PHASE TRANSITION ASYMPTOTICS FOR RANDOM WALKS ON A STATIONARY RANDOM POTENTIAL

GÉRARD BEN AROUS[1], STANISLAV MOLCHANOV[1] AND ALEJANDRO F. RAMÍREZ[1,2]

ABSTRACT. We describe a universal transition mechanism characterizing the passage to an annealed behavior and to a regime where the fluctuations about this behavior are Gaussian, for the long time asymptotics of the empirical average of the expected value of the number of random walks which branch and annihilate on $\mathbb{Z}^d$, with stationary random rates. The random walks are independent, continuous time rate $2d\kappa$, simple, symmetric, with $\kappa \geq 0$. A random walk at $x \in \mathbb{Z}^d$, binary branches at rate $v_+(x)$, and annihilates at rate $v_-(x)$. The random environment w has coordinates $w(x) = (v_-(x), v_+(x))$ which are i.i.d. We identify a natural way to describe the annealed-Gaussian transition mechanism under mild conditions on the rates. Indeed, we introduce the exponents $F_\theta(t) := \frac{H_1((1+\theta)t)-(1+\theta)H_1(t)}{\theta}$, and assume that $\frac{F_{2\theta}(t)-F_\theta(t)}{\theta \log(\kappa t+e)} \to \infty$ for $|\theta| > 0$ small enough, where $H_1(t) := \log\langle m(0,t)\rangle$ and $\langle m(0,t)\rangle$ denotes the average of the expected value of the number of particles $m(0,t,w)$ at time t and an environment of rates w, given that initially there was only one particle at 0. Then the empirical average of $m(x,t,w)$ over a box of side $L(t)$ has different behaviors: if $L(t) \geq e^{\frac{1}{d}F_\epsilon(t)}$ for some $\epsilon > 0$ and large enough t, a law of large numbers is satisfied; if $L(t) \geq e^{\frac{1}{d}F_\epsilon(2t)}$ for some $\epsilon > 0$ and large enough t, a CLT is satisfied. These statements are violated if the reversed inequalities are satisfied for some negative ϵ. As corollaries, we obtain more explicit statements under regularity conditions on the tails of the random rates, including examples in the four universality classes defined in [HKM(2005)]: potentials which are unbounded of Weibull type, of double exponential type, almost bounded, and bounded of Fréchet type. For them we also derive sharper results in the non-annealed regime. Our results indicate the presence of a mean field type phase transition mechanism, generalizing the law of large numbers and the CLT proved in [BBM(2005)] and in [BMR(2005)].

1. INTRODUCTION

This paper is a continuation of Ben Arous, Molchanov and Ramírez [BMR(2005)], where a transition mechanism between the quenched and annealed large time asymptotics for the survival probability of a random walk among random traps was studied, and of Ben Arous, Bogachev and Molchanov [BBM(2005)] where such transition mechanism was first proposed in the context of sums of i. i. d. random exponentials. Here we place ourselves in the context of the parabolic Anderson model, which can

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be associated to a system of continuous time branching and annihilating random walks on the lattice $\mathbb{Z}^d$, and where the branching and annihilation take place according to a stationary random environment. An important quantity of this system corresponding to the survival probability of a random walk among random traps, which is well-studied in the mathematical physics literature is the expected value $m(x,t,w)$ of the total number of random walks at time t, given that initially there was a single one at site x, and the random environment was w. Indeed, this quantity satisfies the *parabolic Anderson problem*, which is a Cauchy problem for the discrete Laplacian heat equation with random potential. To study the long time asymptotic behavior of this quantity, one can either fix an almost sure realization of the environment w, which is called the *quenched regime*, or first take an average with respect to the environment, the so called *annealed regime*. Usually these two approaches give completely different behaviors ([GM(1990)], [GM(1998)], [S(1998)]). Following [BMR(2005)], here we address the question of the relevance of these approaches, obtaining results that indicate that in the context of the parabolic Anderson problem there is a universal transition mechanism describing a broad range of different long time asymptotic behaviors depending on the spatial scale at which the system is averaged, interpolating between the quenched and the annealed regimes. This mechanism is analogous to the phase transition description of mean-field statistical mechanics magnetization models such as the Random Energy Model ([D(1981)], [BKL(2002)]).

In this work we identify natural conditions which describe precisely the scales at which there is a passage to an annealed behavior of an averaged first moment, and at which the fluctuations about this behavior are Guassian. The results we obtain are general, in the sense that they are valid for a large class of product distributions for the random environment, under mild conditions on the branching and annihilation rates. They include examples in the four universality classes recently introduced by van den Hofstad, König and Mörters [HKM(2005)] describing all the cases of the random environment. For example, potentials which are unbounded of Weibull type, of double exponential type, almost bounded and bounded of Fréchet type. Furthermore, our results include and generalize both theorems 2.1 and 2.2 of [BBM(2005)], parts (*ii*) and (*iii*) of theorem 1 and part (*i*) of theorem 2 of [BMR(2005)].

Let us describe briefly the results of [BMR(2005)]. In the sequel, given two functions $f, g : [0,\infty) \to \mathbb{R}$, we will use the notation $f \sim g$ for $\lim_{t\to\infty} f/g = 1$ while $f \ll g$ for $\lim_{t\to\infty} f/g = 0$. A random environment of obstacles denoted $w = \{w(x) : x \in \mathbb{Z}^d\} \in \{0,1\}^{\mathbb{Z}^d}$ was placed on the lattice, where $w(x) = 1$ represents the presence of an obstacle, while $w(x) = 0$ the absence. The law of such an environment was given by a Bernoulli product probability measure μ defined on $\{0,1\}^{\mathbb{Z}^d}$, of parameter $p \in [0,1]$. We called $p(x,t,w)$ the probability that a continuous time symmetric simple random walk moving on the random environment w (independent of the law of the random walk) does not hit a site having an obstacle in the time interval $[0,t]$, starting from site x at time $t = 0$. A well known result of Antal [A(1994)] (see also his Ph.D. thesis [A(1995)]) states that a.s. on the event that the origin belongs to an infinite non-percolating set of sites without obstacles, $\log p(0,t,w) \sim -c_1 t/(\log t)^{2/d}$ for an appropriate constant c_1, depending only on the dimension d and p (an analogous result for Brownian motion was proved by Sznitman before [S(1998)]). This is the *quenched* large time asymptotics of the survival probability. On the other hand, a previous classical result of Donsker and Varadhan [DV(1975)] states that $\log\langle p(0,t)\rangle \sim -c_2 t^{d/(d+2)}$ for another constant c_2 also depending only on d and p (an analogous for Brownian motion was proved by them in [DV(1979)]). Here $\langle p(0,t)\rangle$ is the average of $p(0,t,w)$ with respect to the random environment. This result is the so called *annealed* behavior of the survival probability. Now, if we define for each $L > 0$ a box Λ_L of side $(2L+1)$ centered at the origin, it was shown in [BMR(2005)], that the *averaged survival probability* $p_L(0,t,w) := \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} p(x,t,w)$ has a large time asymptotic behavior which interpolates between the quenched and the annealed one. Indeed, there are constants $0 < \gamma_1 < \gamma_2$ such that if $L(t) \geq e^{\gamma t^{d/(d+2)}}$, with $\gamma > \gamma_1$, a law of large numbers is satisfied and the averaged survival probability decreases like the annealed one; if $L(t) \geq e^{\gamma t^{d/(d+2)}}$, with $\gamma > \gamma_2$, also a CLT is satisfied. Furthermore, if $L(t) \ll t$, the averaged survival probability decreases like the quenched one; if $t \ll L(t)$ and $\log L(t) \ll t^{d/(d+2)}$ an intermediate regime is satisfied.

A random walk on hard core random obstacles can be viewed as a random walk subject to a negative random potential which can have the values 0 or $-\infty$. It is then natural to consider a random potential taking both negative and positive finite values, which is the context of this paper: $v_+ := \{v_+(x) : x \in \mathbb{Z}^d\}$ is an environment representing the branching rate with $v_+(x) \in [0,\infty)$, while $v_- := \{v_-(x) : x \in \mathbb{Z}^d\}$ represents the annihilation rate, with $v_-(x) \in [0,\infty]$ so that we admit the possibility of hard core obstacles. Then $w = (v_-, v_+)$ is the random environment corresponding to the random obstacles. We assume that the random variables $\{w(x) : x \in \mathbb{Z}^d\}$, with $w(x) := (v_-(x), v_+(x))$, have a law μ so that they are i.i.d. Each random walk moves independently of the others according to a continuous time simple symmetric rate $2d\kappa$ dynamics for some $\kappa \geq 0$ (we admit the possibility that $\kappa = 0$). A random walk at a site $x \in \mathbb{Z}^d$, branches at rate $v_+(x)$, disappearing and producing two new independent offsprings, and annihilates at rate $v_-(x)$, disappearing: $v_-(x) = \infty$ means the annihilation is instantaneous. In the recent paper [HKM(2005)], it is shown that under regularity assumptions on the tail of the law of the *effective field* $v(0) := v_+(0) - v_-(0)$, exactly four universality classes of environments can occur. Their assumptions are formulated in terms of the *cumulant generating function*,

$$H(t) = \log\left\langle e^{v(0)t}\right\rangle, \qquad t \geq 0, \tag{1}$$

of the law of the effective potential $v(0)$. Their basic assumption is that this function is defined and finite for every $t \geq 0$. Then, under two regularity assumptions on H they show that four universality classes can occur: (1) a first class where v is unbounded and has "heavy tails" at infinity, and which includes Weibull-type tails; (2) a second class of unbounded potentials with "lighter" tails which includes the double exponential law; (3) a class containing bounded and unbounded potentials; (4) a class of bounded potentials including those which have Fréchet-type tails near their essential supremum, and the degenerate case of random walks on hard core random obstacles with $\mu[v(0) = -\infty] = p < 1$. In this paper we generalize parts (*ii*) and (*iii*) of theorem 1 of [BMR(2005)] describing the passage to an annealed and Gaussian regime, to the previously described system of random walks on the random environment w, under mild conditions which include cases in these four universality classes.

Throughout this article the following will be assumed.

Assumption (E). *The law of the effective potential is such that* $\mu[v(0) = -\infty] < 1$ *and*

$$\left\langle \left(\frac{v^+(0)}{\log_+ v(0)} \right)^d \right\rangle < \infty.$$

Here $\langle \cdot \rangle$, denotes the average with respect to the environment, $v^+(0)$ is the positive part of $v(0)$, while $\log_+ x$ is defined as $\log x$ if $x > e$ and 1 otherwise. Assumption **(E)** ensures the finiteness of the positive exponential moments of $v(0)$ and that μ-a.s. the stochastic process of random walks on the random environment w can be constructed on infinite volume ([GM(1990)]), as a limit of processes defined on finite boxes corresponding to continuous time Markov branching processes, as defined in Athreya and Ney in [AN(1972)]. Furthermore, if $\zeta(t)$ denotes the total number of random walks at time t on a random environment w, given that initially there was only a single one at site x, and E_x^w the expectation defined by its law, condition **(E)** ensures the existence of the first moment $m(x,t,w) = E_x^w[\zeta(t)]$. This is the content of proposition 1 of this paper. This first moment will be the central object of our study. We will see that assumption **(E)** ensures the finiteness for $t \geq 0$ of the annealed first moments $\langle m(0,t) \rangle := \int m(0,t,w) d\mu$, with which we will state our main assumptions. We will need to define the *growth functions* $\{H_1(t) : t \geq 0\}$ by

$$H_1(t) := \log \langle m(0,t) \rangle. \tag{2}$$

Our main assumption will be formulated with the help of a family of functions $\{F_\theta : \theta \in \mathbb{R}\}$ which we call the *intermittency exponents*, defined for every $\theta \neq 0$ and $t \geq 0$ as,

$$F_\theta(t) := \frac{H_1((1+\theta)t) - (1+\theta)H_1(t)}{\theta}. \tag{3}$$

Assumption (MI). *For* $|\theta| > 0$ *small enough,*

$$\lim_{t \to \infty} \frac{F_{2\theta}(t) - F_\theta(t)}{\theta \log(\kappa t + e)} = \infty.$$

As it will be shown in corollary 3, this assumption implies the occurrence of the so called *intermittent* behavior of the random field w [GM(1990)]. It encompasses examples falling in the four universality classes of [HKM(2005)].

We will show that it is possible to formulate an assumption directly in terms of the cumulant generating function H, which is sufficient for **(MI)** to be satisfied, and includes the first class of [HKM(2005)]. For this we need to define the *cumulant exponents*, parametrized by $\theta \neq 0$, for $t \geq 0$ as,

$$G_\theta(t) := \frac{H((1+\theta)t) - (1+\theta)H(t)}{\theta}. \tag{4}$$

By Jensen's inequality it can be seen that $G_\theta(t) \geq 0$. We will see that condition **(MI)** is satisfied whenever the following happens.

Assumption (SI). *For* $|\theta| > 0$ *small enough,*

$$\lim_{t\to\infty} \frac{G_{2\theta}(t) - G_\theta(t)}{\theta(\kappa t + 1)} = \infty.$$

Condition **(SI)** includes the first universality class of [HKM(2005)] and can be viewed as a strong intermittency requirement. It implies $H(t)/t \gg 1$. Hence, using the bounds $e^{H(t)-2d\kappa t} \leq \langle m(0,t)\rangle \leq e^{H(t)}$ (see for example theorem 3.1 of Gärtner and Molchanov [GM(1990)]), we see that if **(SI)** is satisfied, we have the asymptotics

$$\log\langle m(0,t)\rangle \sim H(t), \tag{5}$$

which is much faster than the a.s. one (see [GM(1998)]).

To construct the transition mechanism, we define for each $L \geq 0$, an averaged first moment over a box Λ_L by $m_L(0,t,w) := \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} m(x,t,w)$. Our main result, theorem 1, describes under which conditions does the averaged first moment asymptotics behave like the annealed one, and when are the fluctuations about this asymptotics Gaussian. The interest of these results are their generality. Namely, their are valid only under the assumptions **(E)** and **(MI)**. Part (i) of theorem 1, states that if for some $\epsilon > 0$ we have $L(t) \geq e^{\frac{1}{d}F_\epsilon(t)}$ eventually in t, the law of large numbers $\frac{m^L}{\langle m\rangle} \sim 1$ is satisfied in probability: hence we have the annealed behavior $\log m^L(0,t) \sim \log\langle m(0,t)\rangle$. On the other hand, if for some $\epsilon > 0$ we have $L(t) \leq e^{\frac{1}{d}F_{-\epsilon}(t)}$ eventually in t, in probability $\frac{m^L}{\langle m\rangle} \ll 1$. Part (ii) says that if for some $\epsilon > 0$ we have $L(t) \geq e^{\frac{1}{d}F_\epsilon(2t)}$ eventually in t, then $\frac{m^L-(2L+1)^d\langle m\rangle}{Var_\mu m^L}$ converges in distribution to a centered normal law of unit variance $\mathcal{N}(0,1)$, where Var_μ denotes the variance. Also, if for some $\epsilon > 0$ we have $L(t) \leq e^{\frac{1}{d}F_{-\epsilon}(2t)}$ eventually in t, in probability $\frac{m^L-(2L+1)^d\langle m\rangle}{Var_\mu m^L} \ll 1$. This discussion is summarized in the table below.

Annealed behavior	$d\log L(t) \geq F_\epsilon(t)$	$m^L/\langle m\rangle \sim 1$
Non-Annealed behavior	$d\log L(t) \leq F_{-\epsilon}(t)$	$m^L/\langle m\rangle \ll 1$
Gaussian behavior	$d\log L(t) \geq F_\epsilon(2t)$	$\frac{m^L-(2L+1)^d\langle m\rangle}{Var_\mu m^L} \to \mathcal{N}(0,1)$
Non-Gaussian behavior	$d\log L(t) \leq F_\epsilon(2t)$	$\frac{m^L-(2L+1)^d\langle m\rangle}{Var_\mu m^L} \ll 1$

Table 1: Large time asymptotic behavior of the averaged first moments

Under an additional regularity assumption on the intermittency exponents (assumption **(RI)** of subsection 2.4) it will be shown in corollary 2 that there exist two constants γ_1 and γ_2, called *transition exponents* and a function $J(t):[0,\infty)\to[0,\infty)$ with $J(t)\gg 1$, called the *growth exponent*, describing more explicitly the transition of theorem 1. Indeed, in this case, a law of large numbers is satisfied when $d\log L(t)\geq \gamma J(t)$ for some $\gamma>\gamma_1$; the CLT when $d\log L(t)\geq\gamma J(t)$ for some $\gamma>\gamma_2$. Furthermore, if $d\log L(t)\leq\gamma J(t)$ for some $\gamma<\gamma_1$, the law of large numbers is not satisfied, while if $d\log L(t)\leq\gamma J(t)$ for some $\gamma<\gamma_2$, the CLT is not satisfied. Propositions 4, 5, 6 and 7 give the explicit value of γ_1, γ_2 and J in the case of unbounded potentials with Weibull type tails, unbounded potentials with double exponential type tails, potentials in the third universality class of [HKM(2005)] and bounded potentials with Fréchet type tails including the pure hard core case. Table 2 below summarizes those results.

POTENTIAL	$-\log\mu[v(0)>x]$	$J(t)$	γ_1	γ_2
Weibull	$x^\rho,\rho>1$	$H(t)$	$\frac{1}{1-\rho}$	$2^{1-\gamma_1}\gamma_1$
Double exponential	$e^{x/\rho}$	t	ρ	2ρ
Third class example	e^{x^2}	$\frac{t}{2\sqrt{\log t}}$	1	2
Pure hard core	$--$	$c_2 t^{\frac{d}{d+2}}$	$\frac{2}{d+2}$	$2^{1-\gamma_1}\gamma_1$

Table 2: Transition and growth exponents in the four universality classes. In the pure hard core case $\mu[v(0)=-\infty]=p$ and c_2 is a constant depending on p and d

Also, in theorem 2, we obtain sharper upper bounds for the order of magnitude of the averaged first moments in the non-annealed regime for the examples treated in propositions 4, 5, 6 and 7. This theorem, generalizes **Case 3** of theorem 2 of [BMR(2005)].

The special case in which assumption **(SI)** is satisfied expressed as corollary 1. This includes the case $\kappa=0$, corresponding to sums of i.i.d. random exponentials where condition **(MI)** reduces to,

$$\lim_{t\to\infty}\frac{G_{2\theta}(t)-G_\theta(t)}{\theta}=\infty, \tag{6}$$

for $|\theta|>0$ large enough, and $H_1(t)=H(t)$. Corollary 1 is a result complementary to theorem of [BBM(2005)], generalizing theorems 2.1 and 2.2 of [BBM(2005)] where Weibull and Fréchet type tails are assumed on $v(0)$.

The proof of theorem 1 has as one of the main ingredients a coarse graining technique necessary to reduce the asymptotics of the averaged first moments, to a sum of independent random exponentials. This technique, was introduced in [BMR(2005)], but here it has the difficulty that the terms of the sum defining the averaged probabilities are not uniformly bounded with respect to the time variable

t or the scale L (whereas in [BMR(2005)], such a bound existed having the value 1). This requires more careful estimates on these quantities, which are performed, via the Feynman-Kac formula and spectral estimates. Once the reduction to a sum of i. i. d. exponentials is achieved, an analysis based on von Bahr-Esseen inequality finishes the proof (see also [BMR(2005)] and [BBM(2005)]).

Besides this introduction, this paper has four other sections. The main results are stated in section 2. We first introduce in subsection 2.1 the main definitions. In subsection 2.2 we formulate proposition 1, stating that a growth of the form $\limsup_{|x|\to\infty} \frac{v_+(x)}{|x|\log|x|} = 0$ is enough to ensure well defined first moments for the total number of particles. When this proposition is combined with proposition 2 of [GM(1990)], one concludes that under the condition **(E)**, the reaction-diffusion process on the lattice $\mathbb{Z}^d$ is such that the total number of particles at any given time has a finite first moment, for initial conditions with a finite total number of particles. In particular, there is no explosion, and the process is well defined. We then state theorem 1 in section 2.3. Corollary 1, under the assumption **(SI)** is stated and proved next. The applications of theorem 1 are given in subsection 2.4. First, the regularity condition **(RI)**, is introduced. This is applied to the case of unbounded effective potentials with Weibull type tails, through proposition 4, using the Kasahara exponential Tauberian theorem [BGT(1989)]. Next, corollary 2 is applied to the case of unbounded potentials with double exponentially decaying type tails, through proposition 5. Then, we treat the case of potentials falling in the third universality class (almost bounded) of [HKM(2005)] through proposition 6. We end subsection 2.4 considering the case of bounded potentials with tails near their upper-bound which are of the Fréchet type (proposition 7). In subsection 2.5, we state theorem 2, which improves the upper bounds describing the order of magnitude of the empirical average for the examples considered in subsection 2.4. The proof of proposition 1, is the content of the third section. In section 4, the truncated first moments are introduced. These are the first moments of a reaction-diffusion process defined on a finite box, with Dirichlet boundary conditions. They are then used to approximate some important quantities related to the averaged first moments. Then, several important estimates for the moments and correlations of the first moments are derived. The proof of theorems 1 and 2 are given in section 5. In subsection 5.1, the partition analysis method of [BMR(2005)] is recalled. This and together with the estimates of section 4, and the von Bahr-Esseen inequality, is subsequently applied to prove theorem 1. The paper finishes with subsection 5.7, where theorem 2 is proved.

2. Notation and results

Here we will state the results of this paper. After introducing most of the notation and giving the main definitions in the first subsection, in subsection 2.2 we state proposition 8, which ensures that a.s. there is no explosion for the reaction-diffusion process under assumption **(E)**. Then, the principal result of this paper, theorem 1, is stated in subsection 2.3, together with corollary 1. In subsection 2.4, we state corollary 2, giving the form of theorem 1, under certain regularity assumptions. Here we will consider applications of this results to several specific examples of distributions of the effective potential. We end the presentation of our results with subsection 2.5, where we state theorem 2.

2.1. **Definition of the reaction-diffusion process.** We begin defining a reaction-diffusion model corresponding to a set of random walks on the lattice $\mathbb{Z}^d$ branching and annihilating at rates depending on the their position. Consider the set of natural numbers $\mathbb{N}$ endowed with the discrete topology. Define the set $\Omega := \mathbb{N}^{\mathbb{Z}^d}$ representing the configuration of particles on the lattice. In this paper we will be interested only on the subset of configurations $\Omega_0 \subset \Omega$ characterized by the property that $\{x : \eta(x) > 0\}$ has finite cardinality whenever $\eta \in \Omega_0$. Let $v_+ := \{v_+(x) : x \in \mathbb{Z}^d\}$ and $v_- := \{v_-(x) : x \in \mathbb{Z}^d\}$, where $v_+(x) \in [0, \infty)$ and $v_-(x) \in [0, \infty]$. Here $v_+(x)$ and $v_-(x)$ represent the rate at which particles branch and annihilate at site x, respectively. Note that we admit the value ∞ for the annihilation rate: this represents a hard core obstacle, where particles are instantly annihilated. Call an ordered pair $w := (v_-, v_+) \in W$, with coordinates $w(x) = (v_-(x), v_+(x))$, a *field configuration*, where $W := ([0, \infty) \times [0, \infty])^{\mathbb{Z}^d}$. We will denote the set of *hard core obstacle sites* by $\mathcal{G}(w) := \{x \in \mathbb{Z}^d : v_-(x) = \infty\}$. Given $r \in [0, \infty)$ and $x \in \mathbb{Z}^d$ we will call $\Lambda(x, r) := \{y \in \mathbb{Z} : \|x - y\| \leq r\}$ the ball of radius r centered at x under the norm $\|x\| := \sup_{1 \leq i \leq d} |x_i|$, where x_i are the coordinates of x. We will furthermore use the notation Λ_r in place of $\Lambda(0, r)$. In this subsection we will construct a stochastic process as the limit as $L \to \infty$ of processes defined on the boxes Λ_L. Throughout the sequel, given a subset $U \subset \mathbb{Z}^d$, we will denote by U^c the complement of U and $\delta U := \{x \in U^c : \inf_{y \in U} |x - y| = 1\}$, where $|\cdot|$ denotes the Euclidean distance. So, for each finite subset $U \subset \mathbb{Z}$, we want to consider a process with state space $\Omega_0^w := \{\eta \in \Omega_0 : \eta(x) = 0 \, , x \in \mathcal{G}(w)\}$ defined formally by the infinitesimal generator,

$$
\begin{aligned}
L_U f(\eta) \quad := \quad & \sum_{x \in U \cap \mathcal{G}(w)^c} \sum_{y \in \mathcal{G}(w)^c : ||x-y||=1} \kappa \eta(x) (f(\eta^{x,y}) - f(\eta)) \\
+ \quad & \sum_{x \in U \cap \mathcal{G}(w)^c} \sum_{y \in \mathcal{G}(w) : ||x-y||=1} \kappa \eta(x) (f(\eta^{x,-}) - f(\eta)) \\
+ \quad & \sum_{x \in U \cap \mathcal{G}(w)^c} v_+(x) \eta(x) (f(\eta^{x,+}) - f(\eta)) \\
+ \quad & \sum_{x \in \mathcal{G}(w)^c} v_-(x) \eta(x) (f(\eta^{x,-}) - f(\eta)),
\end{aligned} \tag{7}
$$

acting on an appropriate dense subset $\mathcal{D}_U$ of the space of real bounded functions $B(\Omega_0^w)$, defined on Ω_0^w, endowed with the uniform norm. In the above expression, $\kappa \geq 0$, $\eta^{x,y}$ is the configuration where a particle from site x has jumped to site y so that $\eta^{x,y}(z) = \eta(z)$ if $z \neq x, y$, $\eta^{x,y}(x) = \eta(x) - 1$, and $\eta^{x,y}(y) = \eta(y) + 1$; $\eta^{x,+}$ is the configuration where there is an extra particle at site x and $\eta^{x,-}$ the configuration where one particle at site x has disappeared. It is a well known fact that it is possible to construct a strong Markov process, denoted by $\eta^U := \{\eta^U(t) : t \geq 0\}$, corresponding to an infinitesimal generator of the form (7), and taking values on the Skorokhod space $\mathcal{S} := D([0,\infty); \Omega_0^w)$. In fact, such a process falls in the category called *$|U|$-dimensional continuous time Markov branching process* by Athreya and Ney (see chapter V, sections 7.1-7.2 of Athreya-Ney [AN(1972)]). Furthermore, it can be shown that a.s. the expected value of each coordinate of such a process $\{\eta^U(t) : t \geq 0\}$, is finite, ensuring that there cannot be infinitely many particles produced in a finite time (see section 7.1 of [AN(1972)]). Let us now call $\mathcal{P}(\Omega_0^w)$ the set of probability measures defined on Ω_0^w endowed with the Borel σ-algebra associated to the subspace topology of Ω_0^w as a subset of Ω with the product topology. Then, for each field configuration $w \in W$ and probability measure $\nu \in \mathcal{P}(\Omega_0^w)$ denote by $P_\nu^{U,w}$ the law of the process $\{\eta^U(t) : t \geq 0\}$ defined on $\mathcal{S}$ endowed with its Borel σ-field $\mathcal{B}(\mathcal{S})$. We will call this process the *reaction-diffusion process on U* with field w and initial condition ν. In the particular case in which $U = \Lambda_n$ we will use the obvious notations L_n and $P_\nu^{n,w}$. Furthermore, we will call the process on Λ_n, the *reaction-diffusion process at scale n*. Now, note that using the natural coupling [Lg(1985)] and Kolmogorov's extension theorem, it is possible to define for each field configuration $w \in W$ and initial condition $\nu \in \mathcal{P}(\Omega_0^w)$ a probability measure Q_ν^w on the product space $\mathcal{S}^{\mathbb{N}}$, endowed with its Borel σ-algebra induced by the product topology, in such a way that if $\eta^n \in \mathcal{S}$ denotes the n-th coordinate of an element $\eta \in \mathcal{S}^{\mathbb{N}}$, $\eta^n(t) \in \Omega_0^w$ its value at time $t \geq 0$ and $\eta^n(t,x) \in \mathbb{N}$ the value at time t of the x-coordinate of $\eta^n(t)$, then,

(i) for every $A \in \mathcal{B}(\mathcal{S})$ and $n \geq 1$,

$$Q_\nu^w[\eta^n \in A] = P_\nu^{n,w}[A].$$

In particular, for every $B \in \mathcal{B}(\Omega_0)$ we have that, $Q_\nu^w[\eta^n(0) \in B] = \nu[B]$.

(ii) for every $n \geq 1$,

$$Q_\nu^w[\eta^{n+1}(t) \geq \eta^n(t)] = 1.$$

(iii) for each $n \geq 1$ define the first exit time from the box Λ_n as

$$T_n := \inf\{t \geq 0 : \sup_{x \in \Lambda_n^c} \eta^n(x,t) > 0\}.$$

Then, for every $n \geq 1$,

$$Q_\nu^w[\eta^n(t) = \eta^{n+1}(t), T_n > t] = Q_\nu^w[T_n > t].$$

Let us now remark that due to property (ii), for every $t \geq 0$ and $x \in \mathbb{Z}^d$ the limit,

$$\eta(t,x) := \lim_{n \to \infty} \eta^n(t,x),$$

exists, possibly taking the value ∞. Define $\eta(t) := \{\eta(t,x) : x \in \mathbb{Z}^d\}$. We denote the stochastic process $\{\eta(t) : t \geq 0\}$, taking values on the space $\bar{\mathbb{N}}^{\mathbb{Z}^d}$, where $\bar{\mathbb{N}}$ is the Alexandrov compactification of the natural numbers, and distributed according to the measure Q_ν^w, the *reaction-diffusion process with field w and initial law ν*. We will denote by E^w_ν the corresponding expectation. In addition, for each $t \geq 0$, we define the *total number of particles at time* t by

$$\zeta(t) := \sum_{x\in\mathbb{Z}^d} \eta(t,x).$$

Also, whenever it is true that,

$$Q^w_\nu \left[\eta(x,t) < \infty, \ \forall\, x \in \mathbb{Z}^d, \ t \geq 0 \right] = 1,$$

we will say that with probability one there is *no explosion*. In the sequel we define for each $x \in \mathbb{Z}^d$ the probability measure δ_x on $(\Omega^w_0, \mathcal{B})$ which assigns probability 1 to configurations with one particle at site x and none elsewhere. We will be interested in initial configurations where $\nu = \delta_x$. In such a case we will use the notation P^w_x instead of $P^w_{\delta_x}$ and E^w_x for the corresponding expectation. In the case where $x \in \mathcal{G}(w)$, we adopt the convention that P^w_x is the probability measure which has a unique atom at the configuration $\eta \equiv 0$ ($\eta(x) = 0$ for every $x \in \mathbb{Z}^d$).

Let us denote by $\mathcal{P}(W)$ the set of probability measures defined on the space $W = ([0,\infty) \times [0,\infty])^{\mathbb{Z}^d}$ endowed with its natural σ-algebra. In the sequel we will take fields v_+ and v_- which are random, assigning a probability measure $\mu \in \mathcal{P}(W)$ in such a way that the field configuration $\{w(x) : x \in \mathbb{Z}^d\}$ has independent coordinates with respect to μ. Furthermore, we will use the notation $\langle\cdot\rangle$ to denote expectation with respect to this law and $Var_\mu(\cdot)$ variance. Now, let us define the *quenched first moment* on $\mathbb{Z}^d$ of the total number of particles at time t starting from site x as, $m(x,t,w) := E^w_x[\zeta(t)]$, and the *annealed first moment* on $\mathbb{Z}^d$ of the total number of particles at time t starting from site x as, $\langle m(x,t)\rangle := \int m(x,t,w)d\mu$. Furthermore, we call the sets $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ and $\{\langle m(x,t)\rangle : x \in \mathbb{Z}^d\}$, the *fields of quenched first moments* and the *field of annealed first moments*, respectively. Depending on the context, we might write m or $m(x,t)$ in place of $m(x,t,w)$, dropping the dependence on the field configuration w, and $\langle m\rangle$ instead of $\langle m(x,t)\rangle$.

The quantity which will give us a transition mechanism between of the quenched first moments is the *averaged first moment* at scale L and time t defined for a reaction-diffusion process starting from site x as,

$$m^L(x,t,w) := \frac{1}{|\Lambda(x,L)|} \sum_{y\in\Lambda(x,L)} m(y,t,w).$$

2.2. **Results giving conditions for no explosion.** Here we will give a criteria on the field configuration w, stated as proposition 1, which ensures that there is no explosion in the reaction-diffusion process with field w.

Proposition 1. *Consider the reaction diffusion process with field w and initial law ν. Assume that,*

$$\limsup_{|x|\to\infty} \frac{v_+(x)}{|x|\log|x|} = 0. \tag{8}$$

Then for every $t \geq 0$ we have that

$$E^w_\nu[\zeta(t)] < \infty.$$

Hence, there is no explosion.

We state below with the name of proposition 2, a result of Gärtner-Molchanov [GM(1990)] (lemma 2.5) giving a sufficient condition on the law μ in order that the first moment of the total number of particles $\zeta(t)$ at time t, exists μ-a.s., and hence that there is no explosion. Given a field configuration $w = (v_+, v_-)$, we now need to introduce the *effective field* $\{v(x) : x \in \mathbb{Z}^d\}$, defined by $v(x) := v_+(x) - v_-(x)$. Furthermore, set $\log_+ x = \log x$ if $x > e$ and $\log_+ x = 1$ otherwise, while define the positive part $x^+ := \max(0, x)$.

Proposition 2. *Consider the reaction diffusion process with field w and initial law ν. Assume that the field configuration w is distributed according to a product probability measure $\mu \in \mathcal{P}(W)$. Suppose that condition* **(E)** *is satisfied. Then μ-a.s. it is true that,*

$$\limsup_{|x| \to \infty} \frac{v_+(x)}{|x| \log |x|} = 0,$$

and therefore, μ-a.s. there is no explosion for the reaction-diffusion process with field w and arbitrary initial law in $\mathcal{P}(\Omega_0)$.

Note that the last statement of proposition 2, follows from proposition 1.

2.3. **The Gaussian-Annealed transition results.** Here we will state the main result of this paper, which shows how under different growth of scales, the averaged first moment has an asymptotic behavior where a law of large numbers is satisfied, and a central limit theorem can describe the fluctuations around this law of large numbers. We will assume condition the existence of the annealed first moments (67). We will also need to consider the *growth functions* $\{H_1(t) = \log\langle m(0,t)\rangle : t \geq 0\}$ already defined in display (2) of the introduction and the *intermittency exponents* $\{F_\theta : \theta \in \mathbb{R}\}$, defined in display (3). Let us now state the main result of this paper.

Theorem 1. *For each $x \in \mathbb{Z}^d$ consider a reaction-diffusion process with initial law δ_x and field $w = (v_+, v_-)$ distributed according to a product measure $\mu \in \mathcal{P}(W)$. Consider the intermittency exponents $\{F_\theta : \theta \in \mathbb{R}\}$ defined in display (3) and the growth functions $\{H_1(t) : t \geq 0\}$ defined in display (2). Assume that conditions* **(E)** *and* **(MI)** *are satisfied. Then the following statements are true,*

(i) **Law of large numbers**. *Assume that there is an $\epsilon > 0$ such that eventually in t, $L(t) \geq \exp\left\{\frac{1}{d} F_\epsilon(t)\right\}$. Then in μ-probability we have*

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \sim 1, \tag{9}$$

as $t \to \infty$. Furthermore, assume that there is an $\epsilon > 0$ such that eventually in t, $L(t) \leq \exp\left\{\frac{1}{d} F_{-\epsilon}(t)\right\}$. Then in μ-probability we have

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \ll 1, \tag{10}$$

as $t \to \infty$.

(ii) **Central limit theorem**. *Assume that there is an* $\epsilon > 0$ *such that eventually in* t, $L(t) \geq \exp\left\{\frac{1}{d}F_\epsilon(2t)\right\}$. *Then,*

$$\lim_{t\to\infty} \frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{Var_\mu(m(0,t))}} = N(0,1), \tag{11}$$

where $N(0,1)$ *is a centered normalized normal law and the convergence is in the sense of distributions. Furthermore, assume that there is an* $\epsilon > 0$ *such that eventually in* t, $L(t) \leq \exp\left\{\frac{1}{d}F_{-\epsilon}(2t)\right\}$. *Then in* μ*-probability we have that,*

$$\frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{Var_\mu(m(0,t))}} \ll 1. \tag{12}$$

To provide a better insight on the meaning of theorem 1, we will see the form that it takes under the stronger assumption **(SI)**, which includes the first universality class of [HKM(2005)]. This will be formulated as a corollary, which in the case $\kappa = 0$ generalizes theorem 2.1 and 2.2 of [BBM(2005)] to include distributions μ of the field, which not necessarily have regularly varying log-tails.

Corollary 1. *For each* $x \in \mathbb{Z}^d$ *consider a reaction-diffusion process with initial law* δ_x *and field* $w = (v_+, v_-)$ *distributed according to a product measure* $\mu \in \mathcal{P}(W)$. *Consider the cumulant intermittency exponents* $\{G_\theta : \theta \in \mathbb{R}\}$ *defined in display (4) and the cumulant generating function* $\{H(t) : t \geq 0\}$ *defined in display (1). Assume that conditions* **(E)** *and* **(SI)** *are satisfied. Then the following statements are true,*

(i) **Law of large numbers**. *Assume that there is an* $\epsilon > 0$ *such that eventually in* t, $L(t) \geq \exp\left\{\frac{1}{d}G_\epsilon(t)\right\}$. *Then in* μ*-probability we have*

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \sim 1,$$

as $t \to \infty$. *In particular,* $\frac{\log m^L(0,t,w)}{H(t)} \sim 1$. *Furthermore, assume that there is an* $\epsilon > 0$ *such that eventually in* t, $L(t) \leq \exp\left\{\frac{1}{d}G_{-\epsilon}(t)\right\}$. *Then in* μ*-probability we have*

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \ll 1,$$

as $t \to \infty$.

(ii) **Central limit theorem**. *Assume that there is an* $\epsilon > 0$ *such that eventually in* t, $L(t) \geq \exp\left\{\frac{1}{d}G_\epsilon(2t)\right\}$. *Then,*

$$\lim_{t\to\infty} \frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{Var_\mu(m(0,t))}} = N(0,1),$$

where $N(0,1)$ is a centered normalized normal law and the convergence is in the sense of distributions. Furthermore, assume that there is an $\epsilon > 0$ such that eventually in t, $L(t) \leq \exp\left\{\frac{1}{d}G_{-\epsilon}(2t)\right\}$. Then in μ-probability we have that,

$$\frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{Var_\mu(m(0,t))}} \ll 1.$$

The proof of corollary 1 in the case $\kappa = 0$ follows from the observation that $G_\theta(t) = F_\theta(t)$ in this case. The case $\kappa > 0$ is a direct consequence of the fact that $e^{H(t)-2d\kappa t} \leq \langle m(0,t)\rangle \leq e^{H(t)}$, stated in theorem 3.1 of [GM(1990)], and the observation that $G_\theta(t) \geq 0$ for $\theta > 0$ and $G_\theta(t) \leq 0$ for $\theta < 0$, which follows from Jensen's inequality. Now, the following proposition, which will be proved in section 4, shows that the condition,

$$\lim_{t\to\infty} tH''(t) = \infty, \tag{13}$$

is sufficient for assumption **(SI)** to be true. This condition implies a kind of domination of the branching over the annihilation.

Proposition 3. *Consider the cumulant exponents $\{G_\theta : \theta \in \mathbb{R}\}$. Assume that condition (13) is satisfied. Then, condition* **(SI)** *is satisfied. Furthermore: (i) for every $\theta \neq 0$, there is a $t_0 \geq 0$, such that the function $G_\theta(t)$ is monotone in t, for $t \geq t_0$; (ii) there is a $t_1 \geq 0$, such that the function $G_\theta(t)$ is monotone in θ for $t \geq t_1$.*

Condition (13) implies that the branching dominates the annihilation, in the sense that $H(t)/t \to \infty$ as $t \to \infty$, which implies that the essential supremum of the random variable $v(0)$ is infinite (see [GM(1990)]).

2.4. **Regularity assumptions on the intermittency exponents.** . For the purpose of applications, it will be important to identify cases where the assumptions in theorem 1 can be formulated in a more explicit way. As it will be shown, the following assumption on the intermittency exponents, turns out to fall in one of these situations.

Assumption (RI). *The intermittency exponents $\{F_\theta(t) : \theta \neq \mathbb{R}\}$ satisfy the mild intermittency condition* **(MI)**. *In addition, there exist two increasing functions $f_1, f_2 : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ and a function $J(t) : [0,\infty) \to [0,\infty)$, such that for $\theta \neq 0$ small enough,*
(i) $F_\theta(t) \sim f_1(\theta)J(t)$, and $F_\theta(2t) \sim f_2(\theta)J(t)$.
(ii) There exists two constants γ_1 and γ_2, such that $\lim_{\theta\to 0} f_1(\theta) = \gamma_1$ and $\lim_{\theta\to 0} f_2(\theta) = \gamma_2$.

Throughout the sequel, the constants γ_1 and γ_2 will be called *transition exponents* and the function J *growth exponent.* It will be shown that there exist several important cases of random fields which fall in this category. Furthermore, the following corollary of theorem 1 shows the convenience of assumption **(RI)**.

Corollary 2. *For each $x \in \mathbb{Z}^d$ consider a reaction-diffusion process with initial law δ_x and field $w = (v_+, v_-)$ distributed according to a product measure $\mu \in \mathcal{P}(W)$. Suppose that assumptions* **(E)** *and* **(RI)** *are satisfied with transition exponents γ_1 and γ_2 and growth function J. Then the following statements are true,*

(i) **Law of large numbers**. *Assume that there is a $\gamma > \gamma_1$ such that eventually in t, $\log L(t) \geq \frac{1}{d}\gamma J(t)$. Then in μ-probability we have*

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \sim 1,$$

as $t \to \infty$. Furthermore, assume that there is a $0 < \gamma < \gamma_1$ such that eventually in t, $\log L(t) \leq \frac{1}{d}\gamma J(t)$. Then in μ-probability we have

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \ll 1,$$

as $t \to \infty$.

(ii) **Central limit theorem**. *Assume that there is a $\gamma > \gamma_2$ such that eventually in t, $\log L(t) \geq \frac{1}{d}\gamma J(t)$. Then,*

$$\lim_{t\to\infty} \frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{Var_\mu(m(0,t))}} = N(0,1),$$

where $N(0,1)$ is a centered normalized normal law and the convergence is in the sense of distributions. Furthermore, assume that there is a $0 < \gamma < \gamma_2$ such that eventually in t, $\log L(t) \leq \frac{1}{d}\gamma J(t)$. Then in μ-probability we have that,

$$\frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{Var_\mu(m(0,t))}} \ll 1.$$

In the next subsection we will apply corollary 2 to four situations each one falling in one of the universality classes described be van den Hofstad, König and Mörters in [HKM(2005)]. These classes encompass all possible situations under three conditions. The first condition ensures the existence of the positive moments defining the cumulant generating functions (1). For example, assumption **(E)**. The second and third condition avoids different qualitative behaviors of the potential at different scales. Le us formulate next the second condition of [HKM(2005)].

Assumption (H). *The function $H(t)/t$ is in the de Haan class.*

A function f is said to be in the de Haan class if for some regularly varying function $g : (0,\infty) \to \mathbb{R}$, we have that $(f(\lambda t) - f(t))/g(t)$ converges to a limit different from 0 as $t \to \infty$, for $\lambda > 0$. Let us recall that a function h is regularly varying at infinity with index ρ, if for any $a > 0$ we have $\lim_{x\to\infty} h(ax)/h(x) = a^\rho$. This property will be stated as $h \in R_\rho$. Whenever **(H)** is satisfied, then $H(t)$ is regularly varying with index $\gamma \geq 0$. Furthermore, in proposition 1.1 of [HKM(2005)], it is proven that under assumption **(H)** there exist a function $\hat{H} : (0,\infty) \to \mathbb{R}$ and a continuous function $k(t) : (0,\infty) \to (0,\infty)$ such that,

$$\lim_{t\to\infty} \frac{H(ty) - yH(t)}{k(t)} = \hat{H}(y) \neq 0, \tag{14}$$

for $y \in (0,1) \cup (1,\infty)$. It is also shown that $k(t)$ is regularly varying of index γ. We can now recall the third assumption of [HKM(2005)].

Assumption (K). *The limit* $k^* = \lim_{t\to\infty} \frac{k(t)}{t}$ *exists in* $[0,\infty]$.

Under assumptions **(E)**, **(H)** and **(K)**, the four universality classes defined in [HKM(2005)] are:

(1) $\gamma > 1$, or $\gamma = 1$ and $k^* = \infty$.
(2) $\gamma = 1$ and $k^* \in (0,\infty)$.
(3) $\gamma = 1$ and $k^* = 0$.
(4) $\gamma < 1$.

In what follows we exhibit examples in each one of these classes satisfying assumption **(RI)** so that corollary 2 can be applied, and the transition exponents can be explicitly written. In the sequel, following [HKM(2005)], we will call the third class the class of *almost bounded potentials.*

2.4.1. *Unbounded potentials with Weibull type tails.* Our first application of corollary 1 will be to an example falling in the first universality class of [HKM(2005)]. We will assume that the essential supremum of $v(0)$ is ∞, and the tails at ∞ of $v(0)$ follow a Weibull-type law, $\mu[v(0) > x] = \exp\{-h(x)\}$ for $x > 0$, with $h \in R_\rho$ for some $1 < \rho < \infty$. In the terminology of [BBM(2005)] in the context of i.i.d. random exponentials ($\kappa = 0$ in our situation), this is called *Case* **B**. The following proposition shows that assumption **(RI)** is satisfied in this situation, and hence corollary 2, which generalizes theorems 2.1 and 2.2 of [BBM(2005)] in case **B** form $\kappa = 0$ to $\kappa > 0$.

Proposition 4. *Suppose that the essential supremum of the effective potential* $v(0)$ *is* ∞ *with Weibull-type tails* $\mu[v(0) > x] = \exp\{-h(x)\}$ *for* $x > 0$*, and* $h \in R_\rho$ *for some* $1 < \rho < \infty$*. Then assumption* **(RI)** *is satisfied with transition exponents*

$$\gamma_1 = \frac{1}{\rho - 1}, \qquad \gamma_2 = 2^{\frac{\rho}{\rho-1}} \frac{1}{\rho - 1},$$

and growth exponent,

$$J(t) = H(t).$$

Let us now prove proposition 4. Note that under the conditions on the tail of the law of $v(0)$, the cumulant generating function $H(t)$ of $v(0)$ is well defined, smooth, non-decreasing and tends to infinity as $t \to \infty$. Furthermore, by the Kasahara exponential Tauberian theorem (see Bingham, Goldie and Teugels [BGT(1989)], Theorem 4.12.7), we know that $H \in R_{\rho'}$, where the index ρ' is defined by the equation,

$$\frac{1}{\rho}+\frac{1}{\rho'}=1.$$

From this observation it is easy to check that assumption **(SI)** of corollary 1 is satisfied, and that for every $\epsilon \neq 0$, the cumulant growth exponents $G_\epsilon(t)$ and $G_\epsilon(2t)$ satisfy

$$\frac{G_\epsilon(t)}{H(t)} \sim \frac{(1+\epsilon)^{\rho'}-(1+\epsilon)}{\epsilon}$$

and

$$\frac{G_\epsilon(2t)}{H(t)} \sim 2^{\rho'}\frac{(1+\epsilon)^{\rho'}-(1+\epsilon)}{\epsilon}.$$

Now note that $\frac{(1+\epsilon)^{\rho'}-(1+\epsilon)}{\epsilon}$ is increasing in ϵ and converges to $\gamma_1=\frac{1}{\rho-1}$ as $\epsilon \to 0$, and similarly $2^{\rho'}\frac{(1+\epsilon)^{\rho'}-(1+\epsilon)}{\epsilon}$ is increasing in ϵ and converges to $\gamma_2 = 2^{\frac{\rho}{\rho-1}}\gamma_1$ as $\epsilon \to 0$.

2.4.2. *Unbounded potentials with double exponential type tails.* Here we consider the second universality class so that H is regularly varying with index $\gamma = 1$ and $k^* \in (0,\infty)$. As shown in [HKM(2005)] in proposition 1.1, this is equivalent to the existence of a constant $\rho \in (0,\infty)$ such that

$$\lim_{t\to\infty}\frac{H(yt)-yH(t)}{t}=\rho y\log y, \tag{15}$$

for all $y \in (0,1)\cup(1,\infty)$ (this and $0<\rho<\infty$ is called assumption **(H)** in [GM(1998)]). Furthermore, under assumption (15) it is true that,

$$\lim_{t\to\infty}\frac{H(t)}{t}=\infty.$$

As pointed out in page 22 of [GM(1990)], this second universality class includes the case of unbounded potentials which are double exponentially distributed with parameter ρ, $0<\rho<\infty$,

$$\mu[v(0)>x]=\exp\left\{-e^{x/\rho}\right\},$$

for $x \in \mathbb{R}$. We then have the following interesting proposition.

Proposition 5. *Suppose that assumption (15) is satisfied for some* $\rho \in (0,\infty)$*. Then assumption* **(RI)** *is satisfied with transition exponents*

$$\gamma_1=\rho, \qquad \gamma_2=2\rho,$$

and growth exponent,

$$J(t)=t.$$

To prove proposition 5, we quote theorem 1.2 of [GM(1998)], which shows that under assumption (15) we have that,

$$\langle m(0,t)\rangle = \exp\left\{H(t) - 2d\kappa\chi\left(\frac{\rho}{\kappa}\right)t + o(t)\right\},$$

for $\kappa > 0$, where $\chi(x) := \frac{1}{2}\inf_{p\in\mathcal{P}(\mathbb{Z})}[S(p) + \rho I(p)]$ for $x \geq 0$, $\mathcal{P}(\mathbb{Z})$ is the space of probability measure on $\mathbb{Z}$, $S : \mathcal{P}(\mathbb{Z}) \to [0,\infty)$ is the Donsker-Varadhan functional defined by $S(p) := \sum_{x\in\mathbb{Z}}\left(\sqrt{p(x+1)} - \sqrt{p(x)}\right)^2$, while $I : \mathcal{P}(\mathbb{Z}) \to [0,\infty)$ is the entropy functional defined by $I(p) := -\sum_{x\in\mathbb{Z}} p(x)\log p(x)$. On the other hand, from the previous discussion we can conclude that, for every $\epsilon \neq 0$,

$$\frac{F_\epsilon(t)}{t} \sim \rho(1+\epsilon)\frac{\log(1+\epsilon)}{\epsilon},$$

and

$$\frac{F_\epsilon(2t)}{t} \sim 2\rho(1+\epsilon)\frac{\log(1+\epsilon)}{\epsilon}.$$

From these limiting behaviors, we see that assumption **(MI)** is satisfied. Furthermore, form the fact that $(1+\epsilon)\frac{\log(1+\epsilon)}{\epsilon}$ is increasing for ϵ small enough and converges to 1 as $\epsilon \to 0$, we obtain the transition exponents at ρ and 2ρ of proposition 5.

2.4.3. *Almost bounded potentials.* We now focus on the third universality class, where $\gamma = 1$ and $k^* = 0$. As shown in theorem 1.4 of [HKM(2005)], in this case it is true that,

$$\log\langle m(0,t)\rangle \sim \frac{H(t\alpha_t^{-d})}{\alpha_t^{-d}}, \tag{16}$$

where $\alpha_t : [0,\infty) \to [0,\infty)$ is the so called *scaling function*, which is implicitly defined for all $t > 0$ sufficiently large by the equation,

$$\frac{k(t\alpha_t^{-d})}{t\alpha_t^{-d}} = \frac{1}{\alpha_t^2}. \tag{17}$$

As shown in proposition 1.2 of [HKM(2005)], this function is unique up to asymptotic equivalence. Furthermore, for the third universality class, it is a slowly varying function (regularly varying of index 0).

Proposition 6. *Suppose that H is regularly varying of index 1 and that (14) is satisfied for $k(t)$ such that $k(t) \ll t$. Then assumption* **(RI)** *is satisfied with transition exponents*

$$\gamma_1 = \rho, \qquad \gamma_2 = 2\rho,$$

and growth exponent,

$$J(t) = \frac{t}{\alpha_t^2}. \tag{18}$$

The proof of proposition 6 follows now applying the definition of α_t through (17), the asymptotics (16) and (14).

A specific example of a distribution falling in the third universality class is given by the *squared double exponential* law. In other words, an effective potential $v(0)$ which is unbounded, with law,

$$\mu[v(0) > x] = \exp\left\{-e^{x^2}\right\},$$

for $x \geq 0$. As it can be deduced from the discussion of example 1.4.3 of [HKM(2005)], in this case the scale function is given by $\alpha_t \sim 2^{1/2}(\log t)^{1/4}$ and $k(t) \sim \frac{t}{2\sqrt{\log t}}$. Furthermore, $\rho = 1$ and the growth exponent can be chosen as $J(t) \sim t/(2(\log t)^{1/2})$ (see table 2 of the introduction).

2.4.4. *Bounded potentials with Fréchet type tails.* We now continue with an example falling in the universality class (4) of [HKM(2005)]. We will consider the case in which the essential supremum of $v(0)$ is 0 and the tails are of Fréchet-type: $\mu[v(0) > -x] = \exp\left\{-h(x^{-1}\right\}$ for $x > 0$, with $h \in R_\rho$ for some $0 < \rho < \infty$. Using the terminology of [BBM(2005)] in the context of i.i.d. random exponentials, this is *Case* **A**. To state appropriately this result, we will need to recall the work of Biskup and König [BK(2001)], who studied the asymptotics of the annealed and quenched first moments in the case of product environments μ such that the cumulant generating function $H(t)$ is in the so called γ-class for some $\gamma \in [0, 1)$. We say that H *is in the* γ*-class* if the essential supremum of $v(0)$ is 0 and if there is a non-decreasing function $\alpha_t \in (0, \infty)$ and a function $\tilde{H} : [0, \infty) \to (-\infty, 0]$, $\tilde{H} \neq 0$, such that

$$\lim_{t\to\infty} \frac{\alpha_t^{d+2}}{t} H\left(\frac{t}{\alpha_t^d} y\right) = \tilde{H}(t), \tag{19}$$

for $y \geq 0$, uniformly on compact sets in $(0, \infty)$. We will denote the function α_t the *scale function.* It is not difficult to show, using the de Bruijn exponential Tauberian theorem [BGT(1989)], that if $v(0)$ is in case **A** for some $\rho > 0$, then H is in the ρ'-class for $\rho' = \frac{\rho}{\rho+1}$, or $\frac{1}{\rho'} - \frac{1}{\rho} = 1$. We can now sate the following proposition.

Proposition 7. *Suppose that the essential supremum of the effective potential* $v(0)$ *is* 0 *with Fréchet-type tails* $\mu[v(0) > -x] = \exp\{-h(x^{-1})\}$ *for* $x > 0$*, and* $h \in R_\rho$ *for some* $0 < \rho < \infty$*. Then assumption* **(RI)** *is satisfied with transition exponents*

$$\gamma_1 = \left(\frac{1}{d+2+2\rho}\right)^2, \qquad \gamma_2 = 2^{1-\gamma_1}\gamma_1,$$

and growth exponent,

$$J(t) = \chi \frac{t}{\alpha_t^2},$$

for some constant $\chi \in (0, \infty)$.

As a consequence of proposition 7 we can now apply corollary 2, generalizing case **A** of theorems 2.1 and 2.2 of [BBM(2005)] from $\kappa = 0$ to $\kappa \geq 0$. In contrast to proposition 4, where there is no change in the value of the transition exponents γ_1 and γ_2 from $\kappa = 0$ to $\kappa > 0$, here there is.

Let us now prove proposition 7. In proposition 2.1 of [BK(2001)], it is shown that whenever μ is in the ρ'-class, then the scaling function $\alpha_t \in R_\nu$, for,

$$\nu := \frac{1-\rho'}{d+2-d\rho'}.$$

Furthermore, theorem 1.2 of [BK(2001)] states that when μ is in the ρ'-class, there exists a $\chi \in (0,\infty)$ such that,

$$\log\langle m(0,t)^\beta\rangle \sim -\chi \frac{\beta t}{\alpha_{\beta t}^2},$$

for every $\beta \in (0,\infty)$. Then condition **(MI)** of theorem 1 is satisfied, and for every $\epsilon \neq 0$, the growth exponents $F_\epsilon(t)$ and $F_\epsilon(2t)$ satisfy

$$\frac{F_\epsilon(t)}{t/\alpha_t^2} \sim \frac{(1+\epsilon)-(1+\epsilon)^{1-\nu^2}}{\epsilon}$$

and

$$\frac{F_\epsilon(2t)}{t/\alpha_t^2} \sim 2^{1-\nu^2}\frac{(1+\epsilon)-(1+\epsilon)^{1-\nu^2}}{\epsilon}.$$

As in the proof of proposition 4, we can show that that $\frac{(1+\epsilon)-(1+\epsilon)^{1-\nu^2}}{\epsilon}$ is increasing in ϵ and converges to $\gamma_1 = \nu^2$ as $\epsilon \to 0$, and similarly $2^{1-\nu^2}\frac{(1+\epsilon)-(1+\epsilon)^{1-\nu^2}}{\epsilon}$ is increasing in ϵ and converges to $\gamma_2 = 2^{1-\nu^2}\gamma_1$ as $\epsilon \to 0$.

2.5. **The critical regime.** For the examples discussed in the four previous sections, it is possible to obtain the following improvement of theorem 1.

Theorem 2. *For each $x \in \mathbb{Z}^d$ consider a reaction-diffusion process with initial law δ_x and field $w = (v_+, v_-)$ distributed according to a product measure $\mu \in \mathcal{P}(W)$. Then the following statements are satisfied.*

(i) **Weibull-type**. *Assume that μ has Weibull-type tails so that $\mu[v(0) > x] = \exp\{-h(x)\}$ for $x > 0$, $h \in R_\rho$ for $1 < \rho < \infty$. Then if $d \log L(t) \leq \gamma H(t)$ eventually in t, and $0 < \gamma < \gamma_1 = \frac{1}{\rho-1}$, we have that for every $\delta > 0$ in μ-probability,*

$$\frac{m_L(0,t)}{e^{(a_W(\gamma)+\delta)H(t)}} \ll 1,$$

where

$$a_W(\gamma) := \frac{\rho}{\rho-1}[(\rho-1)\gamma]^{1/\rho} - \gamma.$$

(ii) **Double exponential-type**. *Assume that μ satisfies assumption (15) for some constant $\rho \in (0,\infty)$. Then if, $d\log L(t) \leq \gamma t$ eventually in t, and $\gamma < \gamma_1 = \rho$, we have that for every $\delta > 0$ in μ-probability,*

$$\frac{m_L(0,t)}{\exp\left\{\frac{H((a_D(\gamma)+\delta)t)}{a_D(\gamma)+\delta}\right\}} \ll 1,$$

where

$$a_D(\gamma) := \gamma e^{\frac{1}{\rho}(\gamma-\rho)}.$$

(iii) **Almost bounded potentials**. *Assume that μ is such that H is regularly varying of index 1 and that (14) is satisfied for $k(t)$ such that $k(t) \ll t$. Consider the growth exponent $J(t)$ as defined in display (18). Then if, $d\log L(t) \leq \gamma J(t)$ eventually in t, and $\gamma < \gamma_1 = 1$, we have that for every $\delta > 0$ in μ-probability,*

$$\frac{m_L(0,t)}{\exp\left\{\frac{H((a_A(\gamma)+\delta)t)}{a_D(\gamma)+\delta}\right\}} \ll 1,$$

where

$$a_A(\gamma) := \gamma e^{\frac{1}{\rho}(\gamma-\rho)}.$$

(iv) **Fréchet-type**. *Assume that μ is such that essup$v(0) = 0$ and is of Fréchet-type so that $\mu[v(0) > -x] = \exp\left\{-h(x^{-1})\right\}$ for $x > 0$, and $h \in R_\rho$ for some $\rho \in (0,\infty)$. Then if $d\log L(t) \leq \gamma J(t)$ eventually in t, and $\gamma < \gamma_1 = \nu^2$, we have that for every $\delta > 0$ in μ-probability,*

$$\frac{m_L(0,t)}{e^{-(a_F(\gamma)-\delta)J(t)}} \ll 1,$$

where

$$a_F(\gamma) := (1-\nu^2)\left(\frac{\gamma}{\nu^2}\right)^{-\frac{\nu^2}{1-\nu^2}} + \gamma.$$

Let us remark that part (iv) of theorem 2 includes as a particular case, **Case 3** of part (i) of theorem 2 of [BMR(2005)]. We believe that the four functions a_W, a_D, a_A, a_F are sharp, in the sense that the quantities of the four parts of theorem 2 diverge if the sign of δ is changed. Also, these four functions have as maximum value 1, which is reached at γ_1.

In the special case in which μ is a double exponential law so that $\log\mu[v(0) > x] = -e^{x^2}$ in part (ii) of theorem 2, the function $\exp\left\{\frac{H((a_D(\gamma)+\delta)t)}{a_D(\gamma)+\delta}\right\}$ takes the form $\exp\left\{t\log\left(\frac{(a_D(\gamma)+\delta)t}{e}\right)\right\}$, showing how the transition mechanism takes place at a logarithmic order in the exponent in contrast to the polynomial one of parts (i) and (iv). The whole picture suggested by theorem 2 seems to indicate the presence of a phase transition type behavior, as it is found in some mean field statistical mechanics models like the Random Energy Model [D(1981)], [BKL(2002)]. Indeed, when combined with part (i) of theorem 1, we conclude that $\log m_L(0,t,w) \sim \bar{a}(\gamma)J(t)$, where $\bar{a}(\gamma)$ equals a_W, a_D, a_A or a_F depending on the potential for $\gamma < \gamma_1$, while $\bar{a}(\gamma) = 1$ for $\gamma > 1$. Thus, there is non-analyticity at $\gamma = 1$ of a quantity playing the role of a "free energy".

3. The conditions for no explosion

In this section we will prove proposition 1. Since the initial conditions $\nu \in \mathcal{P}(V)$ are concentrated on configurations with a finite number of particles, and by translation invariance of the dynamics of the reaction-diffusion process on $\mathbb{Z}^d$, note that it is enough to consider the case where $\nu = \delta_0$.

3.1. **Preliminary lemmas.** Let us consider the reaction-diffusion process at scale n with field w satisfying condition (8), and with the initial condition δ_0. Define the quantities,

$$\zeta^n(t) := \sum_{x\in\Lambda_n} \eta^n(t,x),$$

representing the total number of particles produced at time t and,

$$\bar{\zeta}^n(t) := \sum_{x\in\Lambda_n^c} (\eta^n(t,x) - \eta^n(0,x)),$$

representing the total number of particles which have touched Λ_n^c in the time interval $[0,t]$. From this definition, we can conclude that for $m > n \geq 1$ it is true that,

$$\zeta^m(t) = \zeta^n(t) + \sum_{k=1}^{\bar{\zeta}^n(t)} \zeta^m_{x_k}(t-\tau_k), \tag{20}$$

where for $1 \leq k \leq \bar{\zeta}^n(t)$, $x_k \in \delta\Lambda_n$ is the set of exit sites from Λ_n of the random walks which have touched the set Λ_n^c in the time interval $[0,t]$, $0 \leq \tau_k \leq t$ the exit times of each one of these random walks and $\{\zeta^m_{x_k}(s) : s \geq 0\}$ is a set of independent processes such that $\zeta^m_{x_k}(s) = \delta_{x_k}$ for $s \leq 0$ while $\{\zeta^m_{x_k}(s) : s \geq 0\}$ has the law $P^{m,w}_{x_k}$. Let us now define for each $n \in \mathbb{N}$ the maximum value of the field v_+ on the box Λ_n by $v_n := \max_{x\in\Lambda_n} v_+(x)$.

Lemma 1. *Consider the reaction-diffusion process at scale n with field w and initial condition δ_0. Then,*

(i) *For every $t \geq 0$ and $n \geq 1$,*

$$E_0^w\left[\zeta^n(t)\right] \leq \exp\left\{v_n t\right\}.$$

(ii) *For every $t \geq 0$ and $n \geq \frac{t}{2d}$,*

$$E_0^w\left[\bar{\zeta}^n(t)\right] \leq 4d \exp\left\{(v_n - 2\kappa)\, t - n\log\left(\frac{n}{2e\kappa t}\right)\right\}.$$

(iii) *For every $t \geq 0$ and $m > n \geq 1$,*

$$E_0^w\left[\zeta^m(t)\right] \leq E_0^w\left[\zeta^n(t)\right] + E_0^w\left[\bar{\zeta}^n(t)\right] E_0^w\left[\zeta^{2n+m}(t)\right].$$

The following elementary lemma will be used to prove lemma 1. We will need to define,

$$I(y) := \sup_{\lambda\in\mathbb{R}}\{\lambda y - (\cosh\lambda - 1)\} = y\sinh^{-1}y - \sqrt{1+y^2} + 1, \tag{21}$$

Note that $I : [0,\infty) \to [0,\infty)$ is one to one and that $I(x) > 0$ for $x > 0$.

Lemma 2. *Let $\{X_t : t \geq 0\}$ be a simple symmetric continuous time random walk on $\mathbb{Z}$ of total jump rate $2\kappa > 0$ starting from 0. For a non-negative real x define $\tau_x := \inf\{t \geq 0 : |X_t| \geq x\}$ as the first exit time of this random walk from the interval Λ_x. Then, if P is its law, we have*

$$P[\tau_x < t] \leq 4\exp\left\{-2\kappa t I\left(\frac{x}{2\kappa t}\right)\right\} \leq 4\exp\left\{-2\kappa t - x\log\left(\frac{x}{2e\kappa t}\right)\right\}. \tag{22}$$

The proof of lemma 2 is a simple large deviation estimate and will be omitted. So let us proceed with the proof of lemma 1.

Proof of lemma 1. Part (i). Let us remark that for every bounded non-decreasing function $f : \mathbb{N} \to \mathbb{R}$ which is eventually constant we have,

$$L_n f(\zeta^n) \leq v_n\zeta^n(f(\zeta^n+1) - f(\zeta^n)).$$

For a natural $N \geq 1$ fixed, choose $f(m) = m \wedge N$. Using the fact that $\zeta^n(0) = 1$, we then conclude that,

$$\zeta^n(t)\wedge N - 1 - v_n\int_0^t \zeta^n(s)\theta_{[0,N-1]}(\zeta^n(s))ds, \tag{23}$$

where for $A \subset \mathbb{R}$, θ_A is the indicator function of the set A, is a super-martingale. Hence, since the integrand of the integral in (23) is a positive function, by Fubini's theorem,

$$E_0^w\left[\zeta^n(t)\theta_{[0,N-1]}(\zeta^n(t))\right] \leq 1 + v_n\int_0^t E_0^w\left[\zeta^n(s)\theta_{[0,N-1]}(\zeta^n(s))\right]ds.$$

Therefore, by Gronwall's lemma,

$$E_0^w\left[\zeta^n(t)\theta_{[0,N-1]}(\zeta^n(t)\right] \leq \exp\{v_n t\}.$$

Taking the limit when $N \to \infty$ and using the monotone convergence theorem we conclude the proof of part (i) of the lemma.

Part (ii). Let us note the following identity,

$$E_0^w\left[\bar{\zeta}^n(t)\right] = E_0^w\left[\zeta^n(t)\right]P[\tau_n < t], \tag{24}$$

where τ_n is the first exit time of a simple symmetric continuous time random walk of total jump rate $2d\kappa$, starting from the origin 0, from the box Λ_n and P is its law. Now, from the second inequality of display (22) of lemma 2 applied to each of the d coordinates of such a random walk, we conclude that,

$$P[\tau_n < t] \leq 4d \exp\left\{-2\kappa t - n\log\left(\frac{n}{2e\kappa t}\right)\right\},$$

substituting the corresponding expression back in (24) and using part (i) of the proposition we conclude the proof of the lemma.

Part (iii). From (20) we have that,

$$E_0^w[\zeta^m(t)] = E_0^w[\zeta^n(t)] + E_0^w\left[\sum_{k=1}^{\bar{\zeta}^n(t)} E_0^w[\zeta^m_{x_k}(t-\tau_k)|\bar{\zeta}^n(t)]\right].$$

But, $E_0^w[\zeta^m_{x_k}(t-\tau_k)|\bar{\zeta}^n(t)] \leq E_0^w[\zeta^{2n+m}(t)]$, which concludes the proof. □

3.2. **Proof of proposition 1.** We will now prove proposition 1 with the help of lemma 1. Let $\delta = \frac{1}{5}$ and choose N so that $v_n \leq \frac{\delta}{t}\left(n\log\left(\frac{n}{2e\kappa t}\right) - 4n\log 4\right)$ whenever $n \geq N$. By part (i) of the lemma we have that,

$$E_x^w[\zeta^n(t)] \leq \exp\left\{\delta\left(n\log\left(\frac{n}{2e\kappa t}\right) - 4n\log 4\right)\right\},$$

while by part (ii) we have,

$$E_x^w[\bar{\zeta}^n(t)] \leq 4d\exp\left\{-(1-\delta)\,n\log\left(\frac{n}{2e\kappa t}\right) - 4\delta n\log 4\right\},$$

whenever $n \geq N$. Choosing $m = 2n > N$ in part (iii) of the same lemma, it follows that,

$$E_x^w[\zeta^{2n}(t)] \leq A_n + B_n E_x^w[\zeta^{4n}(t)],$$

where $A_n := e^{\delta n\log\left(\frac{n}{2e\kappa t}\right)}$ and $B_n := 4de^{-(1-\delta)n\log\left(\frac{n}{2e\kappa t}\right)-4\delta n\log 4}$. Repeating the bound for $2n$ and $m = 4n$ and substituting back we get,

$$E_x^w[\zeta^{2n}(t)] \leq A_n + B_n A_{2n} + B_n B_{2n} E_x^w[\zeta^{8n}(t)].$$

Now, by induction on m, we get that,

$$E_x^w[\zeta^{2n}(t)] \leq \sum_{k=0}^{m-1} c_k + c_m \frac{E_x^w\left[\zeta^{n2^{m+1}}(t)\right]}{A_{n2^m}}, \tag{25}$$

where $c_0 := A_n$ and for $k \geq 0$, $c_{k+1} := c_k \frac{B_{n2^k}A_{n2^{k+1}}}{A_{n2^k}}$. Now,

$$\frac{B_{n2^k}A_{n2^{k+1}}}{A_{n2^k}} \leq 2de^{-(1-2\delta)n2^k\log\left(\frac{n2^k}{2e\kappa t}\right)}.$$

Hence, by d'Alambert test and the fact that $1 - 2\delta > 0$ we know that the series $\sum_{k=0}^{\infty} c_k$ is convergent. On the other hand we have,

$$c_m \frac{E_x^w\left[\zeta^{n2^{m+1}}(t)\right]}{A_{n2^m}} \le c_{m-1} e^{\delta n 2^m \log\left(\frac{n2^m}{2e\kappa t}\right) - (1-2\delta) n 2^{m-1} \log\left(\frac{n2^{m-1}}{2e\kappa t}\right)},$$

which tends to 0 since $2\delta < 1-2\delta$ and $c_m < 1$ for m large enough. Taking the limit when $m \to \infty$, then when $n \to \infty$ and using the monotone convergence theorem in inequality (25), we deduce that

$$E_x^w[\zeta(t)] \le \sum_{k=0}^{\infty} c_k < \infty.$$

4. Moment and correlation estimates

Here we will obtain some important bounds for the large time asymptotic behavior of the field of quenched first moments $\{m(x,t,w) : x \in \mathbb{Z}^d\}$, the annealed first moment field $\{\langle m(x,t)\rangle : x \in \mathbb{Z}^d\}$, and their correlations. In the first subsection, we will prove proposition 3.

4.1. **Proof of proposition 3.** Our first lemma states a useful super-additivity and convexity property of the cumulant generating function of the random variable $v(0)$.

Lemma 3. *Consider the cumulant generating function $H(t) : [0,\infty) \to \mathbb{R}$ of the random variable $v(0)$, defined in display (1). Then, the following statements are true.*

(i) *H is super-additive. In other words, if $t_1, \dots, t_n$ are non-negative reals then,*

$$H(t_1 + \cdots + t_n) \ge H(t_1) + \cdots + H(t_n).$$

(ii) *Assume that $\lim_{t\to\infty} tH''(t) = \infty$. Then, for every $\alpha > 1$,*

$$\lim_{t\to\infty} \frac{H(\alpha t) - \alpha H(t)}{t} = \infty. \tag{26}$$

Proof. Part (i). Let $t_1, t_2 \ge 0$. From Hölder's inequality, we have that $\phi(t_1)\phi(t_2) \le \phi(t_1 + t_2)$. Hence, $H(t_1 + t_2) \ge H(t_1) + H(t_2)$. By induction on n we conclude the proof.

Part (ii). From the assumption, note that we can write $H''(t) = \frac{f(t)}{t}$, where $\lim_{t\to\infty} f(t) = \infty$. Integrating the function H'' from t to αt, it follows that,

$$H'(\alpha t) - H'(t) \ge \inf_{s\ge t} f(s) \cdot \log \alpha. \tag{27}$$

Integrating again we obtain for $u > t$ that,

$$H(\alpha t) - \alpha H(t) \ge (t-u)\inf_{s\ge u} f(s) \cdot \alpha \log \alpha + c(u),$$

where $c(u) = H(\alpha u) - \alpha H(u)$. Dividing by t, taking the limit when $t \to \infty$ and then the limit when $u \to \infty$, we obtain (26).

□

Let us now prove parts (i) and (ii) of proposition 3. Note that,

$$\frac{\partial G}{\partial t} = \frac{(1+\theta)H'((1+\theta)t) - (1+\theta)H'(t)}{\theta}.$$

Then, inequality (27) implies part (i) of the lemma. On the other hand we have that,

$$\frac{\partial G}{\partial \theta} = \frac{\theta t H'((1+\theta)t) - H((1+\theta)t) + H(t)}{\theta^2},$$

By the mean value theorem there exists a $\bar{\theta}$ such that $0 < \bar{\theta} < \theta$ and $H((1+\theta)t) - H(t) = \theta t H'((1+\bar{\theta})t)$ and hence $\theta t H'((1+\theta)t) - H((1+\theta)t) + H(t) = \theta t \left[H'((1+\theta)t) - H'((1+\bar{\theta})t)\right]$, which by inequality (27) is positive if $t \geq t_1$, where t_1 is independent of θ and $\bar{\theta}$. This proves that $G_\theta(t)$ is monotone in θ for $t \geq t_1$.

We now show that condition (13) ensures **(SI)**. We will without loss of generality assume that $\kappa > 0$. Let us define real valued functions f, g by $f(x) := H(\theta t + xt) - H(xt)$ and $g(x) := 2x$ for real x. By the generalized mean value theorem applied in the interval $[1, 1+\theta]$, there exists a $\theta_1 \in (0, \theta)$, such that $\frac{f(1+\theta)-f(1)}{g(1+\theta)-g(1)} = \frac{f'(1+\theta_1)}{g'(1+\theta_1)}$. In other words, the expression $G_{2\theta}(t) - G_\theta(t) = \frac{H((1+2\theta)t) - 2H((1+\theta)t) + H(t)}{2\theta}$, equals,

$$\frac{t}{2}\left(H'((1+\theta+\theta_1)t) - H'((1+\theta_1)t)\right) = \frac{\theta t^2}{2} H''((1+\theta_1+\theta_2)t),$$

where in the last equality we have applied the mean value theorem and $\theta_2 \in (0, \theta)$. It therefore follows that there is a function $\bar{\theta} : [0, \infty) \to (0, 2\theta)$ such that,

$$\frac{G_{2\theta}(t) - G_\theta(t)}{t} = \frac{\theta t}{2} H''((1+\bar{\theta}(t))t).$$

Our hypothesis $\lim_{t\to\infty} tH''(t) = \infty$, shows that the expression above tends to ∞ as $t \to \infty$. The proof of the case in which $\theta < 0$ is similar and the details will be omitted.

We now continue in subsection 4.2, defining the truncated quenched first moments, and then describing the parabolic Anderson equation satisfied by the quenched first moments and the corresponding Feynman-Kac representations.

4.2. **Truncated quenched first moments.** In the sequel, given a real function $f(x)$ defined on $\mathbb{Z}^d$, we will define the discrete Laplacian by,

$$\Delta f(x) := \sum_{e \in \mathbb{Z}^d : |e|=1} \left(f(x+e) - f(x)\right). \tag{28}$$

Let us now for each finite set $U \subset \mathbb{Z}^d$ and environment $w \in W$, define the field $w^U := (v_-^U, v_+)$ with $v_-^U(x) = v_-(x)$ for $x \in U$, while $v_-^U(x) = \infty$ for $x \notin U$. We now define for $x \in \mathbb{Z}^d$ and $t \geq 0$,

$$\tilde{m}_U(x, t, w) := m(x, t, w^U).$$

As it will be seen later, this expression satisfies the parabolic Anderson equation with Dirichlet boundary conditions. We will denote this quantity the *truncated quenched first moment* on U at time t for a reaction-diffusion process starting from x. Also, we will call the set $\{\tilde{m}_U(x,t,w) : x \in \mathbb{Z}^d\}$, the *field of truncated first moments* on U at time t. Now, in the particular case in which $U = \Lambda(x,r)$ for some $r > 0$, we will use the notation $\tilde{m}_r(x,t,w)$ instead of $\tilde{m}_U(x,t,w)$. We will refer to this quantity as the *truncated quenched first moment* at scale r at time t for a reaction-diffusion process starting from site x. Furthermore, we will call the sets $\{\tilde{m}_r(x,t,w) : x \in \mathbb{Z}^d\}$, the *field of truncated quenched first moments* at scale r at time t.

4.3. **The parabolic Anderson equations.** Here we will recall the moment equations satisfied by the field of quenched first moments $\{m(x,t,w)\}$ and by the corresponding truncated fields. Following [GM(1990)], we have the proposition.

Proposition 8. *Let $U \subset \mathbb{Z}^d$ be a finite set and $w \in W$ an environment. Consider the field of quenched first moments $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ on $\mathbb{Z}^d$ at time t and the field of truncated quenched first moments $\{\tilde{m}_U(x,t,w) : x \in \mathbb{Z}^d\}$ on U at time t. Then the following statements are true.*

(i) *The field of truncated quenched first moments $\{\tilde{m}_U(x,t,w) : x \in \mathbb{Z}^d\}$ of the total number of particles at time t on U, satisfies the equation,*

$$\frac{\partial \tilde{m}_U}{\partial t} = \kappa\Delta\tilde{m}_U + v(x)\tilde{m}_U, \qquad \text{for} \quad x \in U \cap \mathcal{G}(w)^c$$

$$\tilde{m}_U(x,0,w) = 1, \qquad \text{for} \quad x \in \mathbb{Z}^d.$$

$$\tilde{m}_U(x,t,w) = 0, \qquad \text{for} \quad x \notin U \cap \mathcal{G}(w)^c, t > 0.$$

(ii) *The field of quenched first moments $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ of the total number of particles at time t on $\mathbb{Z}^d$, satisfies the equation,*

$$\frac{\partial m}{\partial t} = \kappa\Delta m + v(x)m, \qquad \text{for} \quad x \in \mathcal{G}(w)^c$$

$$m(x,0,w) = 1, \qquad \text{for} \quad x \in \mathbb{Z}^d.$$

$$m(x,t,w) = 0, \qquad \text{for} \quad x \notin \mathcal{G}(w)^c, t > 0.$$

Proof. Consider the family of functions $\{u_z(x,t) := E_x^w[z^{\zeta(t)}]\}$, parametrized by complex z such that $0 < |z| \leq 1$. It is easy to see that,

$$\frac{\partial u_z}{\partial t} = \kappa\Delta u_z + v_+(x)u_z^2 - (v_+(x) + v_-(x))u_z + v_-(x),$$

$$u_z(x,0) = z,$$

for $x \in \mathcal{G}(w)^c$, while $u_z(x,t) = 1$ for $t \geq 0$ and $x \in \mathcal{G}(w)$. Differentiating the above equation with respect to z we obtain part (i). A similar proof can be carried out for part (ii).

□

4.4. **Bounds on the quenched first moments.** We will now obtain upper and lower bounds for the annealed moments of the quenched first moments. Let us first recall two elementary inequalities. For n natural, let $a_1, \dots, a_n$ be arbitrary real numbers. Then, for $r \geq 1$, we have Jensen's inequality,

$$\left|\sum_{i=1}^{n} a_i\right|^r \leq n^{r-1} \sum_{i=1}^{n} |a_i|^r, \tag{29}$$

while for $0 \leq r \leq 1$ we have,

$$\left|\sum_{i=1}^{n} a_i\right|^r \leq \sum_{k=1}^{n} |a_i|^r. \tag{30}$$

We will also need to introduce for $L \geq 0$ the notation,

$$M_L := \max_{x \in \Lambda_L} |v(x)|. \tag{31}$$

Let us start with the following lemma, which is a variation of theorem 2.1 of [GM(1990)].

Lemma 4. *Consider a finite subset $U \subset \mathbb{Z}^d$ and $\mu \in \mathcal{P}(W)$. Assume that μ satisfies condition* **(E)**. *Then, μ-a.s. for every $x \in \mathbb{Z}^d$ and $t \geq 0$, the quenched first moment $m(x,t,w)$ on $\mathbb{Z}^d$ at time t starting from site x admits the Feynman-Kac representation*

$$m(x,t,w) = E_x\left[e^{\int_0^t v(X_s)ds}\mathbf{1}_{\{\tau_{\mathcal{G}^c(w)} > t\}}\right], \tag{32}$$

and the truncated quenched first moment $\tilde{m}_U(x,t,w)$ on U at time t starting from x also,

$$\tilde{m}_U(x,t,w) = E_x\left[e^{\int_0^t v(X_s)ds}\mathbf{1}_{\{\tau_{U \cap \mathcal{G}^c(w)} > t\}}\right], \tag{33}$$

where in both (32) and (33), $\{X_t : t \geq 0\}$ is a simple symmetric random walk of total jump rate $2d\kappa$ starting from x, of law P_x, E_x is the expectation related to this law, and for $A \subset \mathbb{Z}^d$ we define $\tau_A := \inf\{t \geq 0 : X_t \notin A\}$.

We can now apply lemma 4 to obtain the first estimates on the quenched first moments.

Proposition 9. *Consider a finite subset $U \subset \mathbb{Z}^d$ and $\mu \in \mathcal{P}(W)$. Assume that μ satisfies condition* **(E)**. *Then,*

(i) *For each $x \in \mathbb{Z}^d$, $t \geq 0$ and $\beta > 0$ there exists a constant $c_1(\beta, d)$ such that,*

$$e^{H(\beta t) - 2d\kappa t} \leq \left\langle m(x,t)^\beta \right\rangle \leq c_1(\kappa + t)^d e^{H(\beta t)}, \tag{34}$$

(ii) *For each $x \in U$, $t \geq 0$ and $\beta > 0$,*

$$e^{H(\beta t)-2d\kappa t} \leq \left\langle \tilde{m}_U(x,t)^\beta \right\rangle \leq c_1(\kappa+t)^d e^{H(\beta t)}, \tag{35}$$

where $c_1(\beta, d)$ is the constant of part (i).

(iii) *For each $\beta > 0$, $\gamma > 0$ and $a > 0$ we have that,*

$$\left\langle \left| m(x,t) - \tilde{m}_{\gamma(\kappa t)^a}(x,t) \right|^\beta \right\rangle \leq d4^\beta 2^d (\gamma(\kappa t)^a + 1)^d e^{-2\beta\kappa t I\left(\frac{\gamma(\kappa t)^{a-1}}{2}\right)} e^{H(\beta t)}, \tag{36}$$

where $I : [0,\infty) \to [0,\infty)$ is defined in display (21).

Proof. Part (i). The first inequality of display (34) can be obtained from the Feynman-Kac representation (32), taking only into account the contribution of the path X_s which stays during the whole time interval $[0,t]$ at 0 (page 637 of [GM(1990)]). To prove the second inequality of (34), let us note that by translation invariance it is enough to prove the estimate for $\langle m(0,t) \rangle$. On the other hand,

$$\mathbf{1}_{\{\tau_{G^c(w)} > t\}} e^{\int_0^t v(X_s) ds} \leq \sum_{n=0}^\infty e^{M_{R_n} t} \mathbf{1}_{T_{n-1} \leq t < T_n},$$

where $T_{-1} = 0$, while for n natural T_n is the first exit time of the random walk $\{X_t : t \geq 0\}$ from the box Λ_{R_n}, with $R_n := R_0 2^n$ and $R_0 := \max\{\kappa t, 1\}$, while $M_{R_n} := \max_{x:\|x\| \leq R_n} |v(x)|$ as defined in display (31). It follows that,

$$m(0,t,w) \leq \sum_{n=0}^\infty e^{M_{R_n} t} P[T_{n-1} \leq t]. \tag{37}$$

Let us also remark that since $\beta > 0$, for each natural n we have the following inequality which will be used soon,

$$\langle e^{\beta t M_{R_n}} \rangle \leq (2(R_n+1))^d \exp\{H(\beta t)\}. \tag{38}$$

In fact, $\langle e^{\beta t M_{R_n}} \rangle \leq \sum_{x \in [-R_n, R_n]^d} \left\langle e^{\beta t M_{R_n}} \mathbf{1}_{v(x) = M_{R_n}} \right\rangle$. Let us now consider the case $0 < \beta \leq 1$. Then, by an application of inequality (30) to estimate (37), we conclude that,

$$m(0,t,w)^\beta \leq e^{\beta t M_{R_0}} + \sum_{n=1}^\infty e^{\beta t M_{R_n}} P[T_{n-1} \leq t]^\beta.$$

Taking expectations on both sides of this inequality and applying the estimate (38) and the second inequality of display (22) of lemma 2 for each of the d coordinates of the underlying random walk, we obtain,

$$\left\langle m(0,t)^\beta \right\rangle \leq 2^d (R_0+1)^d e^{H(\beta t)} \left(1 + 4d \sum_{n=1}^\infty 2^{nd} e^{-\beta R_0 2^{n-1} \log\left(\frac{R_0 2^{n-1}}{2e\kappa t}\right)} \right),$$

Now, since $e^{-\beta R_0 2^{n-1}\log\left(\frac{R_0 2^{n-1}}{2e\kappa t}\right)} \le e^{-\beta 2^{n-1}\log\left(\frac{2^{n-1}}{2e}\right)}$, and $R_0+1\le 2R_0$, we see that there is a constant $c_1(\beta,d)$ such that inequality (34) is satisfied for $0<\beta\le 1$.
Let us now consider the case $\beta>1$. Let $\beta'>1$ be defined by $\frac{1}{\beta}+\frac{1}{\beta'}=1$. Then, if we represent the left hand side of (37) as $\sum_{k=0}^\infty e^{tM_{R_n}}P[T_{n-1}<t\le T_n]^{\frac{1}{\beta'}}P[T_{n-1}<t\le T_n]^{\frac{1}{\beta}}$, by Hölder's inequality we get that,

$$m(0,t,w)\le\sum_{n=0}^\infty e^{t\beta M_{R_n}}P[T_{n-1}<t\le T_n]^{\frac{\beta}{\beta'}}.$$

A computation similar to the case $0<\beta\le 1$ finishes the proof of part (i).
Part (ii). The first inequality of display (35) can be deduced by an argument analogous to the one leading to the first inequality of display (34). Now, note from the representations (32) and (33) that $\tilde m_U(x,t,w)\le m(x,t,w)$. Hence, the second inequality of display (35) is a corollary of the second inequality of display (34).
Part (iii). Let us remark from the Feynman-Kac representations (32) for $m(0,t)$ and (33) for $\tilde m_{\gamma(\kappa t)^a}(0,t)$ that,

$$|m(0,t)-\tilde m_{\gamma(\kappa t)^a}(0,t)|\le e^{tM_{\gamma(\kappa t)^a}}P[\tau_{\Lambda_{\gamma(\kappa t)^a}}\le t]. \tag{39}$$

But, as in (38) we can conclude that $\langle e^{\beta t M_{\gamma(\kappa t)^a}}\rangle\le 2^d(\gamma(\kappa t)^a+1)^d\exp\{H(\beta t)\}$. Hence, from the first inequality of display (22) of lemma 2 we see that,

$$\left\langle\left|m(0,t)-\tilde m_{\gamma(\kappa t)^a}(0,t)\right|^\beta\right\rangle\le d4^\beta 2^d(\gamma(\kappa t)^a+1)^d e^{-2\beta\kappa t I\left(\frac{\gamma(\kappa t)^{a-1}}{2}\right)}\exp\{H(\beta t)\}.$$

□

We end this section with important estimates involving the growth functions. Given a subset $U\subset\mathbb{Z}^d$, we define the discrete Laplacian operator with effective potential $v(0)=v_+(0)-v_-(0)\in[-\infty,\infty)$ by its action on functions $f:\mathbb{Z}^d\to\mathbb{R}$, which vanish outside $U_w:=U\cap\mathcal{G}^c(w)$ ($f(x)=0$ for $x\notin U\cap\mathcal{G}^c(w)$) as,

$$(\Delta+v)f(x)=\sum_{e\in\mathcal{B}}(f(x+e)-f(x))+v(x)f(x),$$

where $\mathcal{B}$ is the set formed by the elements of the basis of the free Abelian group $\mathbb{Z}^d$ and its inverses. Defining $L^2(U_w):=\{f:\sum_{x\in\mathbb{Z}^d}f(x)^2<\infty, f(x)=0 \text{ if } x\notin U_w\}$, we can check that $\Delta+v$ is a bounded self adjoint operator on the Hilbert space $L^2(U_w)$ endowed with the inner product $(f,g):=\sum_{x\in\mathbb{Z}^d}f(x)g(x)$. We then define $\{\lambda_n(U,w):0\le n\le\mathcal{U}-1\}$ as the set of eigenvalues of $\Delta+v$ in $L^2(U_w)$ in decreasing order, where $\mathcal{U}$ is the total number of eigenvalues. We will denote by $\psi_n^{U,w}$ the corresponding normalized eigenfunctions. Let $r\ge 0$. In the case in which $U=\Lambda(x,r)$, we will employ the notation $\{\lambda_n(x,r,w)\}$ instead of $\{\lambda_n(U,w)\}$ and $\psi_n^{x,r,w}$ instead of $\psi_n^{U,w}$.

We can now state the following important lemma.

Lemma 5. *Consider the quenched first moment $m(0,t,w)$. Then, for every $\beta > 0$, $a > 1$ and $t \geq 1$, there is a constant $k_1(d,a,\beta)$ such that,*

$$\frac{k_1^{-1}}{(\kappa t)^{da(\beta+1)}+1}\langle m(0,\beta t)\rangle \leq \left\langle m(0,t)^\beta \right\rangle \leq k_1((\kappa t)^{da(\beta+1)}+1)\langle m(0,\beta t)\rangle, \tag{40}$$

and

$$\frac{k_1^{-1}}{(\kappa t)^{da(\beta+1)}+1}\langle \tilde{m}_{(\kappa t)^a}(0,\beta t)\rangle \leq \left\langle \tilde{m}_{(\kappa t)^a}(0,t)^\beta \right\rangle \leq k_1((\kappa t)^{da(\beta+1)}+1)\langle \tilde{m}_{(\kappa t)^a}(0,\beta t)\rangle. \tag{41}$$

Proof. We will only prove (40), being the proof of display (41) analogous. Let us first show that for every real $a > 0$, $\beta > 0$ and $t \geq 0$ there is a constant $c(d,\beta,a)$ such that,

$$m(x,t,w) \leq ([2(\beta\kappa t)^a]+1)^{d/2} e^{t\lambda_0(x,(\beta\kappa t)^a,w)} + 4de^{-2\kappa t I\left(\frac{(\beta\kappa t)^{a-1}}{2}\right)} e^{tM_{(\beta\kappa t)^a}}. \tag{42}$$

Note that by inequality (39) with $\gamma = \beta^a$, we have that $m(x,t,w) \leq 4de^{-2\kappa t I\left(\frac{(\beta\kappa t)^{a-1}}{2}\right)} e^{tM_{(\beta\kappa t)^a}} + \tilde{m}_U(x,t,w)$, with $U = \Lambda(x,(\beta\kappa t)^a,w)$. We then need to estimate the truncated quenched first moment at scale $(\beta\kappa t)^a$, $\tilde{m}_{(\beta\kappa t)^a}(x,t,w)$. First remark the following expansion in terms of the eigenvalues $\{\lambda_n(x,(\beta\kappa t)^a,w)\}$ and the corresponding eigenfunctions $\{\psi_n^{x,(\beta\kappa t)^a,w}\}$,

$$\tilde{m}_{(\beta\kappa t)^a}(x,t,w) = \sum_{n=0}^{\mathcal{U}-1} e^{t\lambda_n(x,(\beta\kappa t)^a,w)}\psi_n^{x,(\beta\kappa t)^a,w}(x)(\psi_n^{x,(\beta\kappa t)^a,w},\mathbf{1}_A), \tag{43}$$

where $A := \Lambda(x,(\beta\kappa t)^a,w)$. Now, by the Cauchy-Schwartz inequality we see that the right-hand side of equality (43) is upper-bounded by $e^{t\lambda_0(x,(\beta\kappa t)^a,w)}$ $\times\left(\sum_{n=0}^{\mathcal{U}-1}(\psi_n^{x,(\beta\kappa t)^a,w},\mathbf{1}_x)^2\sum_{n=0}^{\mathcal{U}-1}(\psi_n^{x,(\beta\kappa t)^a,w},\mathbf{1}_A)^2\right)^{\frac{1}{2}}$ which in turn is upper-bounded by $e^{t\lambda_0(x,(\beta\kappa t)^a,w)}\sqrt{|A|}$, where $\mathbf{1}_x(y)$ equals 1 if $y = x$ and 0 otherwise. Using the fact that $|A| = |\Lambda(x,(\beta\kappa t)^a,w)| \leq ([2(\beta\kappa t)^a]+1)^d$, finishes the proof of (42).
Let us now show that for every finite subset $U \subset \mathbb{Z}^d$, it is true that,

$$\frac{1}{|U|}\sum_{z\in U} m(z,t) \geq \frac{1}{|U|}e^{t\lambda_0(U,w)}. \tag{44}$$

First note the trivial inequality $m(z,t,w) \geq \tilde{m}_U(z,t,w)$. We also have the expansion, $\tilde{m}_U(z,t,w) = \sum_{n=0}^{\mathcal{U}-1} e^{\lambda_n(U,w)t}\psi_n^{U,w}(z)\left(\psi_n^{U,w},\mathbf{1}_U\right)$. Therefore we can see that,

$$\frac{1}{|U|}\sum_{z\in U} m(z,t) \geq \frac{1}{|U|}e^{\lambda_0(U,w)t}(\psi_0^{U,w},\mathbf{1}_U)^2$$

$$\geq \frac{1}{|U|} e^{\lambda_0(U,w)t} \sum_{z \in U} (\psi_0^{U,w})^2(z) = \frac{1}{|U|} e^{\lambda_0(U,w)t},$$

where we have used in the second inequality the fact that $\psi_0^{U,w}(x) \geq 0$ and in the last inequality the normalization condition $\sum_{z \in U} (\psi_0^{U,w})^2(z) = 1$.
Let us now prove the second inequality of (40). By Jensen's inequality (29), in the case $\beta \geq 1$, or inequality (30), in the case $0 < \beta < 1$, applied to (42), note that for some constant $c(a, d, \beta)$,

$$\left\langle m(x,t)^\beta \right\rangle \leq c((\kappa t)^{da\beta/2} + 1) \left\langle e^{\beta t \lambda_0(x,(\beta\kappa t)^a, w)} \right\rangle + 4de^{-2\beta\kappa t I\left(\frac{(\beta\kappa t)^{a-1}}{2}\right)} \left\langle e^{\beta t M_{(\beta\kappa t)^a}} \right\rangle.$$

Now, by the first inequality of display (34) of part (i) of proposition 9 and a computation similar to the one leading to (38), the second term of the right hand side of the above inequality, is upper bounded by,

$$4d2^d((\beta\kappa t)^a + 1)^d \exp\left\{-2\beta\kappa t I\left(\frac{(\beta\kappa t)^{a-1}}{2}\right) + 2d\kappa t\right\} \langle m(x,t)^\beta \rangle. \tag{45}$$

On the other hand, by inequality (44) with $U = \Lambda(0, (\kappa t)^a)$, we see that, $\left\langle e^{\beta t \lambda_0(x,(\beta\kappa t)^a, w)} \right\rangle \leq (2[(\beta\kappa t)^a] + 1)^d \langle m(x, \beta t) \rangle$. It follows that,

$$\left\langle m(x,t,w)^\beta \right\rangle \leq c((\kappa t)^{da(\beta+1)} + 1) \langle m(x, \beta t) \rangle,$$

where we have used the fact that for $a > 1$, the term (45) is negligible with respect to $\langle m^\beta \rangle$. The second inequality of display (40) of part (ii) now follows noting that for $a > 1$ the second term in the right-hand factor of the above inequality is negligible with respect to the first term.
Let us now prove the first inequality of display (40). In the case $0 < \beta \leq 1$, by inequality (30) and the bound (44), we have that,

$$\sum_{x \in U} m^\beta(x,t,w) \geq e^{\beta t \lambda_0(U,w)}. \tag{46}$$

But when $\beta > 1$, by Jensen's inequality we have that $\sum_{x \in U} m^\beta(x,t,w) \geq |U|^{-(\beta-1)} \left(\sum_{x \in U} m(x,t,w)\right)^\beta$, and (46) is still satisfied. Choosing $U = \Lambda(0, (\beta\kappa t)^a)$, and using translation invariance we get,

$$\langle m(0,t)^\beta \rangle \geq (2[(\beta\kappa t)^a] + 1)^{-d} \left\langle e^{\beta t \lambda_0(0,(\beta\kappa t)^a, w)} \right\rangle.$$

Using again the bound (42), neglecting the second term, we finish the proof in the case $0 < \beta \leq 1$. □

An important consequence of lemma 5, is that it shows that assumption **(MI)** implies the so called *intermittency effect*. Let us define for $\theta \neq 0$, the functions,

$$\bar{F}_\theta(t) := \frac{\log\langle m(0,t)^{1+\theta} \rangle - (1+\theta) \log\langle m(0,t) \rangle}{\theta}.$$

Note that by Jensen's inequality, $\bar{F}_\theta(t) \geq 0$.

Corollary 3. *Suppose that assumptions* **(E)** *and* **(MI)** *are satisfied. Then, for every* $\theta \neq 0$,

$$\lim_{t\to\infty} \frac{\bar{F}_{2\theta}(t) - \bar{F}_\theta(t)}{\theta \log(\kappa t + e)} = \infty. \tag{47}$$

4.5. **Correlation and variance estimates on the field of quenched first moments.** In order to prove part (ii) of theorem 1, it will be important to have a control on the variance of the the quenched first moments. In the sequel of this paper, to avoid heavy notation, we will use m_a instead of $m_{(\kappa t)^a}$. Given $x, y \in \mathbb{Z}^d$ and $t \geq 0$, let us define,

$$c(x,y,t) := \langle m(x,t)m(y,t)\rangle - \langle m(x,t)\rangle \langle m(y,t)\rangle ,$$

which we will call the *correlation between sites* x *and* y *at time* t of the field of quenched first moments. Similarly let us define for $a > 0$,

$$c_a(x,y,t) := \langle \tilde{m}_a(x,t)\tilde{m}_a(y,t)\rangle - \langle \tilde{m}_a(x,t)\rangle \langle \tilde{m}_a(y,t)\rangle ,$$

the *correlation between sites* x *and* y *at time* t of the truncated field of quenched first moments at scale $(\kappa t)^a$. Let us begin with the following lemma.

Lemma 6. *Let* $t \geq 0$. *Consider the fields of quenched first moments* $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ *and truncated quenched first moments at scale* $(\kappa t)^a$, $\{\tilde{m}_a(x,t,w) : x \in \mathbb{Z}^d\}$. *Then the following statements are true.*

(i) *The sum of the correlations between site* 0 *and the other sites of the field of quenched first moments, behaves asymptotically as* $t \to \infty$ *like the sum of the correlations between site* 0 *and the other sites of the truncated field of quenched first moments at scale* $(\kappa t)^a$. *In other words,*

$$\sum_{y\in\mathbb{Z}^d} c(0,y,t) \sim \sum_{y\in\mathbb{Z}^d} c_a(0,y,t).$$

(ii) *Let* $\{U_t : t > 0\}$ *be a collection of subsets of the lattice* $\mathbb{Z}^d$ *indexed by* $t > 0$. *Assume that* $|U_t| \sim |U_{t,(\kappa t)^a}|$ *as* $t \to \infty$, *where* $U_{t,r} := \{x \in U_t : dist(x, U_t^c) \geq 2r\}$, *for* $r > 0$. *Then,*

$$Var_\mu \sum_{x\in\Lambda_{U_t}} m(x,t) \sim |U_t| \sum_{y\in\mathbb{Z}^d} c(0,y,t), \tag{48}$$

$$Var_\mu \sum_{x\in\Lambda_{U_t}} \tilde{m}_a(x,t) \sim |U_t| \sum_{y\in\mathbb{Z}^d} c_a(0,y,t), \tag{49}$$

and

$$Var_\mu \sum_{x\in\Lambda_{U_t}} m(x,t) \sim Var_\mu \sum_{x\in\Lambda_{U_t}} \tilde{m}_a(x,t). \tag{50}$$

Proof. Part (i). From the Feynman-Kac representation (32) of lemma 4, note that it is possible to write,

$$m(x,t,w) = E_x\left[e^{\sum_{z\in\mathbb{Z}^d} v(z)\mathcal{L}(z,t)}\mathbf{1}_A\right],$$

where $A := \{\tau_{\mathcal{G}^c(w)} > t\}$, $\mathcal{L}(z,t) := \int_0^t \delta_z(X_s)ds$ is the local time at the point z of the random walk $\{X_t : t \geq 0\}$ starting from x, and $\delta_z : \mathbb{Z}^d \to \{0,1\}$ is the indicator function of the set $\{z\}$. From here, using Fubini's theorem it follows that we have the following representation for the correlations between site x and y of the quenched field of first moments.

$$c(x,y,t) = E_{x,y}\left[\left\langle e^{\sum_{z\in\mathbb{Z}^d} v(\mathcal{L}+\tilde{\mathcal{L}})}\mathbf{1}_{A\cap\tilde{A}}\right\rangle - \left\langle e^{\sum_{z\in\mathbb{Z}^d} v\mathcal{L}}\mathbf{1}_A\right\rangle \left\langle e^{\sum_{z\in\mathbb{Z}^d} v\tilde{\mathcal{L}}}\mathbf{1}_{\tilde{A}}\right\rangle\right] \tag{51}$$

where $\tilde{\mathcal{L}}(z,t) := \int_0^t \delta_z(\tilde{X}_s)ds$ is the local time at the point z of the random walk $\{\tilde{X}_t : t \geq 0\}$, independent of $\{X_t : t \geq 0\}$, starting from y, with law P_y, and $\tilde{A}$ is an identical copy of A, but defined in terms of the random walk $\{\tilde{X}_t : t \geq 0\}$. Furthermore, $E_{x,y} := E_x \otimes E_y$, denotes the expectation with respect to the law of the independent random walks $\{X_t\}$ and $\{\tilde{X}_t\}$. Now, the expression (51) for the correlations can be written in terms of the cumulant generating function defined in display (1),

$$c(x,y,t) = E_{x,y}\left[e^{\sum_{z\in\mathbb{Z}^d} H(\mathcal{L}+\tilde{\mathcal{L}})} - e^{\sum_{z\in\mathbb{Z}^d} H(\mathcal{L})}e^{\sum_{z\in\mathbb{Z}^d} H(\tilde{\mathcal{L}})}\right], \tag{52}$$

where we have used the independence of the coordinates of the effective field $\{v(x) : x \in \mathbb{Z}^d\}$ under μ. Note that the super-additivity of H (part (i) of lemma 3), implies that this expression is non-negative. On the other hand, a reasoning similar to the one leading to the representation (52), this time based on the Feynman-Kac representation (33) of lemma 4, enables us to deduce that,

$$c_a(x,y,t) = E_{x,y}\left[\mathbf{1}_{\{\tau_a>t\}}\mathbf{1}_{\{\tilde{\tau}_a>t\}}\left(e^{\sum_{z\in\mathbb{Z}^d} H(\mathcal{L}+\tilde{\mathcal{L}})} - e^{\sum_{z\in\mathbb{Z}^d} H(\mathcal{L})}e^{\sum_{z\in\mathbb{Z}^d} H(\tilde{\mathcal{L}})}\right)\right], \tag{53}$$

where $\tau_a := \tau_{\Lambda(x,(\kappa t)^a)} = \inf\{t \geq 0 : X_t \notin \Lambda(x,(\kappa t)^a)\}$ and $\tilde{\tau}_a := \tilde{\tau}_{\Lambda(x,(\kappa t)^a)} := \inf\{t \geq 0 : \tilde{X}_t \notin \Lambda(y,(\kappa t)^a)\}$. From (52) and (53) it follows that,

$$\sum_{y\in\mathbb{Z}^d} c(0,y,t) \geq \sum_{y\in\mathbb{Z}^d} c_a(0,y,t). \tag{54}$$

But note that in reality, due to the independence between any pair of truncated quenched first moments at time t at two points at a distance larger than $2(\kappa t)^a$, we have $\sum_{y\in\mathbb{Z}^d} c_a(0,y,t) = \sum_{y\in\Lambda_{2(\kappa t)^a}} c_a(0,y,t)$. Furthermore,

$$\sum_{y\in\mathbb{Z}^d} c(0,y,t) = \sum_{y\in\Lambda_{2(\kappa t)^a}} c(0,y,t) + \sum_{y\notin\Lambda_{2(\kappa t)^a}} c(0,y,t). \tag{55}$$

Now, an application of the first inequality of display (34) of part (i) of proposition 9 and of part (ii) of the same proposition, shows that since $a > 0$, it is true that

$\sum_{y\in\Lambda_{2(\kappa t)^a}} c(0,y,t) \sim \sum_{y\in\Lambda_{2(\kappa t)^a}} c_a(0,y,t)$. And a second application of display (34) and the first inequality in display (22) of lemma 2, shows that $\sum_{y\notin\Lambda_{2(\kappa t)^a}} c(0,y,t) \ll \sum_{y\in\Lambda_{2(\kappa t)^a}} c(0,y,t)$. This, together with inequality (54), ends up the proof of part (i) of lemma 6.

Part (ii). We will first prove (49). Remark that,

$$Var_\mu \sum_{x\in U_t} \tilde{m}_a(x,t) = \sum_{x,y\in U_t} c_a(x,y,t) = \sum_{x,y\in U_t:\|x-y\|\leq 2(\kappa t)^a} c_a(x,y,t),$$

where we used as in the proof of part (i), the fact that $\tilde{m}_a(x,t,v)$ and $\tilde{m}_a(y,t,v)$ are independent for $\|x-y\| \geq 2(\kappa t)^a$. Now, the right-most member of the above inequality is bounded by, $|U_t|\sum_{y:\|y\|\leq 2t^a} c_a(0,y,t)$, which gives us the inequality,

$$Var_\mu \sum_{x\in U_t} \tilde{m}_a(x,t) \leq |U_t| \sum_{y\in\mathbb{Z}^d} c_a(0,y,t).$$

Similarly we can conclude that,

$$Var_\mu \sum_{x\in U_t} \tilde{m}_a(x,t) \geq |U_{t,(\kappa t)^a}| \sum_{y\in\mathbb{Z}^d} c_a(0,y,t).$$

This finishes the proof of the statement of display (49). The statement of display (48) now follows from display (49), part (i) of lemma 6 proved above, and the inequalities,

$$Var_\mu \sum_{x\in U_t} \tilde{m}_a(x,t) \leq Var_\mu \sum_{x\in U_t} m_a(x,t) \leq |U_t| \sum_{y\in\mathbb{Z}^d} c(0,y,t).$$

Finally, note that (50) is a trivial consequence of (48) and (49). □

5. The Annealed and Gaussian regimes

Here we will prove parts (i) and (ii) of theorem 1, making use of the estimates obtained in the previous section, and theorem 2. The main argument which will be used to prove 1 is a renormalization method, which we call *partition analysis*, as developed in [BMR(2005)]. In order to present a self-contained proof, in the first subsection we recall this technique. Then, in the second subsection, we derive some important estimates via the partition analysis technique, which will enable us to reduce the proofs to sums of independent random variables. In the third subsection we prove the law of large numbers, stated in display (9) of part (i) of theorem 1. In the fourth subsection we will prove the negative part (absence of a law of large numbers) stated in display (10) of part (i) of theorem 1. In subsection 5.5, we will prove the central limit theorem stated in display (11) of part (ii) of theorem 1. Then, in subsection 5.6 we prove the absence of a central limit behavior stated in display (12) of part (ii) of the same theorem. Finally, in subsection 5.7, we prove theorem 2.

5.1. **Partition Analysis.** Here we shall follow closely section 5.1 of Ben Arous, Molchanov and Ramírez [BMR(2005)]. For a fixed natural L we consider the box $\Lambda_L = \{x \in \mathbb{Z}^d : \|x\| \leq L\}$. We will define two related but different kinds of partitions of Λ_L. The first one shows that Λ_L can be decomposed into disjoint *partition boxes* $\{\Lambda'_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$, indexed by some set $\mathcal{I}$, so that $\Lambda_L = \bigcup_{\mathbf{i}} \Lambda'_{\mathbf{i}}$. The second one defines a partition of Λ_L in a *strip set* and *main boxes* $\{\Lambda''_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$. In the first case, the index set $\mathcal{I}$ will be partitioned in disjoint subsets $\{\mathcal{I}_K : K \in \mathcal{K}\}$, where the cardinality of $\mathcal{K}$ is 2^d, in such a way that for each $K \in \mathcal{K}$ any pair of elements of the collection of partition boxes $\{\Lambda'_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}_K\}$ is at a large Euclidean distance. In the second partition case, it turns out that the survival probabilities corresponding to sites in the strip set have a total sum which is negligible, while the main boxes happen to be essentially independent. To proceed we will need to introduce some notation defining the corresponding scales and subsets.

Our first parameter is a natural number L' smaller than or equal to L, which will be called the *mesoscopic scale*. By the division algorithm, we know that there exist natural numbers q and $\bar{q}$ such that $2L+1 = qL' + \bar{q}$, with $0 \leq \bar{q} < q$. Note that this last equation can be written in the form

$$2L + 1 = \sum_{i=1}^{q} L'_i, \tag{56}$$

with $L'_i = L' + \theta_{\bar{q}}(i)$ and $\theta_{\bar{q}}(i) = 1$ for $i \leq \bar{q}$ and $\theta_{\bar{q}}(i) = 0$ for $i > \bar{q}$. For our purposes, the relevant fact is that $L' \leq L'_i \leq L' + 1$. In the sequel, for any given pair of real numbers a, b we will use the notation $[a,b]_l$ for $[a,b] \cap \mathbf{Z}$. We now will subdivide the box $[-L, L]_l$ in intervals according to equation (56). Thus, we define $I_1 := [-L, -L + L'_1 - 1]_l$ and for $1 < i \leq q$ we let $I_i := \left[-L + \sum_{j=1}^{i-1} L'_i, -L + \sum_{j=1}^{i} L'_i - 1\right]_l$. Note that $I_q = \left[L - L'_q + 1, L\right]_l$ and $|I_i| = L'_i$. Next, we introduce a second parameter r which is a natural number smaller than or equal to L'. We will call r the *fine scale*. Then, for each I_i we define an interval J_i such that $J_i \subset I_i$, $|J_i| = L' - 2r$ and the endpoints of J_i are at a distance larger than r to the endpoints of I_i. To do so, first let $r_i := r + \theta_{\bar{q}}(i)$. Then define $J_1 := [-L + r, -L + L'_1 - 1 - r_1]_l$ and for $1 < i \leq q$ we let $J_i := \left[-L + \sum_{j=1}^{i-1} L'_i + r, -L + \sum_{j=1}^{i} L'_i - 1 - r_i\right]_l$.

We now proceed to define the partition in Λ_L in partition boxes and define the corresponding decomposition of the index set. First we define the set $\mathcal{I} := \{1, 2, \ldots, q\}^d$, which will correspond to the indexes parameterizing the sub-boxes. For a given element $\mathbf{i} \in \mathcal{I}$, of the form $\mathbf{i} = (i_1, \ldots, i_d)$ with $1 \leq i_k \leq q$, $1 \leq k \leq d$, we define

$$\Lambda'_{\mathbf{i}} := I_{i_1} \times I_{i_2} \times \cdots \times I_{i_d}.$$

We will call such a set a *partition box*. By definition the cardinality $|\Lambda'_{\mathbf{i}}|$ of a partition box satisfies,

$$(L')^d \leq |\Lambda'_{\mathbf{i}}| \leq (L' + 1)^d. \tag{57}$$

Note also that the partition boxes defines a partition of Λ_L so that $\Lambda_L = \bigcup_{\mathbf{i}\in\mathcal{I}} \Lambda'_{\mathbf{i}}$ where the union is disjoint.

Next we define a partition of the index set $\mathcal{I}$. Consider the collection $\mathcal{K}$ of subsets of $\{1,2,\ldots,d\}$. Note that $|\mathcal{K}| = 2^d$. Now given $K \in \mathcal{K}$ we define $\mathcal{I}_K$ as the subset of $\mathcal{I}$ having coordinates which are even for $k \in K$ and odd for $k \notin K$. In other words, if we define $\mathbb{E}$ as the set of even natural numbers and $\mathbb{O}$ as the set of odd natural numbers then,

$$\mathcal{I}_K := \{\mathbf{i} = (i_1,\ldots,i_d) \in \mathcal{I} : i_k \in \mathbb{E} \text{ if } k \in K,\ i_k \in \mathbb{O} \text{ if } k \notin K,\ 1 \le k \le d\}.$$

Note that $\{\mathcal{I}_K : K \in \mathcal{K}\}$ defines a partition of $\mathcal{I}$ so that $\mathcal{I} = \bigcup_{K\in\mathcal{K}} \mathcal{I}_K$ is a disjoint union. Hence,

$$\sum_{K\in\mathcal{K}} \sum_{\mathbf{i}\in\mathcal{I}_K} \sum_{x\in\Lambda'_{\mathbf{i}}} f(x) = \sum_{x\in\Lambda_L} f(x), \tag{58}$$

for every function $f : \Lambda_L \to \mathbb{R}$, which is a consequence of the fact that $\Lambda_L = \bigcup_{K\in\mathcal{K}} \bigcup_{\mathbf{i}\in\mathcal{I}_K} \Lambda'_{\mathbf{i}}$ is a disjoint union. We will refer to such a decomposition as the *parity partition at scale L'* of Λ_L. Furthermore, given $K \in \mathcal{K}$ and any pair of boxes $\Lambda'_{\mathbf{i}}$ and $\Lambda'_{\mathbf{j}}$ with $\mathbf{i},\mathbf{j} \in \mathcal{I}_K$ and $\mathbf{i} \neq \mathbf{j}$ we have that,

$$dist\left(\Lambda'_{\mathbf{i}}, \Lambda'_{\mathbf{j}}\right) \ge L'. \tag{59}$$

Here for any pair of subsets $A, B \subset \mathbb{Z}^d$ we define $dist(A,B) := \inf_{x\in A, y\in B} |x-y|$. In other words (59) expresses the fact that the distance between any pair of partition boxes with different sub-indexes in $\mathcal{I}_K$ is larger than or equal to L'. This completes the description of the partition of Λ_L in partition boxes.

Next, we describe the partition of Λ_L into the strip set and main boxes. Given an $\mathbf{i} \in \mathcal{I}$ we let,

$$\Lambda''_{\mathbf{i}} := J_{i_1} \times J_{i_2} \times \cdots \times J_{i_d}.$$

Such a box will be called a *main box*. Its cardinality is $|\Lambda''_{\mathbf{i}}| = (L'-2r)^d$. Now let,

$$S_L := \Lambda_L - \bigcup_{\mathbf{i}\in\mathcal{I}} \Lambda''_{\mathbf{i}}.$$

Such a set will be called the *strip set*. Note that S_L and $\{\Lambda''_L : \mathbf{i} \in \mathcal{I}\}$ define a partition of Λ_L. We will refer to such a partition as the *strip-box partition at scale L'* of Λ_L. We furthermore remark the following cardinality estimate for the strip set which will be useful later,

$$\frac{|S_L|}{(2L+1)^d} \le \frac{\left((L'+1)^d - (L'-2r)^d\right)}{(L')^d}, \tag{60}$$

where we have used the fact that $|\mathcal{I}| = q^d$.

5.2. **Moment and decoupling inequalities.** Following [BMR(2005)], we recall here some standard inequalities and derive inequalities involving sums of the quenched first moments that will be necessary to perform the partition analysis.

Let us first recall the well-known inequality of von Bahr and Esseen (page 82, exercise 2.6.20, of Petrov [Pt(1996)]).

Lemma 7. *(von Bahr-Esseen). Let $X_1, \ldots, X_n$ be mean zero independent random variables and $S_n := \sum_{k=1}^n X_i$. Then if E denotes the expectation with respect to the joint law of these random variables, and $1 \le r \le 2$, it is true that*

$$E|S_n| \le 2\sum_{k=1}^{n} E|X_k|^r. \tag{61}$$

We continue with the following lemma (analogous to lemma 6 of [BMR(2005)]), which is a consequence of (61) and (30).

Lemma 8. *Let $a > 1$. Consider the field of truncated quenched first moments $\{\tilde{m}_a(x,t,w) : x \in \mathbb{Z}^d\}$ at scale $(\kappa t)^a$. Let $L(t), L'(t) : [0,\infty) \to \mathbb{N}$ be functions such that $(\kappa t)^a \le L'(t) \le L(t)$. Then if $1 \le r \le 2$, it is true that,*

$$\left\langle \left| \sum_{x\in\Lambda_L} (\tilde{m}_a - \langle \tilde{m}_a \rangle) \right|^r \right\rangle \le 2(2L'+2)^{d(r-1)}(2L+1)^d \left\langle |\tilde{m}_a - \langle \tilde{m}_a\rangle|^r \right\rangle. \tag{62}$$

Proof. The proof of this result is analogous to that of lemma 6 of [BMR(2005)], requiring the use of the decomposition (58) with $f(x) = \tilde{m}_1(x,t,w) - \langle \tilde{m}_1\rangle(x,t)$, von Bahr-Esseen inequality (61) and Jensen's inequality (29). □

Next, we have the following estimate analogous to lemma 7 of [BMR(2005)].

Lemma 9. *Let $a > 1$. Consider the field of quenched first moments $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ and of truncated quenched first moments $\{m_a(x,t,w) : x \in \mathbb{Z}^d\}$ at scale $(\kappa t)^a$. Let $L(t) : [0,\infty) \to \mathbb{N}$ be such that $(\kappa t)^a \ll L$. Then the following statements are true.*

(i) *Assume that,*

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L} \tilde{m}_a}{(2L+1)^d \langle \tilde{m}_a \rangle} - 1 \right| \right\rangle \ll 1.$$

Then,

$$\left\langle \left| \frac{m^L}{\langle m \rangle} - 1 \right| \right\rangle \ll 1.$$

(ii) *Asymptotically as $t \to \infty$ we have,*

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L}(m-\langle m\rangle)}{\sqrt{Var_\mu \sum_{x\in\Lambda_L} m}} - \frac{\sum_{x\in\Lambda_L}(\tilde{m}_a-\langle \tilde{m}_a\rangle)}{\sqrt{Var_\mu \sum_{x\in\Lambda_L} \tilde{m}_a}} \right| \right\rangle \ll 1, \tag{63}$$

and

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L}(m-\langle m\rangle)}{\sqrt{Var_\mu \sum_{x\in\Lambda_L} m}} - \frac{\sum_{x\in\Lambda_L}(\tilde{m}_a-\langle \tilde{m}_a\rangle)}{\sqrt{Var_\mu \sum_{x\in\Lambda_L} m}} \right| \right\rangle \ll 1. \tag{64}$$

Proof. The case $\kappa = 0$ is trivial, so we will assume that $\kappa > 0$.
Part (i). A direct calculation shows us that,

$$\left|\frac{m^L}{\langle m\rangle} - \frac{\sum_{x\in\Lambda_L}\tilde{m}_a}{(2L+1)^d\langle \tilde{m}_a\rangle}\right| \le \frac{\sum_{x\in\Lambda_L}|m-\tilde{m}_a|}{(2L+1)^d\langle m\rangle} + \frac{\langle |m-\tilde{m}_a|\rangle}{\langle m\rangle}\left|\frac{\sum_{x\in\Lambda_L}\tilde{m}_a}{(2L+1)^d\langle \tilde{m}_a\rangle}\right|.$$

Now, an application of inequality (36) of part (ii) of proposition 9 and of the first inequality in display (34) of part (i) shows us that, $\frac{\sum\langle|m-\tilde{m}_a|\rangle}{(2L+1)^d\langle m\rangle} \le \varepsilon_1(t)$, where $\varepsilon_1(t) := d42^d((\kappa t)^a+1)^d e^{-2\kappa t\left(I\left(\frac{(\kappa t)^{a-1}}{2}\right)-d\right)}$. Similarly we have that $\frac{\langle|m-\tilde{m}_a|\rangle}{\langle m\rangle} \le \varepsilon_1(t)$. By the triangle inequality it follows that,

$$\left\langle\left|\frac{m^L}{\langle m\rangle} - \frac{\sum_{x\in\Lambda_L}\tilde{m}_a}{(2L+1)^d\langle \tilde{m}_a\rangle}\right|\right\rangle \le 2\varepsilon_1(t) + \left\langle\left|\frac{\sum_{x\in\Lambda_L}\tilde{m}_a}{(2L+1)^d\langle \tilde{m}_a\rangle}-1\right|\right\rangle. \tag{65}$$

Now, for $a>1$, we have $I\left(\frac{(\kappa t)^{a-1}}{2}\right) \to \infty$. Hence, $\varepsilon_1(t)\to 0$ as $t\to\infty$. This clearly implies the statement of part (i) of the lemma.
Part (ii). Let us first note that the left-hand side of display (63) is upper-bounded by,

$$2\frac{\sum\limits_{x\in\Lambda_L}\langle|m-\tilde{m}_a|\rangle}{\sqrt{Var_\mu \sum\limits_{x\in\Lambda_L} m}} + \left\langle \frac{\sum\limits_{x\in\Lambda_L}|\tilde{m}_a-\langle\tilde{m}_a\rangle|}{\sqrt{Var_\mu \sum\limits_{x\in\Lambda_L}\tilde{m}_a}}\right\rangle \left(\frac{\sqrt{Var_\mu \sum\limits_{x\in\Lambda_L}\tilde{m}_a}}{\sqrt{Var_\mu \sum\limits_{x\in\Lambda_L} m}}-1\right). \tag{66}$$

Now, by Cauchy-Schwartz inequality we have that $\left\langle \frac{\sum_{x\in\Lambda_L}|\tilde{m}_a-\langle\tilde{m}_a\rangle|}{\sqrt{Var_\mu\sum_{x\in\Lambda_L}\tilde{m}_a}}\right\rangle \le 1$. On the other hand, by the assumption $t^a \ll L$ we know that the hypothesis of part (ii) of lemma 6 is satisfied for $U_t = \Lambda_{L(t)}$ so that the asymptotics (50) of this lemma holds, and hence the second term of (66) tends to 0 as $t\to\infty$. Furthermore, Cauchy-Schwartz inequality and part (ii) of proposition 9, imply that $\sum_{x\in\Lambda}\langle|m-\tilde{m}_a|\rangle \le (2L+1)^d\varepsilon_2(t)\sqrt{\langle m(0,t)^2\rangle}$, where $\varepsilon_2(t) := d42^d((\kappa t)^a+1)^{d/2}\exp\left\{-2\kappa t I\left(\frac{(\kappa t)^{a-1}}{2}\right)+2d\kappa t\right\}$. In addition, by part (ii) of lemma 6 we have

$Var_\mu \sum_{x\in\Lambda} m \sim (2L+1)^d \sum_{x\in\mathbb{Z}^d} c(0,y,t)$. Now, $\sum_{x\in\mathbb{Z}^d} c(0,y,t) \geq \sqrt{\langle m^2\rangle - \langle m\rangle^2}$. Thus, the first term of (66) is upper-bounded by a quantity which asymptotically behaves as $t\to\infty$ like,

$$\frac{\varepsilon_2(t)}{\sqrt{1-\langle m\rangle^2/\langle m^2\rangle}}.$$

But by corollary 3, and the fact that $\bar{F}_1(t)\geq 0$, we conclude that $\langle m\rangle^2/\langle m^2\rangle \ll 1$. In brief, the first term of (66) is upper-bounded by a quantity which asymptotically as $t\to\infty$ behaves like, $\varepsilon_2(t)$. Obviously, when $a>1$ we have $\varepsilon_2(t)\ll 1$. □

5.3. **The Annealed asymptotics.** Let us now prove the law of large numbers stated in display (9) of part (i) of theorem 1. To simplify the writing of the expressions in the calculations, we will redefine ϵ as 2ϵ, assuming that

$$L(t) \geq \exp\left\{\frac{1}{d}F_{2\epsilon}(t)\right\}, \tag{67}$$

for some $\epsilon>0$, and prove that then in μ-probability it is true that,

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \sim 1. \tag{68}$$

To do so we first remark that by inequality (65) of lemma 9, it is enough to show that for $a=3/2$,

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L} \tilde{m}_a}{(2L+1)^d\langle\tilde{m}_a\rangle} - 1\right|^{1+\epsilon}\right\rangle \ll 1. \tag{69}$$

Remark that the right hand side of display (69) can be rewritten as,

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L} \tilde{m}_a}{(2L+1)^d\langle\tilde{m}_a\rangle} - 1\right|^{1+\epsilon}\right\rangle = \frac{\left\langle \left|\sum_{x\in\Lambda_L}(\tilde{m}_a - \langle\tilde{m}_a\rangle)\right|^{1+\epsilon}\right\rangle}{(2L+1)^{d(1+\epsilon)}\langle\tilde{m}_a\rangle^{1+\epsilon}}. \tag{70}$$

At this point we make use of the parity partition decomposition for Λ_L previously defined to deal with the numerator of the right-hand side of display (70) via inequality (62) of lemma 8 with $r=1+\epsilon$. We will chose a time dependent mesoscopic scale $L'(t)=(\kappa t)^b$, where $0<a<b$. Therefore, by lemma 8, the right-hand side of equality (70) is upper-bounded by

$$\frac{2(2L'+2)^{d\epsilon}\left\langle |\tilde{m}_a - \langle\tilde{m}_a\rangle|^{1+\epsilon}\right\rangle}{(2L+1)^{d\epsilon}\langle\tilde{m}_a\rangle^{1+\epsilon}}. \tag{71}$$

Now, since for any non-negative reals x,y we have $|x-y|^{1+\epsilon}\leq |x|^{1+\epsilon}+|y|^{1+\epsilon}$, it follows that $\left\langle |\tilde{m}_a - \langle\tilde{m}_a\rangle|^{1+\epsilon}\right\rangle \leq \langle\tilde{m}_a^{1+\epsilon}\rangle + \langle\tilde{m}_a\rangle^{1+\epsilon} \leq 2\left\langle\tilde{m}_a^{1+\epsilon}\right\rangle$, where in the last inequality we have used Jensen's inequality. Hence, since $\tilde{m}_a\leq m$, the expression of display (71) is upper bounded by,

$$\frac{2(2L'+2)^{d\epsilon}\left\langle m^{1+\epsilon}\right\rangle}{(2L+1)^{d\epsilon}\langle\tilde{m}_a\rangle^{1+\epsilon}}.$$

But by parts (ii) and (iii) of proposition 9, we can replace in the denominator of the above expression the term $\langle\tilde{m}_a\rangle$ by $\langle m\rangle$. Thus, (71) is upper-bounded by an expression which is asymptotically equivalent to,

$$\begin{aligned}&\frac{2(2L'+2)^{d\epsilon+1}k_1((\kappa t)^{da(2+\epsilon)}+1)e^{H_1((1+\epsilon)t)-(1+\epsilon)H_1(t)}}{(2L+1)^{d\epsilon}}\\&\le c((\kappa t)^{da(2+\epsilon)})+1)(L')^{d\epsilon}e^{-\epsilon(F_{2\epsilon}(t)-F_\epsilon(t))},\end{aligned}\tag{72}$$

where we have used the second inequality of display (40) of lemma 5 in the first inequality, and assumption (67) in the last inequality. Note that when $\kappa=0$ the last expression reduces to $e^{-\epsilon(G_{2\epsilon}(t)-G_\epsilon(t))}$. Then, assumption **(MI)** shows that the first and the second terms of the right-hand side of (72) tend to 0 as $t\to\infty$.

5.4. **The non-Annealed asymptotics.** In this subsection we will prove the asymptotic behavior of display (10). Again, we will redefine ϵ by 2ϵ, assuming that,

$$L(t)\le\exp\left\{\frac{1}{d}F_{-2\epsilon}(t)\right\},\tag{73}$$

for some $\epsilon>0$. Note that it will be enough to show that,

$$\left\langle\left|\frac{m_L(0,t)}{\langle m(0,t)\rangle}\right|^{1-\epsilon}\right\rangle\ll 1.\tag{74}$$

To do so, note from inequality (30) that the left-hand side of display (74) is upper bounded by,

$$(2L+1)^{d\epsilon}\frac{\langle m^{1-\epsilon}(0,t)\rangle}{\langle m(0,t)\rangle^{1-\epsilon}}.\tag{75}$$

Now, by the second inequality of display (40) of lemma 5 applied with $\beta=1-\epsilon$ to the numerator of (75), we see that the left-hand side of display (74) is upper bounded by,

$$(2L+1)^{d\epsilon}k_1((\kappa t)^{da(2-\epsilon)}+1)e^{H_1((1-\epsilon)t)-(1-\epsilon)H_1(t)}.$$

Finally, by assumption (73), this expression is upper bounded by,

$$c((\kappa t)^{da(2-\epsilon)}+1)e^{-\epsilon(F_{-\epsilon}(t)-F_{-2\epsilon}(t))}.$$

Now, the assumption **(MI)**, implies that this expression converges to 0 as $t\to\infty$.

5.5. **The Gaussian asymptotics.** Here we prove the central limit theorem stated in display (11) of part (ii) of theorem 1. We will perform this time a strip-box partition of the box Λ_L into the strip set S_L and the main boxes. Let us fix $a>1$. Let us fix b and c such that $c>b>a$, and choose the mesoscopic scale $L'(t)=(\kappa t)^c$ and the fine scale $r=(\kappa t)^b$. Note that by part (ii) of lemma 9, it is enough to prove that,

$$\frac{\sum_{x\in\Lambda_L}(\tilde{m}_a(x,t,w)-\langle\tilde{m}_a(x,t)\rangle)}{\sqrt{Var_\mu\sum_{x\in\Lambda_L}\tilde{m}_a(x,t)}},$$

converges in distribution to the normal law $\mathcal{N}(0,1)$. To do so, we write,

$$\frac{\sum_{x\in\Lambda_L}(\tilde{m}_a-\langle\tilde{m}_a\rangle)}{\sqrt{Var_\mu\sum_{x\in\Lambda_L}\tilde{m}_a}}=\frac{\sum_{x\in S_L}(\tilde{m}_a-\langle\tilde{m}_a\rangle)}{\sqrt{Var_\mu\sum_{x\in\Lambda_L}\tilde{m}_a}},+\frac{\sum_{\mathbf{i}\in\mathcal{I}}\sum_{x\in\Lambda''_{\mathbf{i}}}(\tilde{m}_a-\langle\tilde{m}_a\rangle)}{\sqrt{Var_\mu\sum_{x\in\Lambda_L}\tilde{m}_a}}. \tag{76}$$

We will first show that the strip component of the decomposition (76) converges to 0 in probability. In fact, note that for t large enough, we have by statement (49) of part (ii) of lemma 6 applied with $U_t=S_{L(t)}$ and $U_t=\Lambda_{L(t)}$, that

$$\frac{Var_\mu\sum_{x\in S_L}\tilde{m}_a}{Var_\mu\sum_{x\in\Lambda_L}\tilde{m}_a}\sim\frac{|S_L|}{(2L+1)^d}\leq\min\{\kappa,(\kappa t)^{-(c-b)}\},$$

where for the last inequality we have used estimate (60). Since $c>b$, this tends to 0 as $t\to\infty$.

Therefore, it is enough to prove that the second term of the right-hand side of equality (76), tends in law to $\mathcal{N}(0,1)$. For this purpose, since the random variables $\left\{\sum_{x\in\Lambda''_{\mathbf{i}}}(\tilde{m}_a-\langle\tilde{m}_a\rangle):\mathbf{i}\in\mathcal{I}\right\}$ are independent, it is enough to verify a version of the Lyapunov condition. Namely, we will show that,

$$\frac{\sum_{\mathbf{i}\in\mathcal{I}}\left\langle\left|\sum_{x\in\Lambda''_{\mathbf{i}}}(\tilde{m}_a-\langle\tilde{m}_a\rangle)\right|^{2+\epsilon}\right\rangle}{\left(\sum_{\mathbf{i}\in\mathcal{I}}Var_\mu\sum_{x\in\Lambda''_{\mathbf{i}}}\tilde{m}_a\right)^{1+\epsilon/2}}\ll 1, \tag{77}$$

and then apply again statement (49) of part (ii) of lemma 6 to conclude that the variance of the denominator of the second term of display can be substituted by $\sum_{\mathbf{i}\in\mathcal{I}}Var_\mu\sum_{x\in\Lambda''_{\mathbf{i}}}\tilde{m}_a$. Now, by the same token, we see that the denominator of the left-hand side of display (77), behaves asymptotically as $t\to\infty$ like

$$(2L+1)^{d(1+\epsilon/2)}\left(\sum_{x\in\mathbb{Z}^d}c_a(0,x,t)\right)^{1+\epsilon/2}. \tag{78}$$

Thus, it is enough to prove the asymptotic behavior (77), with the denominator replaced by (78). Now, note that $\sum_{x\in\mathbb{Z}^d}c_a(0,x,t)$ is lower bounded by $\langle\tilde{m}_a^2(0,t)\rangle-\langle\tilde{m}_a(0,t)\rangle^2$. And by parts (i) and (iii) of proposition 9 and lemma 5, this variance is lower-bounded by an expression which is asymptotically equivalent to,

$$c\left(((\kappa t)^{3da}+1)^{-1}\langle m(0,2t)\rangle - \langle m(0,t)\rangle^2\right). \tag{79}$$

Now, assumption **(MI)** with $\theta = 2$ and the inequality $\langle \tilde{m}_a(0,t)\rangle \leq \langle m(0,t)\rangle$, imply that, $(\log\langle m(0,2t)\rangle - 2\log\langle m(0,t)\rangle \gg \log t$. This shows that the second term $c\langle m(0,t)\rangle^2$ of the expression (79), is negligible with respect to the first one $c((\kappa t)^{3da}+1)^{-1}\langle m(0,2t)\rangle$. Hence, we conclude that it is enough to prove that,

$$((\kappa t)^{3da}+1)\frac{\sum_{\mathbf{i}\in\mathcal{I}}\left\langle\left|\sum_{x\in\Lambda''_{\mathbf{i}}}(\tilde{m}_a - \langle\tilde{m}_a\rangle)\right|^{2+\epsilon}\right\rangle}{(2L+1)^{d(1+\epsilon/2)}\langle m(0,2t)\rangle^{\left(1+\frac{\epsilon}{2}\right)}} \ll 1. \tag{80}$$

By Jensen's inequality (29) and the upper-bound in display (57), we see that, $\sum_{\mathbf{i}\in\mathcal{I}}\left\langle\left|\sum_{x\in\Lambda''_{\mathbf{i}}}(\tilde{m}_a - \langle\tilde{m}_a\rangle)\right|^{2+\epsilon}\right\rangle \leq (L'+1)^{d(1+\epsilon)}(2L+1)^d\left\langle|\tilde{m}_a - \langle\tilde{m}_a\rangle|^{2+\epsilon}\right\rangle$. Using again the fact that for non-negative reals x,y we have $|x-y|^{1+\epsilon}\leq |x|^{1+\epsilon}+|y|^{1+\epsilon}$, it follows that $\left\langle|\tilde{m}_a - \langle\tilde{m}_a\rangle|^{2+\epsilon}\right\rangle \leq \left\langle m^{2+\epsilon}\right\rangle + \langle m\rangle^{2+\epsilon} \leq 2\left\langle m^{2+\epsilon}\right\rangle$, where we have used Jensen's inequality in the last inequality. We see that the left-hand side of (77) is upper-bounded by,

$$((\kappa t)^{3da}+1)(L'+1)^{d(1+\epsilon)}\frac{\langle m(0,t)^{2+\epsilon}\rangle}{(2L+1)^{d\frac{\epsilon}{2}}\langle m(0,2t)\rangle^{\left(1+\frac{\epsilon}{2}\right)}}.$$

Finally, by the hypothesis on the growth of L this can be bounded by,

$$((\kappa t)^{3da}+1)^2(L'+1)^{d(1+\epsilon)}e^{\frac{\epsilon}{2}F_{\frac{\epsilon}{2}}(2t)-\frac{\epsilon}{2}F_\epsilon(2t)},$$

which by condition **(MI)**, tends to 0 as $t\to\infty$.

5.6. **The non-Gaussian asymptotics.** Here we will prove the asymptotics of display (12) of part (ii) of theorem 1. It will be necessary to perform a parity partition of Λ_L with a mesoscopic scale $L' = (\kappa t)^b$ for some $b>1$. First note that by display (64) of part (ii) of lemma 9 it will be enough to show that for some $b>a>1$,

$$\left\langle\left|\frac{\sum_{x\in\Lambda_L}(\tilde{m}_a - \langle\tilde{m}_a\rangle)}{\sqrt{Var_\mu\sum_{x\in\Lambda_L}m}}\right|^{2-\epsilon}\right\rangle \ll 1. \tag{81}$$

Now, by display (48) of part (ii) of lemma 6, the denominator $\left(Var_\mu\sum_{x\in\Lambda_L}m\right)^{1-\frac{\epsilon}{2}}$, of the left-hand side of display (81), can be lower-bounded by $(2L+1)^{d\left(1-\frac{\epsilon}{2}\right)}\left(\left\langle m^2\right\rangle - \langle m\rangle^2\right)^{1-\frac{\epsilon}{2}} \geq c(2L+1)^{d\left(1-\frac{\epsilon}{2}\right)}\left\langle m^2\right\rangle^{1-\frac{\epsilon}{2}} \geq c((\kappa t)^{3da}+1)^{-1}(2L+1)^{d\left(1-\frac{\epsilon}{2}\right)}\langle m(0,2t)\rangle^{1-\frac{\epsilon}{2}}$, where in the second to last inequality we used the assumption **(MI)** and lemma 5 and in the last inequality assumption 5. On the other hand, inequality (62) of lemma 8, applied with $r = 2-\epsilon$, Jensen's inequality and $\tilde{m}_a\leq m$, shows us that the numerator of the left-hand side of display (81) is upper-bounded by $4(2L'+1)^{d(1-\epsilon)}(2L+1)^{d\frac{\epsilon}{2}}\langle m^{2-\epsilon}\rangle$. Using the upper-bound

$\langle m^{2-\epsilon}(0,t)\rangle \leq k_1((\kappa t)^{3da}+1)\langle m(0,(2-\epsilon)t)\rangle$ of lemma 5, the definition of the intermittency exponents $\{F_\theta\}$ in (3), and the assumption $L(t) \leq e^{\frac{1}{2}F_{-\epsilon}(2t)}$, we hence see that it is enough to show that,

$$(2L'+1)^{d(1-\epsilon)}((\kappa t)^{3da}+1)^2 e^{\frac{\epsilon}{2}F_{-\epsilon}(2t)} e^{-\frac{\epsilon}{2}F_{-\frac{\epsilon}{2}}(2t)} \ll 1.$$

But this is a consequence of assumption **(MI)**.

5.7. **Proof of theorem 2.** Let us first prove part (i) for Weibull-type tails. Note that it is enough to find an $x > 0$ such that,

$$\left\langle \left(\frac{m_L(0,t)}{e^{(a(\gamma)+\delta)H(t)}} \right)^x \right\rangle \ll 1.$$

Now, this expression is upper bounded by,

$$e^{-(x(a+\delta)H(t)-H(xt)-(1-x)\gamma H(t))+o(H(t))},$$

where we have used the asymptotics (5). Since $H \in R_{\rho'}$ for $\rho' = \frac{\rho}{\rho-1}$, we see that the above expression is upper bounded by,

$$e^{-f_W(x,a+\delta)H(t)+o(H(t))},$$

where $f_W(x,b) := xb - x^{\rho'} - (1-x)\gamma$ for $x>0, b>0$. This function has a unique root at $x_0 = \left(\frac{\rho-1}{\rho}(b+\gamma)\right)^{\rho-1}$ when $b = a(\gamma)$. Choosing $x = x_0$, we get the upper bound,

$$e^{-\delta x_0 H(t)+o(H(t))} \ll 1.$$

This proves part (i) of theorem 2. The proof of part (iv) for Fréchet-type tails is completely analogous to the previous argument, so it will be omitted. To prove part (ii), note that in analogy to the proof of part (i), it is enough to show that,

$$e^{-\left(x\frac{H((a+\delta)t)}{a+\delta}-H(xt)-(1-x)\gamma t\right)+o(t)} \ll 1. \qquad (82)$$

Now, by supposition (15) we have $\frac{H(xt)-xH((a+\delta)t)/(a+\delta)}{t} \sim -\rho x \log\frac{x}{a+\delta}$. Thus, the expression of display (82), is upper-bounded by,

$$e^{-f_D(x,a+\delta)t+o(t)},$$

where $f_D(x,b) := \rho x \log\frac{x}{b} - (1-x)\gamma$ for $x>0, b>0$. But this function has a single root at $x_0 = ae^{\frac{1}{\rho}(\gamma-1)}$ when $b = a$. Hence, we obtain the upper bound,

$$e^{-x_0\log\left(1+\frac{\delta}{a}\right)+o(t)} \ll 1.$$

This proves part (ii) of theorem 2. The proof of part (iii) for almost bounded potentials is analogous to the proof of part (ii) so it will be omitted.

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(Gérard Ben Arous) Courant Insitute of Mathematical Sciences, New York University, NY, NY 10012, USA

(Stanislav Molchanov) Department of Mathematics, University of North Carolina-Charlotte, 376 Fretwell Bldg. 9201 University City Blvd., Charlotte, NC 28223-0001, USA

(Alejandro Ramírez) Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago 6904411, Chile
URL: http://www.mat.puc.cl/~aramirez
E-mail address: benarous@cims.nyu.edu, smolchan@math.uncc.edu,aramirez@mat.puc.cl