaneko. They are models of discrete time dynamical systems on lattice spaces. They act on a product space formed by an interval or a manifold on each site of the lattice $\mathbb{Z}^d$. The evolution at each step of time is the composition of a chaotic dynamics applied independently on each site and of a coupling between sites.

Such systems present a competition between the chaos of the local map and the coupling which tends to organize spatially the system. They present many interesting features as spatio-temporal chaos, intermittency or phase transitions (see [14, 15] for an overview of physical studies and numerical simulations).

We consider in this paper the case of a weak coupling between expanding maps of the circle. We work on the state space $\mathcal{X} = (S^1)^{\mathbb{Z}^d}$ and take as local dynamics an expanding map of the circle, i.e. $f : S^1 \to S^1$ which is $\mathcal{C}^{1+\alpha}$ and such that

$$|f'(x)| \geq \lambda > 1 \quad \forall\, x \in S^1.$$

The coupling can be chosen from a wide class (see Section 2.2 for the needed assumptions) but the simplest example to be considered is a diffusive coupling between nearest

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neighboors

$$(G_\varepsilon(x))_i = (1-\varepsilon)x_i + \frac{\varepsilon}{2d}\sum_{j\sim i} x_j$$

(or more precisely a smooth modification of this example on the circle).

The coupled map lattice is then the map $F = F_\varepsilon = G_\varepsilon \circ F_0$, where F_0 is the uncoupled map defined by $(F_0(x))_i = f(x_i)$.

We study the limit behavior of the spatio-temporal empirical measures associated to the coupled map F,

$$R_T(x) = \frac{1}{T|\Lambda_T|}\sum_{\substack{0\le t<T\\ i\in\Lambda_T}} \delta_{S^i\circ F^t(x)} \in \mathcal{M}^1(\mathcal{X}),$$

where S denotes the spatial shifts (defined by $(S^i x)_k = x_{k+i}$), $\Lambda_T = [-T,T]^d$ and $\mathcal{M}^1(\mathcal{X})$ is the space of probability measures on $\mathcal{X}$.

We prove in Theorem 2.2 that if the coupling is small enough (for small ε in the explicit example G_ε considered here), R_T satisfies under initial measure $\overline{m}$, the product of Lebesgue measures on the circles, a Large Deviations Principle with rate function

$$I_{\text{st}}(\mu) = \begin{cases} -h_{(F,S)}(\mu) - \int_{\mathcal{X}} \varphi\, d\mu & \text{if } \mu \text{ is invariant by } F \text{ and } S, \\ +\infty & \text{otherwise,} \end{cases}$$

with $h_{(F,S)}$ the metric entropy associated to the $(d+1)$-dimensional dynamical system (F,S) and φ a potential associated to the dynamics (see Section 3 for its exact definition). This result means, roughly, that

$$\overline{m}\{x \,:\, R_T(x) \sim \mu\} \sim \exp\Big(T|\Lambda_T|\Big(h_{(F,S)}(\mu) + \int_{\mathcal{X}} \varphi\, d\mu\Big)\Big).$$

This implies in particular that R_T converges exponentially fast to the set of equilibrium measures associated to φ,

$$\mathrm{EQ}(\varphi) = \{\nu \in \mathcal{M}^1(\mathcal{X}) \,:\, h_{(F,S)}(\nu) + \int_{\mathcal{X}} \varphi\, d\nu = 0\}.$$

This result is linked to previous papers of Jiang and Pesin [13, 12]. Generalizing previous results of Bunimovich-Sinai [4] and Pesin-Sinai [28], they caracterized the spatio-temporal chaos for a weak coupling between expanding or Anosov maps by the uniqueness of the equilibrium measure associated to φ.

Our result puts hence the emphasis on the variational principle associated to this potential and shows by a new way that $-\varphi$ really plays in this context the role of the logarithm of Jacobian. Our result is indeed a generalization to the case of coupled map lattices of well known results for single site hyperbolic dynamical systems [33, 24, 25, 20, 7] or Gibbs measures on shift spaces [9, 23, 5, 8]. This offers the perspective of generalizing other linked properties as the Gibbs characterization (as defined by Haydn and Ruelle in [10, 30]) of equilibrium measures or the multifractal analysis (see [27]).

It has to be noticed that (except for the construction of the potential φ) our Large Deviations Principle is independent of previous results of Jiang and Pesin. We use neither the coding by a shift space, nor the uniqueness of the equilibrium measure. This allows us to work under less restrictive assumptions, although we still need a weak coupling assumption for many steps of the proof.

The most important and demanding part of the proof is a Volume Lemma result (Theorem 2.1): we show that the partial sum of the potential φ governs the size under $\overline{m}$ of the set of points whose orbit stays near a given one under fixed time and space translations. The proof of this result relies on a property of expansivity for the coupled map and a sharp analysis of inverse branches.

Using this to prove large deviations is then a natural generalization of the methods of L.S. Young [33] and Y. Kifer [20] for single site maps.

Another approach has been developped to characterize spatio-temporal chaos under stronger regularity assumptions, via spectral properties of an adapted transfer operator. We refer the reader to [31, 1] for the most recent results and detailed bibliography. It makes it possible to study the asymptotic behavior of the temporal empirical measure, but in this case Thermodynamic Formalism can not be used and results are less complete (see [3] for such results).

The paper is organized as follows:

We give our precise Assumptions and Results in Section 2. In Section 3 we recall the derivation of the potential we are interested in, done in [13, 12]. In Section 4, we analyze precisely the inverse branches of the coupled map and deduce a preserved expanding property. Section 5 is then devoted to the proof of the Volume Lemma and Sections 6 and 7 to the proof of the Large Deviations Principle.

For the sake of comprehension, some facts on convergence of subsets of $\mathbb{Z}^d$ and a reminder on Thermodynamic Formalism are given in Appendix.

2 Settings and results

2.1 The state space

We work on the state space $\mathcal{X} = (S^1)^{\mathbb{Z}^d}$ (with $d \geq 1$), equipped with the reference measure $\overline{m} = m^{\otimes \mathbb{Z}^d}$ where m is the Lebesgue measure on the circle.
On the circle $S^1 = \mathbb{R}/\mathbb{Z}$, the distance is $d(x,y) = \min_{k \in \mathbb{Z}} |x + k - y| \leq 1/2$. We put on $\mathcal{X}$ two distances constructed from this one:

- $d(x,y) = \sup_{i \in \mathbb{Z}^d} d(x_i, y_i)$, which is compatible with the differentiable structure of $\mathcal{X}$ defined by partial derivatives;

- $d_\rho(x,y) = \sup_{i \in \mathbb{Z}^d} \rho^{|i|} d(x_i, y_i)$ where we take for $i \in \mathbb{Z}^d$ the norm $|i| = \max_{1 \leq k \leq d} |i_k|$ and $\rho < 1$ is a fixed parameter. The main interest of d_ρ is that $(\mathcal{X}, d_\rho)$ is a compact space, hence we can use the thermodynamic formalism to describe the system.

We denote by S^k the spatial shift of vector $k \in \mathbb{Z}^d$ on $\mathcal{X}$: if $x = (x_i)_{i \in \mathbb{Z}^d}$ then $(S^k x)_i = x_{i+k}$.
For $N \in \mathbb{N}$, we write $\Lambda_N = [-N, N]^d \subset \mathbb{Z}^d$.

2.2 The coupled map

Let the uncoupled expanding map be $F_0 = \otimes_{i\in\mathbb{Z}^d} f_i$ where $f_i = f : S^1 \to S^1$ is $\mathcal{C}^{1+\alpha}$ and expanding, i.e. satisfies

$$1 < \gamma \le |f'(x)| \le M \qquad \forall x \in S^1, \tag{1}$$

and f' hence $\log|f'|$ is α-Hőlder continuous,

$$\big|\log|f'(x)| - \log|f'(y)|\big| \le C_1 d^\alpha(x,y) \qquad \forall x,y \in S^1. \tag{2}$$

We define also the coupling map $G : \mathcal{X} \to \mathcal{X}$ as a $\mathcal{C}^2$ map (for the distance d) commuting with all the spatial translations $(S^k)_{k\in\mathbb{Z}^d}$ and which satisfies the following estimates

$$\left|\frac{\partial G_i}{\partial x_j} - \delta_{i,j}\right| \le \mathcal{E}\,\theta^{2|i-j|} \qquad \forall i,j \in \mathbb{Z}^d, \tag{3}$$

$$\left|\frac{\partial^2 G_i}{\partial x_j \partial x_k}\right| \le \mathcal{E}\,\theta^{2\max(|i-j|,|i-k|)} \qquad \forall i,j,k \in \mathbb{Z}^d, \tag{4}$$

with $\mathcal{E} > 0$ and $0 < \theta < 1$.
We denote $\mathcal{K} = \mathcal{E}\sum_{i\in\mathbb{Z}^d}\theta^{|i|}$ and $\mathcal{K}_2 = \mathcal{E}\sum_{i\in\mathbb{Z}^d}\theta^{2|i|}$.

The first derived estimates are

$$d_i(G(x)-x, G(y)-y) \le \mathcal{E}\sum_{k\in\mathbb{Z}^d}\theta^{2|i-k|}d_k(x,y) \qquad \forall i \in \mathbb{Z}^d,\ x,y \in \mathcal{X}, \tag{5}$$

$$\left|\frac{\partial G_i}{\partial x_j}(x) - \frac{\partial G_i}{\partial x_j}(y)\right| \le \mathcal{E}\sum_{k\in\mathbb{Z}^d}\theta^{2|i-k|}d_k(x,y) \qquad \forall i,j \in \mathbb{Z}^d,\ x,y \in \mathcal{X}. \tag{6}$$

The associated coupled map is then

$$F = G \circ F_0.$$

We say that F satisfies Assumption $(\mathcal{H})$ if it is the composition of two such maps whose parameters satisfy the two conditions

$$\begin{cases} \theta < \rho & (H1) \\ \gamma - M\mathcal{K} > 1 & (H2) \end{cases}$$

The first assumption is essentially technical, to get functions regular enough for the distance d_ρ. $(H2)$ expresses the preservation of the expanding property for the coupled map and implies two essential estimates

$$\tilde{\gamma} = \gamma - M\mathcal{K}_2 > 1, \tag{7}$$

$$\mathcal{K} < 1. \tag{8}$$

Remark: These coupling maps are similar to those given in previous papers on this type of system (they are called short range maps in [13, 12]).

2.3 Volume Lemma

We define for $T \in \mathbb{N}$ and E a finite subset of $\mathbb{Z}^d$

$$B_x(T,E;\delta) = \{y\,:\, d_\rho\left(S^i \circ F^t(x), S^i \circ F^t(y)\right) < \delta \quad \forall 0 \le t \le T,\, i \in E\} \tag{9}$$

the ball associated to a distance which describes the dynamics of F and the spatial shifts S. It contains points whose orbit stays near a given orbit under fixed space and time translations. The Volume Lemma describes the measure of this ball in terms of local derivatives along the orbit of x:

Theorem 2.1. *If F satisfies Assumption $(\mathcal{H})$, then there exists a potential function $\varphi : \mathcal{X} \mapsto \mathbb{R}$ Hőlder continuous for the distance d_ρ, such that for any $x \in \mathcal{X}$, $0 < \delta < \frac{1}{2M}$, E a finite subset of $\mathbb{Z}^d$ and $T \ge 1$, we have*

$$C_2(T,E,\delta,\rho)\exp\Big(\sum_{\substack{0\le t<T\\ i\in E}} \varphi\circ S^i \circ F^t(x)\Big) \le \overline{m}(B_x(T,E;\delta))$$

$$\le C_3(T,E,\delta)\exp\Big(\sum_{\substack{0\le t<T\\ i\in E}} \varphi\circ S^i \circ F^t(x)\Big), \tag{10}$$

with

$$\lim_{\substack{T\to\infty\\ n\to\infty}} \frac{1}{T|E_n|}\log C_2(T,E_n,\delta,\rho) = \lim_{\substack{T\to\infty\\ n\to\infty}} \frac{1}{T|E_n|}\log C_3(T,E_n,\delta) = 0 \quad \forall\delta < \frac{1}{2M}, \theta<\rho<1, \tag{11}$$

and for any sequence E_n converging to $\mathbb{Z}^d$ in the sense of Van Hove (see Definition A.1).

Remarks: 1. The potential function φ is defined in Section 3.2 following readily the construction given in [12] and [13]. From this definition and the role it plays in the Volume Lemma (see for example [19] for an equivalent result in the case of a single map), $-\varphi$ can be called the "logarithm of Jacobian per site" of the map F.

2. The speeds of convergence in time and space are completely independent.

3. This result is in fact true not only under Lebesgue measure but also for any probability measure μ which is locally absolutely continuous with respect to it, with a Radon-Nikodym derivative satisfying with $0 < A < B$

$$A^{|E|} \le \left.\frac{d\mu}{d\overline{m}}\right|_E \le B^{|E|} \qquad \forall E \subset \mathbb{Z}^d.$$

A direct consequence of this result, or of Proposition 6.1, concerns the topological pressure (see Appendix B.2 for the definition) of the potential φ:

Corollary 2.1. *If F satisfies Assumption $(\mathcal{H})$, the topological pressure of the potential φ under the dynamical system (F,S) is null,*

$$P_{(F,S)}(\varphi) = 0.$$

This was already stated in [11, 12] in various contexts. It takes here a particular importance since it ensures with the Gibbs Variational Principle B.3 that the rate function I (defined in (13) below) is non negative.

2.4 Large Deviations Principle

We can use the previous Volume Lemma to prove a spatio-temporal Large Deviations Principle for the empirical process

$$R_{T,E}(x) = \frac{1}{T|E|} \sum_{\substack{0 \le t < T \\ i \in E}} \delta_{S^i \circ F^t(x)} \in \mathcal{M}^1(\mathcal{X}), \tag{12}$$

under the initial measure $\overline{m}$ (and, more generally, under the same probability measures as for Volume Lemma, see Remark 3 after Theorem 2.1).

We introduce the function I defined on $\mathcal{M}^1(\mathcal{X})$ by

$$I(\nu) = \begin{cases} -h_{(F,S)}(\nu) - \int_{\mathcal{X}} \varphi \, d\nu & \text{if } \nu \in \mathcal{M}^1_{\text{inv}}(\mathcal{X}), \\ +\infty & \text{otherwise,} \end{cases} \tag{13}$$

where $\mathcal{M}^1_{\text{inv}}(\mathcal{X})$ is the set of probability measures which are invariant under F and spatial shifts, with the weak-star topology ($\mu_k \to \mu$ iff $\int g \, d\mu_k \to \int g \, d\mu$ for any $g \in \mathcal{C}(\mathcal{X})$), and $h_{(F,S)}$ is the metric entropy (see Appendix B).
We have then:

Theorem 2.2. *Assume F satisfies Assumption $(\mathcal{H})$. Then I is a non negative, convex and lower semi-continuous function.*
For any map $s : \mathbb{N} \to \mathbb{N}$ non decreasing and such that $s(T)$ tends to infinity as T tends to infinity, the sequence $(R_{T,\Lambda_{s(T)}})^*(\overline{m})$ of measures on $\mathcal{M}^1(\mathcal{X})$ satisfies a Large Deviations Principle with rate function *I, i.e.:*

1. *For any K closed subset of $\mathcal{M}^1(\mathcal{X})$, we have*

$$\limsup_{T \to \infty} \frac{1}{T|\Lambda_{s(T)}|} \log \overline{m}\{x : R_{T,\Lambda_{s(T)}}(x) \in K\} \le -\inf_{\nu \in K} I(\nu). \tag{Upper Bound}$$

2. *For any O open subset of $\mathcal{M}^1(\mathcal{X})$, we have*

$$\liminf_{T \to \infty} \frac{1}{T|\Lambda_{s(T)}|} \log \overline{m}\{x : R_{T,\Lambda_{s(T)}}(x) \in O\} \ge -\inf_{\nu \in O} I(\nu). \tag{Lower Bound}$$

Remarks: This result remains in fact true for more general sequences of sets: the upper bound is valid for any spatial sequence E_T converging to $\mathbb{Z}^d$ in the sense of Van Hove, the lower bound for any special averaging sequence (see Definition A.2). Proofs are given in Sections 6 and 7 in this general setup.

We may notice that the relative speeds of averaging in time and space can be completely arbitrary (we make no assumption on the function s).

This independence of speeds of convergence in time and space is important but not surprising since we know that for weak coupling there is a semi-conjugacy between (F, S) and shifts of a $(d+1)$ dimensional Gibbs system (see Theorem 2 in [12]). The time direction becomes then a spatial shift like others on the coding space.

This semi-conjugacy allows in fact to deduce a Large Deviations Principle for R_{T,E_T} from the same result for Gibbs systems (see [9, 23, 5, 8]) by a contraction principle (Theorem 4.2.1 of [6]). We could not identify the rate function obtained in this way, hence preferred to develop a direct proof, without using the coding. It has however to be noticed that our analysis of inverse branches in Subsection 4.2 is not far from the construction of a Markov partition for the system.

3 Expansion of the derivative

In this section, we follow [13] to derive the potential φ by a sharp analysis of the derivative of the map F restricted to finite boxes. We give all the steps, referring the reader to Section 5 of [13] or to [2] for the detailed computations.

3.1 Finite box maps

For Λ a finite subset of $\mathbb{Z}^d$ and $\eta \in \mathcal{X}$ a fixed boundary condition, we define

$$\begin{aligned} F_{\Lambda,\eta} \, : \, \mathcal{X}_\Lambda = \left(S^1\right)^\Lambda &\longrightarrow \mathcal{X}_\Lambda \\ x_\Lambda &\longmapsto F(x_\Lambda \vee \eta_{\Lambda^C})|_\Lambda \end{aligned}$$

with $w = x_\Lambda \vee \eta_{\Lambda^C}$ defined by $w_i = x_i$ when $i \in \Lambda$ and $w_i = \eta_i$ otherwise. In fact $F_{\Lambda,\eta} = G_{\Lambda,F_0(\eta)} \circ F_0$ with $G_{\Lambda,\eta} = G(x_\Lambda \vee \eta_{\Lambda^C})$.

$G_{\Lambda,\eta}$ is a $\mathcal{C}^2$ map and if we write $DG_{\Lambda,\eta} = Id_\Lambda + A_{\Lambda,\eta}$ with $A_{\Lambda,\eta} = \left(a_{i,j}\right)_{i,j\in\Lambda}$, we get from estimates (3) and (6) the following estimates for any $i, j \in \Lambda$, $x_\Lambda, y_\Lambda \in \mathcal{X}_\Lambda$

$$|a_{i,j}(x_\Lambda)| \le \mathcal{E}\theta^{2|i-j|}, \tag{14}$$

$$|a_{i,j}(x_\Lambda) - a_{i,j}(y_\Lambda)| \le \mathcal{E} \sum_{k\in\Lambda} \theta^{2|i-k|} d_k(x_\Lambda, y_\Lambda), \tag{15}$$

$$|a_{i,j}^{(\eta)}(x_\Lambda) - a_{i,j}^{(\eta')}(x_\Lambda)| \le \frac{\mathcal{K}}{2}\,\theta^{d(i,\Lambda^C)}, \tag{16}$$

$$|a_{i,j}^{(\Lambda)}(x_\Lambda) - a_{i,j}^{(\Lambda')}(y_{\Lambda'})| \le \frac{\mathcal{K}}{2}\,\theta^{d(i,\Lambda'\setminus\Lambda)}, \tag{17}$$

if $\Lambda \subset \Lambda'$ and $y_{\Lambda'}|_\Lambda = x_\Lambda$.

3.2 Expansion

We get using (8),

$$\|A\|_\infty \le \max_{i\in\Lambda} \left(\mathcal{E} \sum_{j\in\Lambda} \theta^{2|i-j|} \right) \le \mathcal{K}_2 \le \mathcal{K} < 1,$$

hence $\log(Id + A)$ exists and we can write

$$\begin{aligned}\log|\det DF_{\Lambda,\eta}(x_\Lambda)| &= \log|\det DF_0(x_\Lambda)\det DG_{\Lambda,F_0(\eta)}(F_0(x_\Lambda))| \\ &= \sum_{i\in\Lambda}\log|f'(x_i)| + \log|\det(\exp\log(Id+A)(F_0(x_\Lambda)))| \\ &= \sum_{i\in\Lambda}\log|f'(x_i)| + \log\exp(\operatorname{tr}\log(Id+A)(F_0(x_\Lambda))) \\ &= \sum_{i\in\Lambda}\log|f'(x_i)| + \operatorname{tr}\left(-\sum_{t\geq 1}\frac{(-1)^t}{t}A^t(F_0(x_\Lambda))\right) \\ &= \sum_{i\in\Lambda}\left(\log|f'(x_i)| - w_{\Lambda,\eta,i}(x_\Lambda)\right),\end{aligned}$$

where $w_{\Lambda,\eta,i}(x_\Lambda) = \sum_{t\geq 1}\frac{(-1)^t}{t}a^{(t)}_{i,i}(F_0(x_\Lambda))$, denoting $A^t = \left(a^{(t)}_{i,j}\right)$.

Estimates (14) to (17) give analogous results for w under the same condition (8)

$$|w_{\Lambda,\eta,i}(x_\Lambda)| \leq \frac{\mathcal{E}}{1-\mathcal{K}}, \tag{18}$$

$$|w_{\Lambda,\eta,i}(x_\Lambda) - w_{\Lambda,\eta,i}(y_\Lambda)| \leq \frac{M\mathcal{E}}{1-\mathcal{K}}\sum_{k\in\Lambda}\theta^{|i-k|}d_k(x_\Lambda, y_\Lambda), \tag{19}$$

$$|w_{\Lambda,\eta,i}(x_\Lambda) - w_{\Lambda,\eta',i}(x_\Lambda)| \leq \frac{1}{2(1-\mathcal{K})}\,\theta^{d(i,\Lambda^C)}, \tag{20}$$

$$|w_{\Lambda,\eta,i}(x_\Lambda) - w_{\Lambda',\eta,i}(y_{\Lambda'})|, \leq \frac{1}{2(1-\mathcal{K})}\,\theta^{d(i,\Lambda'\setminus\Lambda)}, \tag{21}$$

if $\Lambda \subset \Lambda'$ and $y_{\Lambda'}|_\Lambda = x_\Lambda$.

All these estimates imply that $\psi_i(x) = \lim_{N\to\infty} w_{\Lambda_N,\eta,i}(x|_{\Lambda_N})$ exists, is independent of the boundary conditions, shift invariant (i.e. $\psi_i = \psi_0 \circ S^i$ for all $i \in \mathbb{Z}^d$) and satisfies

$$|\psi_0(x)| \leq \frac{\mathcal{E}}{1-\mathcal{K}}, \tag{22}$$

$$|\psi_0(x) - \psi_0(y)| \leq \frac{M\mathcal{E}}{1-\mathcal{K}}\sum_{k\in\mathbb{Z}^d}\theta^{|i-k|}d_k(x,y), \tag{23}$$

$$|\psi_0(x) - w_{\Lambda,\eta,0}(x|_\Lambda)| \leq \frac{1}{2(1-\mathcal{K})}\,\theta^{d(i,\Lambda^C)}. \tag{24}$$

Assumption $(H1)$ implies moreover with (23) that ψ_0 is Lipschitz continuous for the distance d_ρ.

We define hence

$$\varphi(x) = -\log|f'(x_0)| + \psi_0 \tag{25}$$

as the potential of interest to describe the dynamic of the system (F, S). φ is α-Hőlder continuous for the distance d_ρ.

4 Conservation of the expanding property

We introduce $\emptyset \neq E \subset \Lambda$ two finite subsets of $\mathbb{Z}^d$, a time $T \in \mathbb{N}$ and $x \in \mathcal{X}$ a reference point.
We choose a finite box restriction of F^T to Λ, F_Λ^T with boundary conditions changing with time: $F_\Lambda^t = F_{\Lambda,F^{t-1}(x)} \circ \cdots \circ F_{\Lambda,F(x)} \circ F_{\Lambda,x}$. It implies in particular that

$$F_\Lambda^t(x|_\Lambda) = F^t(x)\big|_\Lambda \qquad \forall\, 0 \leq t \leq T. \tag{26}$$

This will essentially simplify the approximation of F by F_Λ in the proof of the Volume Lemma. We do not mention explicitly the dependence on the boundary conditions following the orbit of x: we have already seen in previous Section that the limit potential does not depend on it.

4.1 Bijectivity of the coupling map

First of all, our assumptions on the coupling map G are sufficient to get:

Proposition 4.1. *Under assumption* $(H2)$, G_Λ *is a* $\mathcal{C}^1$ *diffeomorphism.*

Proof. We get from estimate (5) and the triangle inequality that

$$d_i(G_\Lambda(x), G_\Lambda(y)) \geq d_i(x,y) - \mathcal{E} \sum_{k \in \Lambda} \theta^{2|i-k|} d_k(x,y) \qquad \forall i \in \Lambda,$$

hence if $x \neq y$, let i_0 be such that $d_{i_0}(x,y) = \max_{i\in\Lambda} d_i(x,y) > 0$. Then

$$d_{i_0}(G_\Lambda(x), G_\Lambda(y)) \geq d_{i_0}(x,y) \left(1 - \mathcal{E} \sum_{k\in\Lambda} \theta^{2|i-k|}\right) \geq (1-\mathcal{K}_2) d_{i_0}(x,y) > 0$$

because $\mathcal{K}_2 \leq \mathcal{K} < 1$ by (8). This proves that G_Λ is one-to-one.

We have already noticed that $\|A\|_\infty < 1$, which gives that DG_Λ is invertible, hence that G_Λ is everywhere a local diffeomorphism. The range of G_Λ is then open, and closed by compactness of $\mathcal{X}_\Lambda$, hence its range is the whole space $\mathcal{X}_\Lambda$ because it is connected.

G_Λ is then a bijection and a local diffeomorphism, then a diffeomorphism. □

Remark: G is also a bijection (one-to-one in the same way, surjective taking limit of preimages on finite boxes).

4.2 Inverse branches of F_Λ^T

The single site map $f : S^1 \to S^1$ is of degree $p = \int_{S^1} |f'(x)|\, dx$, an integer between γ and M, and has then locally p inverse branches around each point. We can in fact construct them globally except in one point (see Section 2.4 of [17]).

We will use this to construct inverse branches for F_0 around the orbit of x. Associated to the fact that G is a diffeomorphism, it will give us inverse branches for F_Λ^T.
We denote $\mathcal{C}[\Lambda] = \{0, \ldots, p-1\}^\Lambda$ to enumerate the inverse branches of F_0.

At each time $0 \le t < T$, we construct them around $F^t(x)$. We take

$$A_t = \{y \in \mathcal{X}_\Lambda \, : \, d_i(y, F_0 \circ F^t(x)) < 1/2 \quad \forall i \in \Lambda\}$$

(then $m^\Lambda(A_t) = 1$) and for any site $i \in \Lambda$ we denote $x_0^{(t,i)}$, $x_1^{(t,i)}$, ..., $x_{p-1}^{(t,i)}$ (resp. $a_0^{(t,i)}$, $a_1^{(t,i)}$, ..., $a_{p-1}^{(t,i)}$) the preimages by f of $(F_0 \circ F^t(x))_i$ (resp. $(F_0 \circ F^t(x))_i - 1/2$), indexed such that:

- $x_0^{(t,i)} = F_i^t(x)$,
- $x_0^{(t,i)}, a_1^{(t,i)}, x_1^{(t,i)}, \cdots, a_0^{(t,i)}$ are in this order on the circle.

Then, for all $\beta \in \mathcal{C}[\Lambda]$, we define

$$x_\beta^{(t)} = \left(x_{\beta(i)}^{(t,i)}\right)_{i\in\Lambda} \qquad \text{the preimages by } F_0 \text{ of } F_0 \circ F^t(x),$$
$$A_{\beta,t} = \prod_{i\in\Lambda} \left(a_{\beta_i}^{(t,i)}, a_{\beta(i)+1}^{(t,i)}\right),$$

satisfying the following straightforward properties:

- $x_0^{(t)} = F^t(x)$,
- $x_\beta^{(t)} \in A_{\beta,t} \quad \forall \beta \in \mathcal{C}[\Lambda]$,
- $m^\Lambda\Big(\bigcup_{\beta\in\mathcal{C}[\Lambda]} A_{\beta,t}\Big) = 1$,
- F_0 is a bijection from $A_{\beta,t}$ onto A_t.

We denote $F_{0,t,\beta}^{-1}$ its inverse characterized by $F_{0,t,\beta}^{-1}(y) = A_{\beta,t} \cap F_0^{-1}(y)$ for any $y \in A_t$. These inverse branches satisfy a contraction property, which has to be precisely described:

Lemma 4.1. *For all $y, z \in A_t$, there exists $\phi_{y,z}$ permutation of $\mathcal{C}[\Lambda]$, with $y, z \mapsto \phi_{y,z}$ measurable, such that $\forall \beta, \tilde\beta \in \mathcal{C}[\Lambda], \forall i \in \Lambda$, if $\beta(i) = \tilde\beta(i)$, then*

$$\frac{1}{M} d_i(y,z) \le d_i\left(F_{0,t,\tilde\beta}^{-1}(y), F_{0,t,\phi_{y,z}(\beta)}^{-1}(z)\right) \le \frac{1}{\gamma} d_i(y,z). \tag{27}$$

If y or z equals $F_0 \circ F^t(x)$, then $\phi_{y,z} = Id$.

Proof. The left inequality is obvious, because $d_i(F_0(\tilde y), F_0(\tilde z)) \le M d_i(\tilde y, \tilde z)$ is always true. For the contraction rate, we have to be careful because the partition is adapted to $F^t(x)$ but not to all other points. What has to be understood is how $d_i(y,z)$ is realized at each site $i \in \Lambda$:

- if the shortest arc from y_i to z_i (defining the distance) does not contain $(F_0 \circ F^t(x))_i - 1/2$ (case (i) of Figure 1), then $\phi_{y,z}(\beta)(i) = \beta(i)$;
- otherwise, $\phi_{y,z}(\beta)(i) = \beta(i) \pm 1$, depending on the order of the three points y, z and $(F_0 \circ F^t(x))_i - 1/2$ (cases (ii) and (iii) of the Figure)) but not on β.

This defines $\phi_{y,z}$ as a one-to-one map, and if we are interested in site i, the inverse maps β and $\tilde\beta$ are indistinguishable, hence

$$d_i\left(F_{0,t,\tilde\beta}^{-1}(y), F_{0,t,\phi_{y,z}(\beta)}^{-1}(z)\right) = d_i\left(F_{0,t,\beta}^{-1}(y), F_{0,t,\phi_{y,z}(\beta)}^{-1}(z)\right) \le \frac{1}{\gamma} d_i(y,z).$$

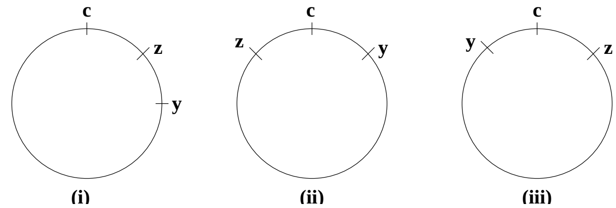

Figure 1: The three cases, where $c = F_0 \circ F^t(x) - \frac{1}{2}$. If f preserves the direction on the circle (i.e. $f' > 0$), (ii) corresponds to $\phi_{y,z}(\beta)(i) = \beta(i) + 1$, (iii) to $\phi_{y,z}(\beta)(i) = \beta(i) - 1$, and this is reversed otherwise.

If y or z is equal to $F_0 \circ F^t(x)$, we always are in the first case.
It is not hard to check that $\phi_{y,z}$ depends on y and z only through the distance and the order of their coordinates in the open sets $S^1 \setminus \{(F_0 \circ F^t(x))_i - 1/2\}$, which are measurable maps of y and z. $\square$

We have also, from the left inequality of (27) applied with $y = (F_0 \circ F^t(x))_i$ and z tending to $(F_0 \circ F^t(x))_i - 1/2$, that

$$\left\{ y \,:\, d_i(F^t(x), y) < \frac{1}{2M} \right\} \subset \bigcup_{\substack{\beta \in \mathcal{C}[\Lambda] \\ \beta(i)=0}} A_{\beta,t}. \tag{28}$$

We can then describe the inverse branches of F_Λ^T, with

$$\mathcal{C}[T,\Lambda] = \{0, \ldots, p-1\}^{[1,\ldots,T] \times \Lambda},$$
$$\mathcal{C}[T,\Lambda,E] = \{\alpha \in \mathcal{C}[T,\Lambda] \,:\, \alpha_{t,i} = 0 \quad \forall 1 \le t \le T,\, i \in E\}.$$

Then:

Proposition 4.2. *We associate in a unique way to each $\alpha \in \mathcal{C}[T,\Lambda]$ an open subset $\mathcal{A}_\alpha(x)$ of $\mathcal{X}_\Lambda$ such that:*

- *$\mathcal{A}_\alpha(x) \cap \mathcal{A}_{\alpha'}(x) = \emptyset$ if $\alpha \neq \alpha'$;*
- *$m^\Lambda(\cup \mathcal{A}_\alpha(x)) = 1$;*
- *There exists $\mathcal{A} \subset \mathcal{X}_\Lambda$ with $m^\Lambda(\mathcal{A}) = 1$ such that for all $\alpha \in \mathcal{C}[T,\Lambda]$, F_Λ^T is one-to-one from $\mathcal{A}_\alpha(x)$ onto $\mathcal{A}$. We denote $F_{\Lambda,\alpha}^{-T}$ its inverse.*

Moreover

$$\left\{ y \in \mathcal{X}_\Lambda \,:\, d_i(F^t(x), F_\Lambda^t(y)) < \frac{1}{2M} \quad \forall 0 \le t < T,\, i \in E \right\} \subset \bigcup_{\alpha \in \mathcal{C}[T,\Lambda,E]} \mathcal{A}_\alpha(x). \tag{29}$$

Proof. We define

$$\mathcal{A} = \bigcap_{t=0}^{T-1} F^{T-1-t} \circ G(A_t)$$

to avoid any problem of definition ($m^\Lambda(\mathcal{A}) = 1$ by preservation of total measure by F_0 and G, and by finite intersection) and

$$F_{\Lambda,\alpha}^{-T} = F_{0,0,\alpha(0,\cdot)}^{-1} \circ G^{-1} \circ F_{0,1,\alpha(1,\cdot)}^{-1} \circ G^{-1} \circ \cdots \circ F_{0,T-1,\alpha(T-1,\cdot)}^{-1} \circ G^{-1},$$

which is well defined on $\mathcal{A}$. All properties are then easily deduced from those of $F_{0,i,\beta}^{-1}$'s with

$$\begin{aligned} \mathcal{A}_\alpha(x) &= F_{\Lambda,\alpha}^{-T}(\mathcal{A}) \\ &= \bigcap_{t=0}^{T-1} F^{-t}\left(A_{t,\alpha(t,\cdot)}\right) \bigcap F^{-T}(\mathcal{A}). \end{aligned}$$

□

Remark: 1. $\mathcal{A}_\alpha(x)$ can be really complicated sets, due to the perturbation term G and the non compatibility of inverse branches. But we avoid problems using the contraction property as described in Lemma 4.1.
2. In fact, this construction (except the inclusion (29)) requires only the local Markov structure of expanding maps and the bijectivity of the coupling.

Notation: In the following, when $\alpha \in \mathcal{C}[T,\Lambda]$ and $0 < t < T$, the notation $F_{\Lambda,\alpha}^{-t}$ denotes in fact $F_\Lambda^{T-t} \circ F_{\Lambda,\alpha}^{-T}$, so that

$$F_{\Lambda,\alpha}^{-t} = F_{0,T-t,\alpha(T-t,\cdot)}^{-1} \circ G^{-1} \circ F_{\Lambda,\alpha}^{-t+1}. \tag{30}$$

4.3 Expanding property

We can then use the weak coupling assumptions and the inverse branch analysis of F_Λ to get a sharp form of the preservation of the expanding property when we replace F_0 by F_Λ:

Proposition 4.3. *Suppose F satisfies Assumption $(H2)$, $y \in \mathcal{A}$ satisfies $d_i(F^T(x), y) \leq \delta < 1/2$ for any $i \in E \subset \Lambda$, and that $\alpha \in \mathcal{C}[T,\Lambda,E]$. Then*

$$d_i\left(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y)\right) \leq \frac{\delta}{\tilde{\gamma}^t} + \lambda \cdot \theta^{d(i,E^C)} \qquad \forall\, 0 \leq t \leq T,\, i \in E, \tag{31}$$

where $\lambda = \frac{M\mathcal{K}}{2(\gamma - M\mathcal{K} - 1)}$ and θ, M, $\mathcal{K}$ and $\tilde{\gamma} = \gamma - M\mathcal{K}_2$ are defined Section 2.2.

Remark: This Proposition gives a complete decoupling of temporal expanding property and spatial weak coupling, uniformly in time and space.

Proof. We know that G_Λ is invertible, and by the estimate (5) on the coupling and the triangle inequality, we have, for $y, z \in \mathcal{X}_\Lambda$ and $i \in \Lambda$,

$$d_i(y,z) \leq d_i(G_\Lambda(y), G_\Lambda(z)) + \mathcal{E} \sum_{k \in \Lambda} \theta^{2|i-k|} d_k(x,y),$$

then for each $1 \le t \le T$ and $i \in \Lambda$,

$$d_i(G_\Lambda^{-1} \circ F^{T-t+1}(x), G_\Lambda^{-1} \circ F_{\Lambda,\alpha}^{-t+1}(y)) \le d_i(F^{T-t+1}(x), F_{\Lambda,\alpha}^{-t+1}(y)) + \mathcal{E} \sum_{k \in \Lambda} \theta^{2|i-k|} d_k(G_\Lambda^{-1} \circ F^{T-t+1}(x), G_\Lambda^{-1} \circ F_{\Lambda,\alpha}^{-t+1}(y)).$$

For the inverse of F_0, we can use Lemma 4.1, with the permutation $\phi = \mathrm{Id}$ because one of the points is on the orbit of x, and identity (30) to get for all $i \in E$ (because $\alpha \in \mathcal{C}[T, \Lambda, E]$)

$$\frac{1}{M} d_i(G_\Lambda^{-1} \circ F^{T-t+1}(x), G_\Lambda^{-1} \circ F_{\Lambda,\alpha}^{-t+1}(y)) \le d_i(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y)) \le \frac{1}{\gamma} d_i(G_\Lambda^{-1} \circ F^{T-t+1}(x), G_\Lambda^{-1} \circ F_{\Lambda,\alpha}^{-t+1}(y)).$$

Combining these two estimates gives for any $i \in E$ and $1 \le t \le T$

$$d_i(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y)) \le \frac{1}{\gamma} d_i(F^{T-t+1}(x), F_{\Lambda,\alpha}^{-t+1}(y)) + \frac{M\mathcal{E}}{\gamma} \sum_{k \in E} \theta^{2|i-k|} d_k(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y)) + \frac{M\mathcal{E}}{2\gamma} \sum_{k \in \Lambda \setminus E} \theta^{2|i-k|}. \quad (32)$$

We want now to go from this time to time estimate to a global one (in time and space). We will estimate this term from above by a double sequence which can be entirely solved by a generating function method.

For this we analyze the behavior of all points at a given distance of E^C. With $E^{(l)}$ as defined in Appendix A, we denote for $0 \le t \le T$ and $l \ge 0$

$$v(l,t) = \sup_{i \in E^{(-l)}} d_i(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y))$$

(and $v(l,t) = 0$ if $E^{(-l)} = \emptyset$).

If $i \in E^{(-l)}$, for any $0 \le k \le l$, we have the inclusion $i + \Lambda_k \subset E^{(k-l)} \subset E$, then (32) becomes for $t \ge 1$

$$\begin{aligned} d_i(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y)) &\le \frac{1}{\gamma} d_i(F^{T-t+1}(x), F_{\Lambda,\alpha}^{-t+1}(y)) \\ &\quad + \frac{M\mathcal{E}}{\gamma} \sum_{k=0}^{l} \sum_{|h|=k} \theta^{2|h|} d_{i+h}(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y)) + \frac{M\mathcal{E}}{2\gamma} \sum_{k>l} \sum_{|h|=k} \theta^{2|h|} \\ &\le \frac{1}{\gamma} v(l,t-1) + \frac{M\mathcal{E}}{\gamma} \sum_{k=0}^{l} \sum_{|h|=k} \theta^{2|h|} v(l-k,t) + \frac{M\mathcal{E}}{2\gamma} \sum_{k>l} \sum_{|h|=k} \theta^{2|h|}. \end{aligned}$$

Hence for $l \ge 0$ and $1 \le t \le T$

$$v(l,t) \le \frac{1}{\gamma} v(l,t-1) + \frac{1}{\gamma} \sum_{k=0}^{l} \alpha_k v(l-k,t) + \frac{1}{\gamma} \sum_{k>l} \frac{\alpha_k}{2}, \quad (33)$$

with $\alpha_k = M\mathcal{E}c_k\theta^{2k}$ and $c_k = \mathrm{Card}(h \in \mathbb{Z}^d : |h| = k)$. We define then, for $\delta \geq 0$ the double sequence

$$u(l,t) = \begin{cases} \frac{1}{2} & \text{if } l < 0 \\ \delta & \text{if } l \geq 0, t = 0 \\ \frac{1}{\gamma}u(l,t-1) + \frac{1}{\gamma}\sum_{k\geq 0}\alpha_k u(l-k,t) & \text{if } l \geq 0, t > 0 \end{cases}$$

We have the following upper bound for v:

Lemma 4.2. *If $v(l,t)$ satisfies recursive relation (33), $\sup_{l\geq 0} v(l,0) = v(0,0) \leq \delta$, and if $\alpha_0/\gamma < 1$, then*

$$v(l,t) \leq u(l,t) \qquad \forall\, l \geq 0, t \geq 0. \tag{34}$$

Proof. By induction on t, then on l, because $1 - \alpha_0/\gamma > 0$ and

$$\left(1 - \frac{\alpha_0}{\gamma}\right) v(l,t) \leq \frac{1}{\gamma}v(l,t-1) + \frac{1}{\gamma}\sum_{k=1}^{l}\alpha_k v(l-k,t) + \frac{1}{\gamma}\sum_{k>l}\alpha_k u(l-k,t).$$

□

The fact that $\alpha_0/\gamma < 1$ is a direct consequence of the assumption $(H2)$ because $\alpha_0 \leq \sum \alpha_k = M\mathcal{K}_2 \leq M\mathcal{K} < \gamma$. $(H2)$ implies also that assumptions of Proposition C.1 are satisfied with α_k and $\tilde{\alpha}_k = M\mathcal{E}c_k\theta^k$. This Proposition and Lemma 4.2 imply

$$v(l,t) \leq \frac{\delta}{(\gamma - M\mathcal{K}_2)^t} + \lambda \cdot \theta^{l+1}.$$

Optimizing for any $i \in E$, since $i \in E^{(-d(i,E^C)+1)}$, we get the desired estimate (31). □

We can evaluate in the same way the effect of a change of finite box restriction on the inverse iterates of the map:

Proposition 4.4. *If F satisfies Assumption $(H2)$ then for any $y \in \mathcal{A}$, there is a bijection $\phi_y : \mathcal{C}[T,\Lambda,E] \to \mathcal{C}[T,\Lambda\setminus E]$ such that $y \mapsto \phi_y$ is measurable, and for all $\alpha \in \mathcal{C}[T,\Lambda,E]$*

$$d_i\left(F^{-t}_{\Lambda,\alpha}(y), F^{-t}_{\Lambda\setminus E,\phi_y(\alpha)}(y)\right) \leq \lambda \cdot \theta^{d(i,E)} \qquad \forall 0 \leq t \leq T,\, i \in \Lambda\setminus E. \tag{35}$$

Proof. For the coupling, we have exactly the same type of estimate as in the context of Proposition 4.3 for any $i \in \Lambda \setminus E$,

$$d_i(G^{-1}_{\Lambda}(y), G^{-1}_{\Lambda\setminus E}(z)) \leq d_i(y,z) + \mathcal{E}\sum_{k\in\Lambda\setminus E}\theta^{2|i-k|}d_k(y,z) + \frac{\mathcal{E}}{2}\sum_{k\in E}\theta^{2|i-k|}. \tag{36}$$

The inverse branches of F_0 are constructed in Subsection 4.2 independently on each site and around the orbit of x. Since $F^t_\Lambda(x) = F^t_{\Lambda\setminus E}(x) = F^t(x)$, these inverse branches are in fact locally independent of the finite box. We can then use the same method as in the proof of Lemma 4.1 to choose inverse branches such that the contraction property applies well to preimages of y.

At first step, we compare for $i \in \Lambda \setminus E$ the relative positions of the points $\left(G_\Lambda^{-1}(y)\right)_i$, $\left(G_{\Lambda\setminus E}^{-1}(y)\right)_i$ and $\left(F_0 \circ F^{T-1}(x)\right)_i - 1/2$ to define the action of ϕ_y at time $T-1$ (see Figure 1 in the proof of Lemma 4.1) such that

$$\begin{aligned}\frac{1}{M} d_i \left(G_\Lambda^{-1}(y), G_{\Lambda\setminus E}^{-1}(y)\right) \\ &\leq d_i \left(F_{0,T-1,\alpha(T-1,\cdot)}^{-1} \circ G_\Lambda^{-1}(y), F_{0,T-1,\phi_y(\alpha)(T-1,\cdot)}^{-1} \circ G_{\Lambda\setminus E}^{-1}(y)\right) \\ &\leq \frac{1}{\gamma} d_i \left(G_\Lambda^{-1}(y), G_{\Lambda\setminus E}^{-1}(y)\right).\end{aligned}$$

Then, if ϕ_y is well defined for times greater or equal to $T-t+1$, we compare at each $i \in \Lambda\setminus E$ the relative positions of $\left(G_\Lambda^{-1} \circ F_{\Lambda,\alpha}^{-t+1}(y)\right)_i$, $\left(G_{\Lambda\setminus E}^{-1} \circ F_{\Lambda\setminus E,\alpha}^{-t+1}(y)\right)_i$ and $\left(F_0 \circ F^{T-t}(x)\right)_i - 1/2$ to define the action of ϕ_y at time $T-t$ such that for all $\alpha \in \mathcal{C}[T, \Lambda, E]$:

$$\begin{aligned}\frac{1}{M} d_i \left(G_\Lambda^{-1} \circ F_{\Lambda,\alpha}^{-t+1}(y), G_{\Lambda\setminus E}^{-1} \circ F_{\Lambda\setminus E,\phi_y(\alpha)}^{-t+1}(y)\right) \\ &\leq d_i \left(F_{0,T-t,\alpha(T-t,\cdot)}^{-1} \circ G_\Lambda^{-1} \circ F_{\Lambda,\alpha}^{-t+1}(y), F_{0,T-t,\phi_y(\alpha)(T-t,\cdot)}^{-1} \circ G_{\Lambda\setminus E}^{-1} \circ F_{\Lambda\setminus E,\alpha}^{-t+1}(y)\right) \\ &\leq \frac{1}{\gamma} d_i \left(G_\Lambda^{-1} \circ F_{\Lambda,\alpha}^{-t+1}(y), G_{\Lambda\setminus E}^{-1} \circ F_{\Lambda\setminus E,\phi_y(\alpha)}^{-t+1}(y)\right).\end{aligned}$$

We get in the same way as for Lemma 4.1 that ϕ_y is a measurable function of y. This gives then, combined with (36), for any $i \in \Lambda \setminus E$

$$\begin{aligned}d_i(F_{\Lambda,\alpha}^{-t}(y), F_{\Lambda\setminus E,\phi_y(\alpha)}^{-t}(y)) &\leq \frac{1}{\gamma} d_i(F_{\Lambda,\alpha}^{-t+1}(y), F_{\Lambda\setminus E,\phi_y(\alpha)}^{-t+1}(y)) \\ &+ \frac{M\mathcal{E}}{\gamma} \sum_{k\in\Lambda\setminus E} \theta^{2|i-k|} d_k(F_{\Lambda,\alpha}^{-t}(x), F_{\Lambda\setminus E,\phi_y(\alpha)}^{-t}(y)) + \frac{M\mathcal{E}}{2\gamma} \sum_{k\in E} \theta^{2|i-k|}. \quad (37)\end{aligned}$$

We can hence proceed as in the proof of Proposition 4.3, with

$$v(l,t) = \sup_{i\in\Lambda\setminus\left(E^{(l)}\right)} d_i \left(F_{\Lambda,\alpha}^{-t}(y), F_{\Lambda\setminus E,\phi_y(\alpha)}^{-t}(y)\right)$$

and $\delta = 0$. □

4.4 Expansiveness

A first straightforward consequence of the expanding property stated as Proposition 4.3 is the expansiveness of the dynamical system (F, S):

Proposition 4.5. *If $d_\rho(S^i \circ F^t(x), S^i \circ F^t(y)) < \delta_0 = \frac{1}{2M}$ for all $i \in \mathbb{Z}^d$ and $t \in \mathbb{N}$, then*

$$x = y$$

Proof. The inclusion (29) and the Proposition 4.3 can be combined to get that under assumption $(H2)$, if $d_i(F_\Lambda^t(x), F_\Lambda^t(y)) < \delta_0$ for all $0 \leq t \leq T$ and $i \in E$, then we have in fact the better estimate

$$d_i \left(F_\Lambda^t(x), F_\Lambda^t(y)\right) \leq \frac{\delta_0}{\tilde{\gamma}^{T-t}} + \lambda \cdot \theta^{d(i,E^C)}.$$

We can then take $\Lambda = \Lambda_N$ and N tends to infinity which gives the same property for the global map F. But the assumption done for this Proposition clearly implies that $d_i(F^t(x), F^t(y)) < \delta_0$ for all $i \in \mathbb{Z}^d$ and $t \in \mathbb{N}$, hence

$$d_i(x, y) \leq \frac{\delta_0}{\tilde{\gamma}^T} + \lambda \cdot \theta^{d(i, E^C)}$$

for all $E \subset \mathbb{Z}^d$ and $T \in \mathbb{N}$. taking $E = \Lambda_n$ then T and n going to infinity, we can conclude that $x = y$. □

A classical and essential consequence of this property is that the metric entropy $h_{(F,S)}$ associated to the system is an upper semi-continuous function of the probability measures (see Proposition B.1). This (and the continuity of the potential function φ) proves that the rate function I of the Large Deviations Principle defined in (13) is lower semi-continuous and allows to use the Gibbs variational principle for the proof of the Upper Bound.

5 Proof of the Volume Lemma

We begin by proving an intermediate Volume Lemma for the finite box map F_Λ with constraints on the orbit on the smaller box E, then use it to prove Theorem 2.1 for the global system (F, S).

Proposition 5.1. *Under assumption $(\mathcal{H})$, for x, $E \subset \Lambda$, T and $0 < \delta < \frac{1}{2M}$ as in Section 4 with Λ large enough, we have*

$$\begin{aligned} \exp\Big(\sum_{\substack{0 \leq t < T \\ i \in E}} \varphi \circ S^i \circ F^t(x) - T|E|\tilde{C}_2(T, E, \delta) - C_4(\Lambda, T, E)\Big) \\ \leq m^\Lambda \left\{ y \,:\, d_i\left(F^t(x), F^t_\Lambda(y)\right) < \delta \quad \forall 0 \leq t \leq T,\, i \in E \right\} \\ \leq \exp\Big(\sum_{\substack{0 \leq t < T \\ i \in E}} \varphi \circ S^i \circ F^t(x) + T|E|\tilde{C}_3(T, E, \delta) + C_5(\Lambda, T, E))\Big), \end{aligned} \tag{38}$$

with:

$$\lim_{N \to \infty} C_4(\Lambda_N, T, E) = \lim_{N \to \infty} C_5(\Lambda_N, T, E) = 0 \qquad \forall T \geq 1,\, E \subset \mathbb{Z}^d, \tag{39}$$

$$\lim_{\substack{T \to \infty \\ n \to \infty}} \tilde{C}_2(T, E_n, \delta) = \lim_{\substack{T \to \infty \\ n \to \infty}} \tilde{C}_3(T, E_n, \delta) = 0 \qquad \forall \delta < \frac{1}{2M}. \tag{40}$$

for any sequence E_n tending to $\mathbb{Z}^d$ in the sense of Van Hove. Moreover $\tilde{C}_2$ and $\tilde{C}_3$ are continuous in δ.

The essential idea to prove this result is to do a change of variable by F^T_Λ. This must be done with some precautions to ensure we are on domains where this map is injective and to analyze all the terms.

5.1 Proof of the Upper Bound of Proposition 5.1

We decompose $\mathcal{X}_\Lambda$ in the subsets $(\mathcal{A}_\alpha(x))_{\alpha\in\mathcal{C}[T,\Lambda]}$, on each of which F_Λ^T is one-to-one. It has to be noticed that we do not lose anything because $m^\Lambda(\cup\mathcal{A}_\alpha(x)) = 1$ and that since $\delta < \frac{1}{2M}$ the intervals which appear are those corresponding to $\mathcal{C}[T,\Lambda,E]$ (see Proposition 4.2 for these properties)

$$\begin{aligned} & m^\Lambda \left\{y \in \mathcal{X}_\Lambda \,:\, d_i\left(F^t(x), F_\Lambda^t(y)\right) < \delta \quad \forall 0 \le t \le T,\, i \in E\right\} \\ &= \sum_{\alpha\in\mathcal{C}[T,\Lambda,E]} m^\Lambda \left\{y \in \mathcal{A}_\alpha(x) \,:\, d_i\left(F^t(x), F_\Lambda^t(y)\right) < \delta \quad \forall 0 \le t \le T,\, i \in E\right\} \\ &= \sum_{\alpha\in\mathcal{C}[T,\Lambda,E]} \int_{\mathcal{X}_\Lambda} \prod_{\substack{0\le t\le T \\ i\in E}} \mathbb{1}_{\left\{d_i\left(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y)\right)<\delta\right\}} \frac{1}{\left|DF_\Lambda^T(F_{\Lambda,\alpha}^{-T}(y))\right|}\, m^\Lambda(dy), \end{aligned} \tag{41}$$

by a change of variables with F_Λ^T, bijection from $\mathcal{A}_\alpha(x)$ onto $\mathcal{A}$.
We apply then the results of Section 3.2 to get

$$\begin{aligned} \frac{1}{\left|DF_\Lambda^T(F_{\Lambda,\alpha}^{-T}(y))\right|} &= \exp\Big(- \sum_{0\le t<T} \log|DF_{\Lambda,F^t(x)} \circ F_{\Lambda,\alpha}^{t-T}(y)|\Big) \\ &= \exp\Big(\sum_{\substack{0\le t<T \\ i\in\Lambda}} (-\log|f_i'| + w_{\Lambda,i}) \circ F_{\Lambda,\alpha}^{t-T}(y)\Big), \end{aligned}$$

where we denote $w_{\Lambda,i} = w_{\Lambda,F^t(x),i}$ for any t: we do not mention the boundary conditions since all our estimates are uniform in them.
We treat differently the terms corresponding to $i \in E$ and to $i \in \Lambda \setminus E$. In the first case, we want to replace them by $\varphi \circ S^i \circ F^t(x)$ while in the second we want to reconstitute $D(F_{\Lambda\setminus E,\phi_y(\alpha)}^{-T}(y))$ and integrate it to 1 by another change of variables on $\mathcal{X}_{\Lambda\setminus E}$.

Hence, if $i \in E$,

$$\begin{aligned} |(-\log|f_i'| + w_{\Lambda,i}) \circ F_{\Lambda,\alpha}^{t-T}(y) - \varphi \circ S^i \circ F^t(x)| \le{}& |\log|f_i'| \circ F_{\Lambda,\alpha}^{t-T}(y) - \log|f_i'| \circ F^t(x)| \\ &+ |w_{\Lambda,i} \circ F_{\Lambda,\alpha}^{t-T}(y) - w_{\Lambda,i} \circ F^t(x)| \\ &+ |w_{\Lambda,i} \circ F^t(x) - \psi_i \circ F^t(x)|. \end{aligned}$$

The third term is easily estimated by the speed of convergence of $w_{\Lambda,i}$ to ψ_i given in (24). Summing over all times and sites gives

$$\sum_{\substack{0\le t<T \\ i\in E}} |w_{\Lambda,i} \circ F^t(x) - \psi_i \circ F^t(x)| \le \frac{T}{2(1-\mathcal{K})} \sum_{i\in E} \theta^{d(i,\Lambda^C)} = C_5(\Lambda, T, E), \tag{42}$$

then we get $C_5(\Lambda_N, T, E) \to 0$ when N goes to infinity.
For the two other terms, we use the fact that $d_i\left(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y)\right) < \delta$ for all $0 \le t \le T$ and $i \in E$ which implies with Proposition 4.3 that

$$d_i\left(F^{T-t}(x), F_{\Lambda,\alpha}^{-t}(y)\right) \le \frac{\delta}{\tilde{\gamma}^t} + \lambda \cdot \theta^{d(i,E^C)} \qquad \forall 0 \le t \le T,\, i \in E.$$

This combined with the α-Hőlder property of $\log|f'|$ (see (2)) and the concavity of $x \to x^\alpha$ gives

$$\frac{1}{T|E|}\sum_{\substack{0\le t<T\\ i\in E}} |\log|f_i'|\circ F_{\Lambda,\alpha}^{t-T}(y) - \log|f_i'|\circ F^t(x)| \le C_1\Big(\frac{\delta}{T}\sum_{0\le t<T}\frac{1}{\tilde\gamma^{t-T}} + \frac{\lambda}{|E|}\sum_{i\in E}\theta^{d(i,E^C)}\Big)^\alpha, \tag{43}$$

which goes to 0 as T tends to infinity and E tends to $\mathbb{Z}^d$ in the sense of Van Hove, because $\tilde\gamma > 1$ and $1/|E|\sum_{i\in E}\theta^{d(i,E^C)}$ goes to 0 by Proposition A.1.
For $w_{\Lambda,i}$, we use estimate (19) and get, with $\mathcal{K}_{1/2} = \sum_{i\in\mathbb{Z}^d}\theta^{\frac12|k|}$,

$$\begin{aligned}|w_{\Lambda,i}\circ F_{\Lambda,\alpha}^{t-T}(y) - w_{\Lambda,i}\circ F^t(x)| &\le \frac{M\mathcal{E}}{1-\mathcal{K}}\sum_{k\in\Lambda}\theta^{|i-k|}d_k(F_{\Lambda,\alpha}^{t-T}(y),F^t(x))\\ &\le \frac{M\mathcal{K}}{1-\mathcal{K}}\frac{\delta}{\tilde\gamma^{t-T}} + \frac{\lambda M\mathcal{K}_{1/2}}{1-\mathcal{K}}\theta^{\frac12 d(i,E^C)} + \frac{M\mathcal{E}}{2(1-\mathcal{K})}\sum_{k\in E^C}\theta^{|i-k|}.\end{aligned}$$

Then
$$\begin{aligned}&\frac{1}{T|E|}\sum_{\substack{0\le t<T\\ i\in E}}|w_{\Lambda,i}\circ F_{\Lambda,\alpha}^{t-T}(y) - w_{\Lambda,i}\circ F^t(x)|\\ &\le \frac{M\mathcal{K}}{1-\mathcal{K}}\frac{\delta}{T}\sum_{0\le t<T}\frac{1}{\tilde\gamma^{t-T}} + \frac{M\mathcal{K}_{1/2}}{1-\mathcal{K}}\left(\frac12+\lambda\right)\frac{1}{|E|}\sum_{i\in E}\theta^{\frac12 d(i,E^C)},\end{aligned} \tag{44}$$

which goes also to 0 as $T\to\infty$ and $E\to\mathbb{Z}^d$.

In the same way, for $i\in\Lambda\setminus E$, we use the link between behaviors of $F_{\Lambda,\alpha}^{t-T}(y)$ and $F_{\Lambda\setminus E,\phi_y(\alpha)}^{t-T}(y)$ given in Proposition 4.4, writing

$$\begin{aligned}&|(-\log|f_i'| + w_{\Lambda,i})\circ F_{\Lambda,\alpha}^{t-T}(y) - (-\log|f_i'| + w_{\Lambda\setminus E,i})\circ F_{\Lambda\setminus E,\phi_y(\alpha)}^{t-T}(y)|\\ &\le |\log|f_i'|\circ F_{\Lambda,\alpha}^{t-T}(y) - \log|f_i'|\circ F_{\Lambda\setminus E,\phi_y(\alpha)}^{t-T}(y)| + |w_{\Lambda,i}\circ F_{\Lambda,\alpha}^{t-T}(y) - w_{\Lambda\setminus E,i}\circ F_{\Lambda,\alpha}^{t-T}(y)|\\ &\quad + |w_{\Lambda\setminus E,i}\circ F_{\Lambda,\alpha}^{t-T}(y) - w_{\Lambda\setminus E,i}\circ F_{\Lambda\setminus E,\phi_y(\alpha)}^{t-T}(y)|,\end{aligned}$$

and, using Proposition 4.4 instead of Proposition 4.3 and estimate (21) instead of (24)

$$\frac{1}{T|E|}\sum_{\substack{0\le t<T\\ i\in\Lambda\setminus E}}|\log|f_i'|\circ F_{\Lambda,\alpha}^{t-T}(y) - \log|f_i'|\circ F_{\Lambda\setminus E,\phi_y(\alpha)}^{t-T}(y)| \le C_1\frac{\lambda^\alpha}{|E|}\sum_{i\in E^C}\theta^{\alpha d(i,E)}, \tag{45}$$

$$\frac{1}{T|E|}\sum_{\substack{0\le t<T\\ i\in\Lambda\setminus E}}|w_{\Lambda,i}\circ F_{\Lambda,\alpha}^{t-T}(y) - w_{\Lambda\setminus E,i}\circ F_{\Lambda,\alpha}^{t-T}(y)| \le \frac{1}{2(1+\mathcal{K})}\frac{1}{|E|}\sum_{i\in E^C}\theta^{d(i,E)}, \tag{46}$$

$$\frac{1}{T|E|}\sum_{\substack{0\le t<T\\ i\in\Lambda\setminus E}}|w_{\Lambda\setminus E,i}\circ F_{\Lambda,\alpha}^{t-T}(y) - w_{\Lambda\setminus E,i}\circ F_{\Lambda\setminus E,\phi_y(\alpha)}^{t-T}(y)| \le \frac{\lambda M\mathcal{K}_{1/2}}{1-\mathcal{K}}\frac{1}{|E|}\sum_{i\in E^C}\theta^{\frac12 d(i,E)}, \tag{47}$$

all these terms tending to 0 when E tends to $\mathbb{Z}^d$ in the sense of Van Hove by estimate (57).

We take finally for $\bar{C}_3$ the sum of RHS in formulas (43), (44), (45), (46) and (47) and get the global estimate

$$\frac{1}{|DF_\Lambda^T(F_{\Lambda,\alpha}^{-T}(y))|} \leq \frac{1}{|DF_{\Lambda\setminus E}^T(F_{\Lambda\setminus E,\phi_y(\alpha)}^{-T}(y))|} \times \exp\Big(\sum_{\substack{0\leq t<T\\ i\in E}} \varphi\circ S^i\circ F^t(x) + T|E|\bar{C}_3(T,E,\delta) + C_5(\Lambda,T,E)\Big).$$

On the other hand, we get an upper bound for the product of indicator functions in (41) by the terms corresponding to $t=0$, and use the identity

$$\sum_{\alpha\in\mathcal{C}[T,\Lambda,E]} \frac{1}{DF^T\circ F_{\Lambda,\phi_y(\alpha)}^{-T}} = \sum_{\alpha\in\mathcal{C}[T,\Lambda\setminus E]} \frac{1}{DF^T\circ F_{\Lambda,\alpha}^{-T}}$$

due to the bijectivity of ϕ_y from $\mathcal{C}[T,\Lambda,E]$ onto $\mathcal{C}[T,\Lambda\setminus E]$. We can then separate the terms in E and those in $\Lambda\setminus E$ and integrate the last ones by a change of variable,

$$\begin{aligned}
&m^\Lambda\left\{y\in\mathcal{X}_\Lambda \,:\, d_i\left(F_\Lambda^t(x),F_\Lambda^t(y)\right)<\delta \quad \forall\, 0\leq t\leq T,\, i\in E\right\}\\
&\leq \int_{\mathcal{X}_{\Lambda\setminus E}} \sum_{\alpha\in\mathcal{C}[T,\Lambda\setminus E]} \frac{1}{|DF_{\Lambda\setminus E}^T(F_{\Lambda\setminus E,\alpha}^{-T}(y))|} m^{\Lambda\setminus E}(dy)m^E\{y\,:\, d_i(F^T(x),y)<\delta \quad \forall i\in E\}\\
&\qquad \times \exp\Big(\sum_{\substack{0\leq t<T\\ i\in E}} \varphi\circ S^i\circ F^t(x) + T|E|\bar{C}_3(T,E,\delta) + C_5(\Lambda,T,E)\Big)\\
&= m^{\Lambda\setminus E}\Big(\bigcup_{\alpha\in\mathcal{C}[T,\Lambda\setminus E]} \mathcal{A}_\alpha(x)\Big)(2\delta)^{|E|}\\
&\qquad \times \exp\Big(\sum_{\substack{0\leq t<T\\ i\in E}} \varphi\circ S^i\circ F^t(x) + T|E|\bar{C}_3(T,E,\delta) + C_5(\Lambda,T,E)\Big)\\
&= \exp\Big(\sum_{\substack{0\leq t<T\\ i\in E}} \varphi\circ S^i\circ F^t(x) + T|E|\tilde{C}_3(T,E,\delta) + C_5(\Lambda,T,E)\Big),
\end{aligned}$$

where $\tilde{C}_3 = \bar{C}_3 + \frac{1}{T}\log(2\delta)$ satisfies the announced limit.

5.2 Proof of the Lower Bound of Proposition 5.1

For the lower bound, we use the same kind of estimates that for the upper bound, except for the term

$$\prod_{\substack{0\leq t\leq T\\ i\in E}} \mathbb{1}_{\left\{d_i\left(F_{\Lambda,0}^{-t}(F^T(x)),F_{\Lambda,\alpha}^{-t}(y)\right)<\delta\right\}}.$$

Indeed, to insure this, we have to assume that $d_i(F^T(x), y) < \delta$ for i in a set larger than E: we choose L such that

$$\frac{\delta}{\tilde{\gamma}} + \lambda \cdot \theta^L \leq \delta$$

and assume that $E^{(L)} \subset \Lambda$ (this is the sense of Λ large enough in Proposition 5.1). Then, if $d_i(F^T(x), y) < \delta$ for all $i \in E^{(L)}$, Proposition 4.3 implies that when $\alpha \in \mathcal{C}[T, \Lambda, E^{(L)}]$

$$d_i\left(F^{T-t}(x), F^{-t}_{\Lambda,\alpha}(y)\right) \leq \frac{\delta}{\tilde{\gamma}^t} + \lambda \cdot \theta^{d(i,(E^{(L)})^C)} \qquad \forall\, 0 \leq t \leq T,\, i \in E^{(L)},$$

and in particular

$$d_i\left(F^{T-t}(x), F^{-t}_{\Lambda,\alpha}(y)\right) \leq \delta \qquad \forall\, 0 \leq t \leq T,\, i \in E.$$

The assumption $\alpha \in \mathcal{C}[T, \Lambda, E^{(L)}]$ imposes then to restrict the sum in the decomposition of $\mathcal{X}_\Lambda$. This does not perturb the asymptotic estimates since $\frac{|E^{(L)} \setminus E|}{|E|} \to 0$ when E tends to $\mathbb{Z}^d$ in the sense of Van Hove. Then

$$\begin{aligned}
&m^\Lambda \left\{y \in \mathcal{X}_\Lambda \,:\, d_i\left(F^t_\Lambda(x), F^t_\Lambda(y)\right) < \delta \quad \forall\, 0 \leq t \leq T,\, i \in E\right\} \\
&\geq \sum_{\alpha \in \mathcal{C}[T,\Lambda,E^{(L)}]} \int_{\mathcal{X}_\Lambda} \prod_{i \in E^{(L)}} \mathbb{1}_{\{d_i(F^T(x),y)<\delta\}} \exp\Big(\sum_{\substack{0 \leq t < T \\ i \in \Lambda}} (-\log f_i' + w_{\Lambda,i}) \circ f^{t-T}_{\Lambda,\alpha}(y)\Big)\, m^\Lambda(dy) \\
&\geq m^{\Lambda \setminus E^{(L)}}\Big(\bigcup_{\alpha \in \mathcal{C}[T,\Lambda \setminus E^{(L)}]} \mathcal{A}_\alpha(x)\Big) m^{E^{(L)}}\{y \,:\, d_i(F^T(x), y) < \delta \quad \forall\, i \in E^{(L)}\} \\
&\qquad\qquad \times \exp\Big(\sum_{\substack{0 \leq t < T \\ i \in E^{(L)}}} \varphi \circ S^i \circ F^t(x) - T|E^{(L)}|\bar{C}_3(T, E^{(L)}, \delta) - C_5(\Lambda, T, E^{(L)})\Big) \\
&\geq \exp\Big(\sum_{\substack{0 \leq t < T \\ i \in E}} \varphi \circ S^i \circ F^t(x) - T|E|\tilde{C}_2(T, E, \delta) - C_4(\Lambda, T, E)\Big),
\end{aligned}$$

where

$$\tilde{C}_2(T, E, \delta) = \frac{|E^{(L)}|}{|E|} \tilde{C}_3(T, E^{(L)}, \delta) + \frac{|E^{(L)} \setminus E|}{|E|} |\varphi|_\infty$$

tends to 0 as T goes to infinity and E tends to $\mathbb{Z}^d$ in the sense of Van Hove, and $C_4(\Lambda, T, E) = C_5(\Lambda, T, E^{(L)})$.

5.3 Proof of Theorem 2.1

We approximate F by F_{Λ_N} using convergence on a finite box for finite time: for any $0 < \varepsilon < \frac{1}{2M} - \delta$, there exists N_0 such that for all $N \geq N_0$

$$\begin{cases} d_i(F^t_{\Lambda_N}(y), F^t(y)) \leq \varepsilon & \forall\, 0 \leq t \leq T,\, i \in E \text{ and } y \in \mathcal{X}, \\ C_5(\Lambda_N, T, E) \leq \varepsilon. \end{cases}$$

We deduce then from the upper bound of Proposition 5.1 applied to F_{Λ_N}

$$\begin{aligned}\overline{m}(B_x(T,E;\delta)) &\le \overline{m}\left\{y\in\mathcal{X}\,:\,d_i\left(F^t(x),F^t(y)\right)<\delta\quad\forall\,0\le t\le T,\,i\in E\right\}\\ &\le m^{\Lambda_N}\left\{y\in\mathcal{X}_{\Lambda_N}\,:\,d_i\left(F^t_{\Lambda_N}(x),F^t_{\Lambda_N}(y)\right)<\delta+\varepsilon\quad\forall\,0\le t\le T,\,i\in E\right\}\\ &\le\exp\Big(\sum_{\substack{0\le t<T\\ i\in E}}\varphi\circ S^i\circ F^t(x)+T|E|\tilde{C}_3(T,E,\delta+\varepsilon)+C_5(\Lambda_N,T,E)\Big).\end{aligned}$$

We take then $N\to\infty$, $\varepsilon\to 0$ and use continuity of $\tilde{C}_3$ in δ to get the desired upper bound with $C_3=\exp(T|E|\tilde{C}_3)$.

In the same way, for the lower bound, let $\tilde{L}$ be such that $\frac{1}{2}\rho^{\tilde{L}+1}<\delta\le\frac{1}{2}\rho^{\tilde{L}}$, and for any $0<\varepsilon<\delta$ let N_1 such that for all $N\ge N_1$

$$\begin{cases}d_i(F^t_{\Lambda_N}(y),F^t(y))\le\varepsilon\quad\forall\,0\le t\le T,\,i\in E^{(\tilde{L})}\text{ and }y\in\mathcal{X},\\ C_4(\Lambda_N,T,E^{(\tilde{L})})\le\varepsilon.\end{cases}$$

Then

$$\begin{aligned}\overline{m}(B_x(T,E;\delta) &= \overline{m}\left\{y\,:\,\begin{array}{ll}d_i(F^t(x),F^t(y))<\delta & \forall i\in E,\\ d_i(F^t(x),F^t(y))<\delta\rho^{-1} & \forall i\in E^{(1)}\setminus E,\\ \vdots & \vdots\\ d_i(F^t(x),F^t(y))<\delta\rho^{-\tilde{L}} & \forall i\in E^{(\tilde{L})}\setminus E^{(\tilde{L}-1)},\end{array}\quad\forall\,0\le t\le T\right\}\\ &\ge\overline{m}\left\{y\in\mathcal{X}\,:\,d_i\left(F^t(x),F^t(y)\right)<\delta\quad\forall\,0\le t\le T,\,i\in E^{(\tilde{L})}\right\}\\ &\ge m^{\Lambda_N}\left\{y\in\mathcal{X}_{\Lambda_N}\,:\,d_i\left(F^t_{\Lambda_N}(x),F^t_{\Lambda_N}(y)\right)<\delta-\varepsilon\quad\forall\,0\le t\le T,\,i\in E^{(\tilde{L})}\right\}\\ &\ge\exp\Big(\sum_{\substack{0\le t<T\\ i\in E}}\varphi\circ S^i\circ F^t(x)-T|E^{(\tilde{L})}|\tilde{C}_2(T,E^{(\tilde{L})},\delta-\varepsilon)-C_4(\Lambda_N,T,E^{(\tilde{L})})\Big).\end{aligned}$$

Taking $\varepsilon\to 0$, we get the desired lower bound with $C_2=\exp\Big(-T|E^{(\tilde{L})}|\tilde{C}_2(T,E^{(\tilde{L})},\delta)\Big)$. The only dependence of C_2 on the constant ρ defining the distance comes from the choice of $\tilde{L}$.

6 Large deviations upper bound

In these two last Sections, we will use many results from Thermodynamic Formalism. We refer the reader to the Appendix B for all standard definitions and results.

Our proof of the upper bound of the Large Deviations Principle follows, at least for the main steps, the method of Kifer in [20]. It presents no particular difficulty since the space $\mathcal{M}^1(\mathcal{X})$ is compact for the weak-star topology and the Volume Lemma gives the identification of the log Laplace transforms.
For E_T a given sequence of subsets of $\mathbb{Z}^d$, we denote

$$R_T(x)=R_{T,E_T}(x)=\frac{1}{T|E_T|}\sum_{\substack{0\le t<T\\ i\in E_T}}\delta_{S^i\circ F(x)}\in\mathcal{M}^1(\mathcal{X})$$

the associated empirical process.

6.1 Identification of the pressure

The first step in this proof is the identification of the limit of the log-Laplace transforms of the empirical process R_T integrated against any continuous potential V with the topological pressure of $V + \varphi$:

Proposition 6.1. *Under assumption $(\mathcal{H})$, for any sequence $(E_T)_{T\geq 0}$ tending to $\mathbb{Z}^d$ in the sense of Van Hove and $V \in \mathcal{C}(\mathcal{X})$, we have*

$$\limsup_{T\to\infty} \frac{1}{T|E_T|} \log \int_{\mathcal{X}} \exp\left(T|E_T| \int_{\mathcal{X}} V \, dR_T(x) \right) \overline{m}(dx) = P_{(F,S)}(V + \varphi). \tag{48}$$

Corollary 2.1 is immediately deduced from this Proposition, taking $V = 0$.

Proof. For $\delta > 0$ and $T \geq 0$, we take Y a maximal (T, δ)-separated set in $\mathcal{X}$, which means that

$$x, x' \in Y \text{ and } x \neq x' \Longrightarrow x' \notin B_x(T, E_T; \delta)$$

and Y is maximal for this property.
Then $\cup_{x\in Y} B_x(T, E_T; \delta) = \mathcal{X}$ by maximality and, if $x, x' \in Y$ are distinct, then

$$B_x(T, E_T; \delta/2) \cap B_{x'}(T, E_T; \delta/2) = \emptyset.$$

Hence, denoting $\gamma_V(\delta) = \sup\{|V(x) - V(y)| \, : \, d_\rho(x, y) < \delta\}$, a quantity which goes to 0 with δ by continuity, we decompose the integral in small balls and get

$$\begin{aligned}
&\sum_{x\in Y} \exp\Big(\sum_{\substack{0\leq t<T\\ i\in E_T}} (V \circ S^i \circ F^t(x) - \gamma_V(\delta/2)) \Big) \overline{m}(B_x(T, E_T; \delta/2)) \\
&\qquad \leq \int_{\mathcal{X}} \exp\Big(\sum_{\substack{0\leq t<T\\ i\in E_T}} V \circ S^i \circ F^t(x) \Big) \overline{m}(dx) \\
&\qquad \leq \sum_{x\in Y} \exp\Big(\sum_{\substack{0\leq t<T\\ i\in E_T}} (V \circ S^i \circ F^t(x) + \gamma_V(\delta)) \Big) \overline{m}(B_x(T, E_T; \delta)).
\end{aligned}$$

We use then the Volume Lemma, take logarithm and divide by $T|E_T|$ to get

$$\begin{aligned}
&\frac{1}{T|E_T|} \log \Big[\sum_{x\in Y} \exp\Big(\sum_{\substack{0\leq t<T\\ i\in E_T}} (V+\varphi)\circ S^i \circ F^t(x) \Big) \Big] - \gamma_V(\delta/2) - \frac{1}{T|E_T|} \log C_2(T, E_T, \delta/2, \rho) \\
&\qquad \leq \frac{1}{T|E_T|} \log \int_{\mathcal{X}} \exp\left(T|E_T| \int_{\mathcal{X}} V \, dR_T(x) \right) \overline{m}(dx) \\
&\leq \frac{1}{T|E_T|} \log \Big[\sum_{x\in Y} \exp\Big(\sum_{\substack{0\leq t<T\\ i\in E_T}} (V + \varphi) \circ S^i \circ F^t(x) \Big) \Big] + \gamma_V(\delta) + \frac{1}{T|E_T|} \log C_3(T, E_T, \delta).
\end{aligned}$$

We take now successively the supremum on maximal (T, δ)-separated sets, the limsup when T goes to infinity (makes the terms C_2 and C_3 disappear) and the limit $\delta \to 0$. We get hence the desired result directly from the definition of topological pressure. $\square$

6.2 Proof of the upper bound

For $\delta > 0$ and $V \in \mathcal{C}(\mathcal{X})$ fixed, $\mathcal{M}^1(\mathcal{X})$ is compact and any closed subset F can be included in a finite union of balls of the type $\beta_\nu(V;\delta) = \{\mu \,:\, |\int V\,d\mu - \int V\,d\nu| < \delta\}$,

$$F \subset \bigcup_{l=1}^{d} \beta_{\nu_l}(V;\delta) \quad \text{with } \nu_l \in F. \tag{49}$$

By the Chebychev inequality

$$\overline{m}\{x \,:\, R_T(x) \in \beta_\nu(V;\delta)\} \leq e^{T|E_T|(\delta - \int_{\mathcal{X}} V\,d\nu)} \int_{\mathcal{X}} e^{T|E_T|R_T(x)}\,\overline{m}(dx),$$

then using Proposition 6.1 we have for such an open ball

$$\limsup_{T\to\infty} \frac{1}{T|E_T|} \log \overline{m}(R_T \in \beta_\nu(V;\delta)) \leq \delta - \int_{\mathcal{X}} V\,d\nu + P_{(F,S)}(V+\varphi).$$

The inclusion (49) implies now for F closed

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T|E_T|} \log \overline{m}(R_T \in F) &\leq \max_{1\leq l\leq d} \left(\limsup_{T\to\infty} \frac{1}{T|E_T|} \log \overline{m}(R_T \in \beta_{\nu_l}(V;\delta)) \right) \\ &\leq \max_{\nu\in F} \left(\delta - \int_{\mathcal{X}} V\,d\nu + P_{(F,S)}(V+\varphi) \right).\end{aligned}$$

We can then make δ tend to 0, optimize on V continuous and use a minimax type result (available because F is compact) to get

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T|E_T|} \log \overline{m}(R_T \in F) &\leq \max_{\nu\in F} \left(\inf_{V\in\mathcal{C}(\mathcal{X})} \left(P_{(F,S)}(V+\varphi) - \int_{\mathcal{X}} V\,d\nu \right) \right) \\ &= \sup_{\nu\in F} \left(h_{(F,S)} - \int_{\mathcal{X}} \varphi\,d\nu \right) = -\inf_{\nu\in F} I(\nu),\end{aligned}$$

where we used the dual Gibbs variational principle (because h is upper semi-continuous).

7 Large deviations lower bound

The large deviations lower bound is a local property in the sense that it is equivalent to prove

$$\begin{aligned}&\liminf_{T\to\infty} \frac{1}{T|E_T|} \log \overline{m}\{x \,:\, R_T(x) \in O\} \geq -I(\nu) = h_{(F,S)}(\nu) + \int_{\mathcal{X}} \varphi\,d\nu \quad \forall\,\nu \in O \text{ open} \\ \Longleftrightarrow\ &\liminf_{T\to\infty} \frac{1}{T|E_T|} \log \overline{m}\{x \,:\, R_T(x) \in \beta_\nu(V_1,\ldots,V_K;\delta)\} \geq h_{(F,S)}(\nu) + \int_{\mathcal{X}} \varphi\,d\nu\end{aligned}$$

for all $\nu \in \mathcal{M}^1(\mathcal{X})$, $V_1,\ldots,V_K \in \mathcal{C}(\mathcal{X})$ and $\delta > 0$, denoting $\beta_\nu(V_1,\ldots,V_K;\delta) = \{\mu \,:\, |\int_{\mathcal{X}} V_k\,d\mu - \int_{\mathcal{X}} V_k\,d\nu| < \delta \quad \forall\, 1 \leq k \leq K\}$, because this gives a basis of the weak-star topology on $\mathcal{M}^1(\mathcal{X})$.

The idea for the lower bound is a geometric estimate, which comes from [33] and is better expressed for an ergodic probability ν: we decompose the set $\{x : R_T(x) \in \beta_\nu(V_1,\ldots,V_K;\delta)\}$ in small balls $B_x(T,E_T;\delta)$. We need approximately $e^{(T|E_T|h_{(F,S)}(\nu))}$ of them (by a metric version of Shannon-McMillan-Breiman Theorem, stated as Theorem B.2) and each is approximately of size $e^{(T|E_T|\int_{\mathcal{X}}\varphi\,d\nu)}$ under $\overline{m}$ (by the Volume Lemma and the Ergodic Theorem).

We will write it directly for convex combinations of ergodic measures. We need for this a strong mixing result, the Specification Property. We obtain the general case by an approximation argument.

7.1 Specification property

This strong quantitative mixing property is again a consequence of the preservation of expanding property.

Proposition 7.1. *If F satisfies (H2), then for all $\delta > 0$, there exists $p(\delta) \in \mathbb{N}$ such that for any $T_1,\ldots,T_L \in \mathbb{N}$, $x^1,\ldots,x^L \in \mathcal{X}$ and $p_1,\ldots,p_{L-1} \geq p(\delta)$, there exists $x \in \mathcal{X}$ such that*

$$\begin{aligned} d\left(F^t(x), F^t\left(x^1\right)\right) &< \delta && \forall\, 0 \leq t \leq T_1,\\ d\left(F^{t+T_1+p_1}(x), F^t\left(x^2\right)\right) &< \delta && \forall\, 0 \leq t \leq T_2,\\ &\;\vdots && \quad\vdots\\ d\left(F^{t+\sum_{l=1}^{L-1}(T_l+p_l)}(x), F^t\left(x^L\right)\right) &< \delta && \forall\, 0 \leq t \leq T_L. \end{aligned}$$

Proof. We work in this proof with the global map F and the topology associated to the distance $d(x,y) = \sup_{i\in\mathbb{Z}^d} d_i(x,y)$. Let

$$V_x(T;\delta) = \{y : d(F^t(x),F^t(y)) < \delta \quad \forall\, 0 \leq t \leq T\}$$

be the dynamic neighborhood around the orbit of x. We want to show that

$$V_{x^1}(T_1;\delta) \cap F^{-T_1-p_1}(V_{x^2}(T_2;\delta)) \cap \cdots \cap F^{-\sum_{l=1}^{L-1}(T_l+p_l)}(V_{x^L}(T_L;\delta)) \neq \emptyset.$$

By a simple induction argument, it is sufficient to show that for all $x \in \mathcal{X}$, $T \geq 0$, $0 < \delta < \frac{1}{2M}$, $p \geq p(\delta)$ and A such that $\mathrm{Int}(A) \neq \emptyset$, we have

$$V_x(T;\delta) \cap F^{-T-p}(\mathrm{Int}(A)) \neq \emptyset \Longleftrightarrow \mathrm{Int}(F^T(V_x(T;\delta)) \cap F^{-p}(A)) \neq \emptyset.$$

We can proceed as in the proof of Proposition 4.2 in the infinite dimensional case to get that for any $\alpha \in \mathcal{C}[T,\mathbb{Z}^d] = \{0,\ldots,p-1\}^{[1,\ldots,T]\times\mathbb{Z}^d}$, there exists $\mathcal{A}_\alpha(x)$ defining an infinite open partition of $\mathcal{X}$ ($\cup\mathcal{A}_\alpha(x) = \mathcal{X}$) such that F^T is injective on $\mathcal{A}_\alpha(x)$ with inverse branch F_α^{-T}.
As in Subsection 4.2, if $\delta < \frac{1}{2M}$ then $V_x(T;\delta) \subset \mathcal{A}_0(x)$ and $F^T(V_x(T;\delta)) = \{y : d(F^T(x),y) < \delta\}$ is a product of intervals of size 2δ around $F^T(x)$.
In the same way, F_0^{-T} is a contraction around the orbit of x,

$$d(F^{T-t}(x), F_0^{-t}(y)) \leq \frac{1}{\tilde{\gamma}^t} d(F^T(x), y).$$

Then, if we construct the inverse branches of F^p around the orbit of $F^T(x)$, we know that almost all points of $\mathcal{X}$ have a preimage by F^p at distance less than $\frac{1}{2\tilde{\gamma}^p}$ of $F^T(x)$ (because F_0^{-p} is $\frac{1}{\tilde{\gamma}^p}$ contracting for the metric d). We choose then $p(\delta)$ such that $\frac{1}{\tilde{\gamma}^{p(\delta)}} < 2\delta$ and get the Specification Property. □

7.2 Proof of the lower bound

7.2.1 If $\nu \notin \mathcal{M}^1_{\mathbf{inv}}(\mathcal{X})$

In this case $I(\nu) = +\infty$, hence there is nothing to do.

7.2.2 If $\nu = \sum_{l=1}^L a_l \nu_l$ with $\nu_l \in \mathcal{M}^1_{\mathbf{erg}}(\mathcal{X})$ and $\sum_{l=1}^L a_l = 1$

For $\eta > 0$, $T \geq 1$ and any $1 \leq l \leq L$, we define

$$\hat{R}^l_T(x) = \frac{1}{\lceil a_l T\rceil |E_T|} \sum_{\substack{0 \leq t < \lceil a_l T \rceil \\ i \in E_T}} \delta_{S^i \circ F^t(x)},$$

$$\Gamma^l_T = \left\{ x \,:\, \hat{R}^l_T(x) \in \beta_{\nu_l}(V_1, \ldots, V_K; \delta/4) \text{ and } \int_{\mathcal{X}} \varphi \, d\hat{R}^l_T(x) \geq \int_{\mathcal{X}} \varphi \, d\nu_l - \eta \right\}.$$

Then by application of the Ergodic Theorem, we know that $\nu_l(\Gamma^l_T)$ goes to 1 as T tends to infinity. Hence, for a fixed $0 < b < 1$, we choose T_0 such that for any $T \geq T_0$ and any $1 \leq l \leq L$

$$\nu_l(\Gamma^l_T) \geq b. \tag{50}$$

Using Theorem B.2, we take ε_0 and T_1 such that for all $\varepsilon < \varepsilon_0$ and $T \geq T_1$, then for $1 \leq l \leq L$

$$\frac{1}{\lceil a_l T \rceil |E_T|} \log N^l(\lceil a_l T \rceil, E_T, \varepsilon, b) \geq h_{(F,S)}(\nu_l) - \eta, \tag{51}$$

where N^l denotes the number of balls necessary to cover a set of ν_l measure b (see (59) for the precise definition).
Let now $\varepsilon < \frac{\varepsilon_0}{4}$ and $T \geq \max(T_0, T_1)$. We can then choose for $1 \leq l \leq L$ a set $S^l_T \subset \Gamma^l_L$ which is maximal $(\lceil a_l T \rceil, E_T, 4\varepsilon)$-separated in Γ^l_T. Hence, by maximality, we have

$$\Gamma^l_T \subset \bigcup_{x \in S^l_T} B_x(\lceil a_l T \rceil, E_T; 4\varepsilon),$$

and this gives, combined with estimates (50) and (51),

$$\operatorname{Card}(S^l_T) \geq \exp\left(\lceil a_l T \rceil |E_T| (h_{(F,S)}(\nu_l) - \eta)\right).$$

We use now the Specification Property (Proposition 7.1) to construct from these sets S^l_T a set S_T of points which are typical for ν. Indeed, for any choice of $x^1 \in S^1_T$, $x^2 \in S^2_T$, $\ldots$, $x^L \in E^L_T$, there exists a point which ε-follows the orbits of each x^l during time $\lceil a_l T \rceil$, precisely

$$d_\rho\left(S^i \circ F^{\sum_{m=0}^{l-1} \lceil a_m T \rceil + (l-1)p(\varepsilon) + t}(x), S^i \circ F^t(x^l)\right) < \varepsilon \quad \forall\, 0 \leq t \leq \lceil a_l T \rceil,\ i \in \mathbb{Z}^d.$$

Let S_T be the set of all such constructed points: as S_T^l are $(\lceil a_l T \rceil, E_T, 4\varepsilon)$-separated, then all constructed points are distinct, hence

$$\mathrm{Card}(S_T) = \prod_{l=1}^{L} \mathrm{Card}(S_T^l) \geq \exp\left(|E_T| \sum_{l=1}^{L} \lceil a_l T \rceil (h_{(F,S)}(\nu_l) - \eta) \right).$$

And S_T is $(\hat{T}, E_T, 2\varepsilon)$-separated, with $\hat{T} = \sum_{l=1}^{L} \lceil a_l T \rceil + (L-1)p(\varepsilon)$, which implies

$$B_x(\hat{T}, E_T; \varepsilon) \cap B_y(\hat{T}, E_T; \varepsilon) = \emptyset \quad \forall\, x \neq y \text{ in } S_T. \tag{52}$$

We choose then ε_1 such that $d_\rho(x,y) < \varepsilon_1$ implies that $|\varphi(x) - \varphi(y)| < \eta$ and $|V_l(x) - V_l(y)| < \frac{\delta}{4}$ for all $1 \leq l \leq L$. A direct computation ensures now that there exists T_2 such that for $T \geq T_2$, $\varepsilon < \varepsilon_1$, $1 \leq k \leq K$ and $x \in S_T$, then

$$\int_{\mathcal{X}} \varphi \, d\hat{R}_T(x) \geq \int_{\mathcal{X}} \varphi \, d\nu - 3\eta \quad \text{ and } \quad \left| \int_{\mathcal{X}} V_k \, dR_T(x) - \int_{\mathcal{X}} V_k \, d\nu \right| \leq \frac{3\delta}{4}.$$

The last estimate implies that if $x \in S_T$ then $R_T(x) \in \beta_\nu(V_1, \ldots, V_K; \frac{3\delta}{4})$, and also, with previous estimate on V_k

$$B_x(\hat{T}, E_T; \varepsilon) \subset B_x(T, E_T; \varepsilon) \subset \{y \,:\, R_T(y) \in \beta_\nu(V_1, \ldots, V_K; \delta)\}.$$

We associate this with disjunction of such balls stated in (52), the lower bound of the Volume Lemma and estimates for the cardinal of S_T to get

$$\begin{aligned}
&\overline{m}\{y \,:\, R_T(y) \in \beta_\nu(V_1, \ldots, V_K; \delta)\} \\
&\quad \geq \sum_{x \in S_T} \overline{m}(B_x(\hat{T}, E_T; \varepsilon)) \\
&\quad \geq \sum_{x \in S_T} C_2(\hat{T}, E_T, \varepsilon, \rho) \exp\left(\hat{T} |E_T| \int_{\mathcal{X}} \varphi \, d\hat{R}_T(x) \right) \\
&\quad \geq C_2(\hat{T}, E_T, \varepsilon, \rho) \exp\left(|E_T| \sum_{l=1}^{L} \lceil a_l T \rceil (h_{(F,S)}(\nu_l) - \eta) \right) \exp\left(\hat{T} |E_T| \int_{\mathcal{X}} \varphi \, d\nu - 3\eta \right).
\end{aligned}$$

Then $$\liminf_{T \to \infty} \frac{1}{T|E_T|} \log \overline{m}\{y \,:\, R_T(y) \in \beta_\nu(V_1, \ldots, V_K; \delta)\} \geq h_{(F,S)}(\nu) + \int_{\mathcal{X}} \varphi \, d\nu - 4\eta,$$

because $\frac{1}{T} \sum_{l=1}^{L} \lceil a_l T \rceil h_{(F,S)}(\nu_l)$ tends to $h_{(F,S)}(\nu)$ and $\frac{\hat{T}}{T}$ to 1 as T goes to infinity. It suffices then to let η tend to zero.

7.2.3 If $\nu \in \mathcal{M}^1_{\mathbf{inv}}(\mathcal{X})$

We want to approximate such a probability measure by $\bar{\nu} = \sum a_l \nu_l$ from the previous case with a good control on the entropy. For this we take $\eta > 0$ and fix ε such that

$$\mathrm{dist}_{\mathcal{M}^1(\mathcal{X})}(\tau_1, \tau_2) < \varepsilon \Rightarrow \begin{cases} |\int_{\mathcal{X}} V_k \, d\tau_1 - \int_{\mathcal{X}} V_k \, d\tau_2| < \frac{\delta}{2} & \forall\, 1 \leq k \leq K \\ |\int_{\mathcal{X}} \varphi \, d\tau_1 - \int_{\mathcal{X}} \varphi \, d\tau_2| < \eta & \end{cases}$$

We choose then $\mathcal{P} = \{P_1, \ldots, P_L\}$ a partition of $\mathcal{M}^1(\mathcal{X})$ with diameter less than ε. We know by the ergodic decomposition theorem (Theorem 2.3.3 in [18]) that there exists a probability π on $\mathcal{M}^1(\mathcal{X})$ concentrated on $\mathcal{M}^1_{\text{erg}}(\mathcal{X})$ and such that $\nu = \int_{\mathcal{M}^1(\mathcal{X})} \tau \, \pi(d\tau)$. We take, for $1 \leq l \leq L$, $a_l = \pi(P_l)$ and $\nu_l \in P_l \in \mathcal{M}^1_{\text{erg}}(\mathcal{X})$ such that $h_{(F,S)}(\nu_j) \geq h_{(F,S)}(\tau) - \eta$ for π-almost all $\tau \in P_l$. Then, with $\bar{\nu} = \sum_{l=1}^{L} a_l \nu_l$, we have

$$h_{(F,S)}(\bar{\nu}) \geq h_{(F,S)}(\nu) - \eta,$$
$$\int_{\mathcal{X}} \varphi \, d\nu \geq \int_{\mathcal{X}} \varphi \, d\nu - \eta,$$
$$\beta_{\bar{\nu}}(V_1, \ldots, V_K; \delta/2) \subset \beta_{\nu}(V_1, \ldots, V_K; \delta).$$

This implies

$$\begin{aligned}
\liminf_{T\to\infty} \frac{1}{T|E_T|} \log \overline{m}(y \,:\, R_T(y) \in \beta_{\nu}(V_1, \ldots, V_K; \delta)) \\
&\geq \liminf_{T\to\infty} \frac{1}{T|E_T|} \log \overline{m}(y \,:\, R_T(y) \in \beta_{\bar{\nu}}(V_1, \ldots, V_K; \delta/2)) \\
&\geq h_{(F,S)}(\bar{\nu}) + \int_{\mathcal{X}} \varphi \, d\bar{\nu} \geq h_{(F,S)}(\nu) + \int_{\mathcal{X}} \varphi \, d\nu - 2\eta,
\end{aligned}$$

and we conclude letting ε then η tend to 0.

A Convergence of subsequences of $\mathbb{Z}^d$

We introduce in this Appendix two different notions of convergence for subsets of $\mathbb{Z}^d$, and their main properties.

Definition A.1. *A sequence $(E_n)_{n\geq 0}$ of finite subsets of $\mathbb{Z}^d$* tends to $\mathbb{Z}^d$ in the sense of Van Hove *if $\lim_{n\to\infty} |E_n| = \infty$ and*

$$\lim_{n\to\infty} \frac{|(E_n + i)\Delta E_n|}{|E_n|} = 0 \qquad \forall\, i \in \mathbb{Z}^d \tag{53}$$

(where Δ denotes the symmetric difference of sets, $A\Delta B = (A \setminus B) \cup (B \setminus A)$).

If E is a finite subset of $\mathbb{Z}^d$, we define enlarged and restricted sets in $\mathbb{Z}^d$ by

$$E^{(l)} = \begin{cases} \{j \,:\, d(j,E) \leq l\} & \text{for } l \geq 0, \\ \{j \,:\, d(j,E^C) > -l\} & \text{for } l < 0. \end{cases} \tag{54}$$

We have then two properties of sequences tending to $\mathbb{Z}^d$ in the sense of Van Hove:

Proposition A.1. *If $(E_n)_{n\geq 0}$ tends to $\mathbb{Z}^d$ in the sense of Van Hove, then:*

1. *For all $l \in \mathbb{Z}$, $(E_n^{(l)})_{n\geq 0}$ tends to $\mathbb{Z}^d$ in the sense of Van Hove and*

$$\lim_{n\to\infty} \frac{|E_n^{(l)}|}{|E_n|} = 1. \tag{55}$$

2. For all $\tau < 1$

$$\lim_{n\to\infty} \frac{1}{|E_n|} \sum_{j\in E_n} \tau^{d(j,E_n^C)} = 0. \tag{56}$$

3. For all $\tau < 1$

$$\lim_{n\to\infty} \frac{1}{|E_n|} \sum_{j\in E_n^C} \tau^{d(j,E_n)} = 0. \tag{57}$$

Proof.
1. For $l \geq 1$, we have

$$E_n \subset E_n^{(l)} = \bigcup_{j\in\Lambda_l} (E_n + j),$$

such that $E_n^{(l)} \setminus E_n = \cup_{j\in\Lambda_l}(E_n + j) \setminus E_n$, hence

$$1 \leq \frac{|E_n^{(l)}|}{|E_n|} = 1 + \frac{|E_n^{(l)} \setminus E_n|}{|E_n|} \leq 1 + \sum_{j\in\Lambda_l} \frac{|(E_n + j) \setminus E_n|}{|E_n|} \xrightarrow[n\to\infty]{} 1$$

by definition of the convergence in the sense of Van Hove (see Definition A.1).
In the same way, $(E_n^{(l)} + k) \setminus E_n^{(l)} \subset \cup_{j\in\Lambda_l}(E_n + j + k) \setminus E_n$, then

$$\frac{|(E_n^{(l)} + k) \setminus E_n^{(l)}|}{|E_n^{(l)}|} \leq \frac{|E_n|}{|E_n^{(l)}|} \sum_{j\in\Lambda_l} \frac{|(E_n + j + k) \setminus E_n|}{|E_n|} \xrightarrow[n\to\infty]{} 0.$$

We proceed similarly for $E_n^{(l)} \setminus (E_n^{(l)} + k) = k + (E_n^{(l)} - k) \setminus E_n^{(l)}$, and get that $E_n^{(l)}$ tends to $\mathbb{Z}^d$ in the sense of Van Hove.

For $l \leq -1$, we have the description

$$E_n^{(l)} = \bigcap_{j\in\Lambda_{-l}} (E_n + j) \subset E_n$$

and computations are similar to those for $l \geq 1$.

2. For any $\varepsilon > 0$, we choose $k \geq 0$ such that $\sum_{l\geq k} \tau^l \leq \varepsilon/2$ and write the sum in terms of the subsets $(E_n^{(l)})_{l\leq -1}$

$$\begin{aligned}
\frac{1}{|E_n|} \sum_{j\in E_n} \tau^{d(j,E_n^C)} &= \sum_{l\geq 1} \frac{|E_n^{(1-l)} \setminus E_n^{(-l)}|}{|E_n|} \tau^l \\
&= \sum_{l=1}^{k-1} \frac{|E_n^{(1-l)} \setminus E_n^{(-l)}|}{|E_n|} \tau^l + \sum_{l\geq k} \frac{|E_n^{(1-l)} \setminus E_n^{(-l)}|}{|E_n|} \tau^l \\
&\leq \frac{|E_n \setminus E_n^{(1-k)}|}{|E_n|} + \frac{\varepsilon}{2},
\end{aligned}$$

where we used $\tau < 1$ in the first term and $|E_n^{(1-l)} \setminus E_n^{(-l)}| \leq |E_n^{(1-l)}| \leq |E_n|$ in the second. By (55), the first term goes to 0, hence for n great enough

$$\frac{1}{|E_n|} \sum_{j \in E_n} \tau^{d(j, E_n^C)} \leq \varepsilon.$$

3. We use in this case the fact that $\sum_{l \geq 0} |\Lambda_l| \tau^l = \sum_{l \geq 0} (2l+1)^d \tau^l$ converges. Hence, for $\varepsilon > 0$, we choose $k \geq 0$ such that $\sum_{l \geq k} |\Lambda_l| \tau^l \leq \varepsilon/2$ and decompose E^C in the subsets $(E^{(l)} \setminus E^{(l-1)})_{l \geq 1}$. Then

$$\begin{aligned} \frac{1}{|E_n|} \sum_{j \in E_n^C} \tau^{d(j, E_n)} &= \sum_{l \geq 1} \frac{|E_n^{(l)} \setminus E_n^{(l-1)}|}{|E_n|} \tau^l \\ &\leq \frac{|E_n^{(k-1)} \setminus E_n|}{|E_n|} + \frac{\varepsilon}{2}, \end{aligned}$$

since $|E_n^{(l)} \setminus E_n^{(l-1)}| \leq |E_n^{(l)}| \leq |\Lambda_l||E_n|$. We conclude then as in 2. □

Convergence in the sense of Van Hove is too wide to use some existing results of ergodic theory, in particular the Ergodic Theorem and the Theorem of Shannon-McMillan-Breiman. We need to restrict the class of subsets to get the whole large deviations results:

Definition A.2. $(E_n)_{n \geq 0}$ *is a* special averaging sequence *if it is increasing, it tends to* $\mathbb{Z}^d$ *in the sense of Van Hove and there exists* $R > 0$ *such that*

$$|E_n - E_n| \leq R|E_n| \qquad \forall\, n \geq 0. \tag{58}$$

We will use to apply results from ergodic theory, the following straightforward result

Proposition A.2. *If* $(E_T)_{T \geq 1}$ *is a special averaging sequence in* $\mathbb{Z}^d$*, then* $([0, T-1] \times E_T)_{T \geq 1}$ *is a special averaging sequence in* $\mathbb{N} \times \mathbb{Z}^d$*.*

Remark: We could use some recent results of Lindenstrauss to work with tempered sequences, a notion more general than special averaging sequences. He proves indeed in [21, 22] that the ergodic results we use remain valid in this context.

B Thermodynamic Formalism

We present in our setup the main definitions and the results we need from Thermodynamic Formalism. For a more general approach and all the proofs, we refer to the well-written expository book of G. Keller [18] (and to [26] for proofs of the Ergodic Theorem and of the Shannon-McMillan-Breiman Theorem).

B.1 Entropy

For $\mathcal{A} = \{A_1, \dots, A_K\}$ and $\mathcal{B} = \{B_1, \dots, B_L\}$ finite partitions of $\mathcal{X}$, let

$$\mathcal{A} \vee \mathcal{B} = \{A_k \cap B_l \,:\, 1 \le k \le K, 1 \le l \le L\}.$$

Then, for $\nu \in \mathcal{M}^1_{\mathrm{inv}}(\mathcal{X})$, E_T a sequence tending to $\mathbb{Z}^d$ in the sense of Van Hove, and $\mathcal{A}$ a partition, we define

- $h(\nu|\mathcal{A}) = -\sum_{A \in \mathcal{A}} \nu(A) \log(\nu(A))$ and $\mathcal{A}_T = \bigvee_{\substack{0 \le t < T \\ i \in E_T}} F^{-t} \circ S^{-i}(\mathcal{A})$,
- $h_{(F,S)}(\nu|\mathcal{A}) = \lim_{T \to \infty} \frac{1}{T|E_T|} h(\nu|\mathcal{A}_T)$,
- $h_{(F,S)}(\nu) = \sup\{h_{(F,S)}(\nu|\mathcal{A}) \,:\, \mathcal{A} \text{ finite partition of } \mathcal{X}\}$.

This last quantity is the metric entropy of ν under (F, S), which does not depend on the choice of the sequence $(E_T)_{T \ge 0}$.

Proposition B.1.

1. *$h_{(F,S)}$ is convex affine, $h_{(F,S)}\left(\sum_{l=1}^L a_l \nu_l\right) = \sum_{l=1}^L a_l h_{(F,S)}(\nu_l)$ when $\sum_{l=1}^L a_l = 1$;*
2. *For $\nu \in \mathcal{M}^1_{inv}(\mathcal{X})$ and for any partition $\mathcal{A}$ such that $\nu(\partial\mathcal{A}) = 0$ and $\mathrm{diam}(\mathcal{A}) < \delta_0 = \frac{1}{2M}$, we have*
$$h_{(F,S)}(\nu) = h_{(F,S)}(\nu|\mathcal{A});$$
3. *$h_{(F,S)}$ is upper semi-continuous.*

The two last properties are consequences of the expansiveness of the system stated in Proposition 4.5 (see Theorem 4.5.6 in [18] and its proof).

A well known result about entropy is the Shannon-McMillan-Breiman theorem, which expresses the fact that for an ergodic measure, entropy precisely describes the asymptotic size of elements of the partition:

Theorem B.1 (Shannon-McMillan-Breiman). *If $\nu \in \mathcal{M}^1_{erg}(\mathcal{X})$, $\mathcal{A}$ is a finite partition and $(E_T)_{T \ge 0}$ is a special averaging sequence, then for ν-almost all x*

$$-\frac{\log \nu(\mathcal{A}_T(x))}{T|E_T|} \xrightarrow[T \to \infty]{} h_{(F,S)}(\nu|\mathcal{A}),$$

where $\mathcal{A}_T(x)$ denotes the element of the partition $\mathcal{A}_T$ which contains x.

We use in our proof of the lower bound of Large Deviations a metric equivalent of this theorem, which tells that for an ergodic measure, the metric entropy describes the number of balls necessary to cover a significant set. For $T \ge 0$, $\delta > 0$, $0 < b < 1$ and $(E_T)_{T \ge 0}$ a special averaging sequence, we denote

$$N(T, E_T; \delta, b) = \min\left\{\mathrm{Card}(Y) \,:\, \nu\Big(\bigcup_{x \in Y} B_x(T, E_T; \delta)\Big) > b\right\} \tag{59}$$

(see definition of $B_x(T, E_T; \delta)$ in formula (9)). We call a set Y as in the definition a$(T, E_T; \delta, b)$-covering set for ν.

Theorem B.2. *If $\nu \in \mathcal{M}^1_{erg}(\mathcal{X})$ and $(E_T)_{T\geq 0}$ is a special averaging sequence, then for all $0 < b < 1$*

$$\lim_{\delta\to 0}\liminf_{T\to\infty}\frac{1}{T|E_T|}\log N(T,E_T;\delta,b) = \lim_{\delta\to 0}\limsup_{T\to\infty}\frac{1}{T|E_T|}\log N(T,E_T;\delta,b) = h_{(F,S)}(\nu).$$

This result for the single map case is due to Katok [16]. A proof of our generalization to the lattice setup can be found in [2], where it is adapted from [29].

B.2 Topological pressure

A set $Y \subset \mathcal{X}$ is $(T,E;\delta)$-separated if

$$x, x' \in Y, x \neq x' \quad \Longrightarrow \quad x' \notin B_x(T,E;\delta).$$

It is separated maximal if it is maximal for this separation property.

We define then for $V \in \mathcal{C}(\mathcal{X})$, $(E_T)_{T\geq 0}$ a sequence tending to $\mathbb{Z}^d$ in the sense of Van Hove and $Y \subset \mathcal{X}$ finite

$$P_{(F,S)}(V;T,Y) = \log\sum_{x\in Y}\exp\Big(\sum_{\substack{0\leq t<T\\ i\in E_T}} V\circ S^i\circ F^t(x)\Big).$$

Then

$$\begin{aligned}P_{(F,S)}(V) &= \lim_{\delta\to 0}\limsup_{T\to\infty}\frac{1}{T|E_T|}\sup\{P_{(F,S)}(V;T,Y)\ :\ Y \text{ is } (T,E_T;\delta)\text{-separated}\}\\ &= \lim_{\delta\to 0}\limsup_{T\to\infty}\frac{1}{T|E_T|}\sup\{P_{(F,S)}(V;T,Y)\ :\ Y \text{ is } (T,E_T;\delta)\text{-separated maximal}\}\end{aligned}$$

is the topological pressure of V for the dynamic of (F,S). This definition is independent of the choice of the sequence (E_T). The main result for this quantity is the Gibbs Variational Principle, which expresses it as a variational expression of the entropy:

Theorem B.3 (Gibbs Variational Principle). *For any $V \in \mathcal{C}(\mathcal{X})$*

$$P_{(F,S)}(V) = \sup_{\nu\in\mathcal{M}^1_{inv}(\mathcal{X})}\left(h_{(F,S)}(\nu) + \int_{\mathcal{X}} V\,d\nu\right), \tag{60}$$

and, since $h_{(F,S)}$ is convex affine and upper semi-continuous in our case, for any $\nu \in \mathcal{M}^1_{inv}(\mathcal{X})$

$$h_{(F,S)}(\nu) = \inf_{V\in\mathcal{C}(\mathcal{X})}\left(P_{(F,S)}(V) - \int_{\mathcal{X}} V\,d\nu\right). \tag{61}$$

Definition B.1. *The equilibrium measures associated to the dynamical system (F,S) and to a potential $V \in \mathcal{C}(\mathcal{X})$ are the invariant measures which realize the supremum in the Variational Principle (60).*

C Generating function method for the iteration sequence

For $\delta > 0$, $\gamma > 1$ and (α_k) a sequence of non-negative reals, let $u(l,t)$ be defined for $l \in \mathbb{Z}$ and $t \in \mathbb{N}$ by

$$u(l,t) = \begin{cases} \frac{1}{2} & \text{if } l < 0 \\ \delta & \text{if } l \geq 0, t = 0 \\ \frac{1}{\gamma}u(l,t-1) + \frac{1}{\gamma}\sum_{k\geq 0}\alpha_k u(l-k,t) & \text{if } l \geq 0, t > 0 \end{cases} \tag{62}$$

We have then for such a sequence:

Proposition C.1. *Suppose there exists $\theta < 1$ such that for any $k \geq 0$, $\alpha_k = \theta^k \tilde{\alpha}_k$ and denote $S = \sum_{k\geq 0}\alpha_k$ and $\tilde{S} = \sum_{k\geq 0}\tilde{\alpha}_k$. Then, under the assumption*

$$\gamma - \tilde{S} > 1,$$

we have for all $l \geq 0$ and $t \geq 0$

$$u(l,t) \leq \frac{\delta}{(\gamma - S)^t} + \theta^{l+1}\frac{\tilde{S}}{2(\gamma - \tilde{S} - 1)}. \tag{63}$$

Proof. We solve this equation by a generating function method (see [32] for a general introduction and many useful tools). Let $f(x,y)$ be the formal series defined by

$$f(x,y) = \sum_{\substack{l\geq 0\\ t\geq 1}} u(l,t)x^l y^t.$$

Then the inductive definition of $u(l,t)$ implies for f

$$\begin{aligned} &f(x,y) \\ &= \sum_{\substack{l\geq 0\\ t\geq 1}} \Big(\frac{1}{\gamma}u(l,t-1) + \frac{1}{\gamma}\sum_{k\geq 0}\alpha_k u(l-k,t)\Big)x^l y^t \\ &= \frac{\delta y}{\gamma}\sum_{l\geq 0} x^l + \frac{y}{\gamma}\sum_{\substack{l\geq 0\\ t\geq 1}} u(l,t)x^l y^t + \frac{1}{\gamma}\sum_{\substack{l\geq 0\\ t\geq 1}}\Big(\sum_{k=0}^{l}\alpha_k u(l-k,t)x^l\Big)y^t + \frac{1}{2\gamma}\sum_{\substack{l\geq 0\\ t\geq 1}}\Big(\sum_{k>l}\alpha_k\Big)x^l y^t \\ &= \frac{\delta y}{\gamma}\sum_{l\geq 0} x^l + \frac{1}{2\gamma}\sum_{\substack{l\geq 0\\ t\geq 1}} R_l x^l y^t + \frac{1}{\gamma}\Big(y + \sum_{k\geq 0}\alpha_k x^k\Big)f(x,y) \\ &= \Big(\frac{\delta y}{\gamma}\sum_{l\geq 0} x^l + \frac{1}{2\gamma}\sum_{\substack{l\geq 0\\ t\geq 1}} R_l x^l y^t\Big)\Big(1 - \frac{1}{\gamma}\Big(y + \sum_{k\geq 0}\alpha_k x^k\Big)\Big)^{-1}, \end{aligned}$$

where $R_l = \sum_{k>l} \alpha_k$. We invert formally this expression, using that

$$\begin{aligned}\Big(1-\frac{1}{\gamma}\Big(y+\sum_{k\geq 0}\alpha_k x^k\Big)\Big)^{-1} &= \sum_{n\geq 0}\sum_{u=0}^{n}\binom{n}{u}\frac{1}{\gamma^n}y^u\Big(\sum_{k\geq 0}\alpha_k x^k\Big)^{n-u} \\ &= \sum_{\substack{u\geq 0\\ h\geq 0}}\binom{u+h}{u}\frac{1}{\gamma^{u+h}}y^u\sum_{k_1,\dots,k_h\geq 0}\alpha_{k_1}\cdots\alpha_{k_h}x^{k_1+\cdots+k_h} \\ &= \sum_{\substack{n\geq 0\\ u\geq 0}}\Big(\sum_{h\geq 0}\binom{u+h}{u}\frac{1}{\gamma^{u+h}}\sum_{\substack{k_1,\dots,k_h\geq 0\\ k_1+\cdots+k_h=n}}\alpha_{k_1}\cdots\alpha_{k_h}\Big)x^n y^u.\end{aligned}$$

Hence, using in the upper bound that $R_{l-n} \leq \theta^{l-n+1}\tilde{S}$, we get

$$\begin{aligned} u(l,t) = \frac{\delta}{\gamma}\sum_{n=0}^{l}\Big(\sum_{h\geq 0}\binom{t-1+h}{t-1}\frac{1}{\gamma^{t-1+h}}\sum_{\substack{k_1,\dots,k_h\geq 0\\ k_1+\cdots+k_h=n}}\alpha_{k_1}\cdots\alpha_{k_h}\Big) \\ +\frac{1}{2\gamma}\sum_{\substack{0\leq n\leq l\\ 0\leq u<t}}R_{l-n}\Big(\sum_{h\geq 0}\binom{u+h}{u}\frac{1}{\gamma^{u+h}}\sum_{\substack{k_1,\dots,k_h\geq 0\\ k_1+\cdots+k_h=n}}\alpha_{k_1}\cdots\alpha_{k_h}\Big) \\ \leq \frac{\delta}{\gamma^t}\sum_{h\geq 0}\binom{t-1+h}{t-1}\Big(\frac{S}{\gamma}\Big)^h + \frac{\theta^{l+1}}{2\gamma}\sum_{u\geq 0}\frac{\tilde{S}}{\gamma^u}\sum_{h\geq 0}\binom{u+h}{u}\Big(\frac{\tilde{S}}{\gamma}\Big)^h \\ = \frac{\delta}{(\gamma-S)^t} + \theta^{l+1}\frac{\tilde{S}}{2(\gamma-\tilde{S}-1)}. \end{aligned}$$

□

Remark: We obtained in fact in the course of the proof an exact (but complicated) expression for the sequence $u_{(i,t)}$.

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