SCALING LIMIT FOR TRAP MODELS ON $\mathbb{Z}^d$

GÉRARD BEN AROUS AND JIŘÍ ČERNÝ

Abstract. We give the "quenched" scaling limit of Bouchaud's trap model in $d \geq 2$. This scaling limit is the Fractional-Kinetics process, that is the time change of a d-dimensional Brownian motion by the inverse of an independent α-stable subordinator.

1. Introduction

This work establishes scaling limits for certain important models of trapped random walks on $\mathbb{Z}^d$. More precisely we show that Bouchaud's trap model on $\mathbb{Z}^d$, $d \geq 2$, properly normalised, converges (at the process level) to the Fractional-Kinetics process, which is a self-similar non-markovian continuous process, obtained as the time change of a d-dimensional Brownian motion by the inverse of an independent α-stable subordinator. This is in sharp contrast with the scaling limit for the same model in dimension one (see [FIN02]) where the limiting process is a singular diffusion in random environment. For a general survey about trap models and their motivation in statistical physics we refer to the lecture notes [BČ06b], where we announced the result proved in this paper.

The Bouchaud's trap model on $\mathbb{Z}^d$ is the nearest-neighbour continuous-time Markov process $X(t)$ given by the jump rates

$$c(x,y) = \frac{1}{2d\tau_x} \qquad \text{if } x \text{ and } y \text{ are neighbours in } \mathbb{Z}^d, \tag{1.1}$$

and zero otherwise, where $\{\tau_x : x \in \mathbb{Z}^d\}$ are i.i.d. heavy-tailed random variables. More precisely we assume that for some $\alpha \in (0,1)$

$$\mathbb{P}[\tau_x \geq u] = u^{-\alpha}(1 + L(u)) \qquad \text{with } L(u) \to 0 \text{ as } u \to \infty. \tag{1.2}$$

We will always assume that $X(0) = 0$. The Markov process $X(t)$ waits at a site x an exponentially distributed time with mean τ_x, and then it jumps to one of the neighbours of x with uniform probability. Therefore X is a random time change of a standard discrete-time simple random walk on $\mathbb{Z}^d$. More precisely:

Definition 1.1. Let $S(0) = 0$ and let $S(k)$, $k \in \mathbb{N}$, be the time of the k^{th} jump of X. For $s \in \mathbb{R}$ we define $S(s) = S(\lfloor s \rfloor)$. We call $S(s)$ the *clock process*. Define the embedded discrete-time Markov chain $Y(k)$ by $Y(k) = X(t)$ for $S(k) \leq t < S(k+1)$. Then obviously, Y is a simple random walk on $\mathbb{Z}^d$.

In order to state our main result we need to introduce the limiting Fractional-Kinetics (FK) process.

Date: June 28, 2006.

2000 *Mathematics Subject Classification.* 60K37; 60G52, 60F17, 82D30.

Key words and phrases. Trap model, scaling limit, Lévy process, random walk, fractional kinetics, subordination .

Definition 1.2. Let $B_d(t)$ be a standard d-dimensional Brownian motion started at 0, and let V_α be an independent α-stable subordinator satisfying $\mathbb{E}[e^{-\lambda V_\alpha(t)}] = e^{-t\lambda^\alpha}$. Define the generalised right-continuous inverse of $V_\alpha(t)$ by $V_\alpha^{-1}(s) := \inf\{t : V_\alpha(t) > s\}$. We define the *fractional-kinetics process* $Z_{d,\alpha}$ by

$$Z_{d,\alpha}(s) = B_d(V_\alpha^{-1}(s)). \tag{1.3}$$

This process is well known in the physics literature. See for instance the broad survey by G. Zaslavsky [Zas02] or the recent book [Zas05] about the relevance of this process for chaotic deterministic systems; see also [GM03, Hil00, MK00, MS84, SZK93] for more on this class of processes and references.

We fix a time $T > 0$ and $d \geq 2$ and denote by $D^d([0,T])$ the space of càdlàg functions from $[0,T]$ to $\mathbb{R}^d$. Let $X_N(t)$ be the sequence of elements of $D^d([0,T])$,

$$X_N(t) = \frac{\sqrt{d}X(tN)}{f(N)}, \tag{1.4}$$

where

$$f(N) = \begin{cases} C_2(\alpha)N^{\alpha/2}(\log N)^{(1-\alpha)/2}, & \text{if } d = 2, \\ C_d(\alpha)N^{\alpha/2}, & \text{if } d \geq 3, \end{cases} \tag{1.5}$$

$$C_d(\alpha) = \begin{cases} \left[\pi^{1-\alpha}\alpha^{\alpha-1}\Gamma(1-\alpha)\Gamma(1+\alpha)\right]^{-1/2}, & \text{if } d = 2, \\ \left[G_d(0)^\alpha\Gamma(1-\alpha)\Gamma(1+\alpha)\right]^{-1/2}, & \text{if } d \geq 3, \end{cases} \tag{1.6}$$

and $G_d(0)$ denotes the Green's function of the d-dimensional discrete simple random walk at the origin, $G_d(0) = \sum_{k=0}^{\infty} \mathbb{P}[Y(k) = 0]$, for $d \geq 3$.

Our main result is the following "quenched" scaling limit statement:

Theorem 1.3. *For a.e.* $\boldsymbol{\tau}$*, the distribution of* X_N *converges weakly to the distribution of* $Z_{d,\alpha}$ *on* $D^d([0,T])$ *equipped with the uniform topology.*

This result is a consequence of the following, more detailed statement, i.e. the joint convergence of the clock process and of the position of the embedded random walk. We use $D([0,T], M_1)$ (resp. $D([0,T], U)$) to denote the space $D([0,T])$ equipped with the M_1 (resp. uniform) topology. Define

$$Y_N(t) = \frac{\sqrt{d}}{f(N)}Y(\lfloor tf(N)^2 \rfloor) \qquad \text{and} \qquad S_N(t) = \frac{1}{N}S(\lfloor tf(N)^2 \rfloor). \tag{1.7}$$

Theorem 1.4. *For a.e.* $\boldsymbol{\tau}$*, the joint distribution of* (S_N, Y_N) *converges weakly to the distribution of* (V_α, B_d) *on* $D([0,T], M_1) \times D^d([0,T], U)$*.*

Let us insist on the following important facts:

1. One word of caution is necessary about the nature of this joint convergence. It takes place in the uniform topology for the spatial component but only in the Skorokhod M_1 topology for the clock process (see [Sko56] for the classical reference about the various topologies on $D^d([0,T])$ and [Whi02] for a thorough, more recent, survey). It is important to remark that our statement is not true in the stronger J_1 topology (usually called the Skorokhod topology). Indeed, the main advantage of M_1 topology over J_1 topology, for our purposes, is that existence of "intermediate jumps" forbid convergence in the latter but not in the former. These intermediate jumps are important in our context: they

are caused by the fact that the deep traps giving the main contributions to the clock process are visited at several nearby instants. All these visits are summed up into one jump of the limiting α-stable subordinator V_α.

2. Our scaling limit result is "quenched", that is valid almost sure in the random environment $\boldsymbol{\tau}$, and the limiting process is independent of $\boldsymbol{\tau}$.

3. Our result might be seen as a "triviality" result. Indeed, the fractional kinetics process can be obtained as a scaling limit of a much simpler discrete process, i.e. a Continuous Time Random Walk (CTRW) à la Montroll-Weiss [MW65]. More precisely consider a simple random walk Y on $\mathbb{Z}^d$ and a sequence of positive i.i.d. random variables $\{s_i : i \in \mathbb{N}\}$ satisfying the same condition (1.2) as τ_x's. Define the CTRW $U(t)$ by

$$U(t) = Y(k) \qquad \text{if } t \in \Big[\sum_{i=1}^{k-1} s_i, \sum_{i=1}^{k} s_i\Big). \tag{1.8}$$

It is proved in [SZ97] on the level of fixed-time distributions and in [MS04] on the level of processes that there is a constant C such that

$$CN^{-\alpha/2}U(tN) \xrightarrow{N\to\infty} Z_{d,\alpha}(t). \tag{1.9}$$

The result of Theorem 1.3 shows that the limit of the d-dimensional trap model and its clock process on $\mathbb{Z}^d$ is trivial, in the sense that it is identical with the scaling limit of the much simpler ("completely annealed") dynamics of the CTRW. The necessary scaling is the same as for CTRW if $d \geq 3$, and it requires a logarithmic correction if $d = 2$.

4. As mentioned above the situation is completely different in $d = 1$, where the scaling limit is a singular diffusion in random environment introduced in by Fontes, Isopi and Newman [FIN02] as follows. Let (x_i, v_i) be an inhomogeneous Poisson point process on $\mathbb{R} \times (0, \infty)$ with intensity measure $\mathrm{d}x\, \alpha v^{-1-\alpha} \mathrm{d}v$, and consider the random discrete measure $\rho = \sum_i v_i \delta_{x_i}$ which can be obtained as a scaling limit of the random environment $\boldsymbol{\tau}$. Conditionally on ρ, the F.I.N.-diffusion $Z_\alpha(s)$ is defined as a diffusion process (with $Z_\alpha(0) = 0$) that can be expressed as a time change of the standard one-dimensional Brownian motion B_1 with the speed measure ρ: denoting by $\ell(t, y)$ the local time of the standard Brownian motion B_1, let

$$\phi_\rho(t) = \int_{\mathbb{R}} \ell(t, y)\rho(\mathrm{d}y), \tag{1.10}$$

then $Z_\alpha(s) = B(\phi_\rho^{-1}(s))$.

Observe that both processes, the fractional kinetics and the F.I.N.-diffusion, are defined as a time change of the Brownian motion $B_d(t)$. The clock processes however differ considerably. For $d = 1$, the clock equals $\phi_\rho(t) = \int \ell(t, x)\rho(\mathrm{d}x)$. Therefore, the processes B_1 and ϕ_ρ are *dependent*. In the fractional-kinetics case the Brownian motion B_d and the clock process, i.e. the stable subordinator V_α, are *independent*. The asymptotic independence of the clock process S and the location Y is a very remarkable feature distinguishing $d \geq 2$ and $d = 1$.

Note also that, in constrast with the $d = 1$ case, nothing like a scaling limit of the random environment appears in the definition of $Z_{d,\alpha}$ for $d \geq 2$, and that the convergence holds $\boldsymbol{\tau}$-a.s. The absence of the scaling limit of the environment in the definition of $Z_{d,\alpha}$ translates into the non-markovianity of $Z_{d,\alpha}$. It is, however, considerably easier to control the behaviour of the FK-process than of the F.I.N.-diffusion even if the former is not markovian. Let us mention few elementary properties of the process $Z_{d,\alpha}$.

Proposition 1.5. *(i)* $Z_{d,\alpha}$ *is a.s.* γ*-Hölder continuous for any* $\gamma < \alpha/2$.
(ii) $Z_{d,\alpha}$ *is self-similar,* $Z_{d,\alpha}(t) \overset{\text{law}}{=} \lambda^{-\alpha/2} Z_{d,\alpha}(\lambda t)$.
(iii) $Z_{d,\alpha}$ *is not markovian.*
(iv) The fixed-time distribution of $Z_{d,\alpha}(t)$ *is given by its Fourier transform*

$$\mathbb{E}\big(e^{i\xi\cdot Z_{d,\alpha}(t)}\big) = E_\alpha(-|\xi|^2 t^\alpha/2), \tag{1.11}$$

where $E_\alpha(z) = \sum_{m=0}^{\infty} z^m/\Gamma(1+m\alpha)$ *is the Mittag-Leffler function.*

Proof. Since the Brownian motion is γ-Hölder continuous for $\gamma < 1/2$ and V_α^{-1} is γ-Hölder continuous for $\gamma < \alpha$ (see Lemma III.17 of [Ber96]), fact (i) follows. (ii) can be proved using scaling properties of B_d and V_α. To show (iii) it is enough to observe that the times between jumps of $Z_{d,\alpha}$ have no exponential distribution. Example (b) at page 453 of [Fel71] implies that the Laplace transform of $V_\alpha^{-1}(t)$ is equal to $E_\alpha(-\lambda t^\alpha)$. The result of (iv) then follows by an easy computation. ☐

The name of the FK-process comes from the fact that $Z_{d,\alpha}$ has a smooth density $p(t,x)$ which satisfies the Fractional Kinetics equation (see [Zas02])

$$\frac{\partial^\alpha}{\partial t^\alpha} p(t,x) = \frac{1}{2}\Delta p(t,x) + \delta(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{1.12}$$

The FK-process $Z_{d,\alpha}$ has an obvious aging property due to its very slow clock, namely

$$\mathbb{P}[Z_{d,\alpha}(t_w+s) = Z_{d,\alpha}(t_w) \forall s \le t] = \frac{\sin\alpha\pi}{\pi}\int_0^{t_w/(t_w+t)} u^{\alpha-1}(1-u)^{-\alpha}\,\mathrm{d}u. \tag{1.13}$$

This is simply a restatement of the arcsine law for the stable subordinator V_α since

$$\mathbb{P}[Z_{d,\alpha}(t_w+s) = Z_{d,\alpha}(t_w) \forall s \le t] = \mathbb{P}\big[\{V(t) : t \in \mathbb{R}\} \cap [t_w, t_w+t] = \emptyset\big]. \tag{1.14}$$

This and Theorem 1.3 explain in part the analogous aging result for Bouchaud's trap model

$$\lim_{t_w\to\infty} \mathbb{P}[X(t_w+\theta t_w) = X(t_w)|\boldsymbol{\tau}] = \frac{\sin\alpha\pi}{\pi}\int_0^{1/(1+\theta)} u^{\alpha-1}(1-u)^{-\alpha}\,\mathrm{d}u. \tag{1.15}$$

In fact proving (1.15) requires a slightly more detailed understanding of the discrete clock process (see [BČM06, Čer03, BČ06a]).

At the end of the introduction, we would like to draw reader's attention to the paper [FM06], where the scaling limit of the trap model on a large complete graph is identified. The situation there is slightly different since there is no natural scaling limit of the simple random walk on a large complete graph in the absence of trapping.

The rest of the paper is organised as follows. In Section 2 we recall the coarse graining construction introduced for $d = 2$ in [BČM06] and we state (for all $d \ge 2$) some results related to this construction. Using these results we prove Theorems 1.3 and 1.4 in Section 3. In Section 4 we give the proofs of the claims from Section 2 for $d \ge 3$.

2. Coarse graining

We define in this section the coarse-graining procedure that was used in [BČM06, Čer03] to prove aging (1.15). We also recall some properties of this procedure which we need to prove our scaling limit results.

We use $D_x(r)$ (resp. $B_x(r)$) to denote the ball (box) with radius (side) r centred at x; these sets are understood as subsets of $\mathbb{Z}^d$. We will often use the claim that $D_x(r)$ contains $d^{-1}\omega_d r^d$ sites, where ω_d is the surface of a d-dimensional unit sphere, although it is not precisely true. Any error we introduce by this consideration is negligible for r large.

It follows from the definition of X that the clock process S can be written as

$$S(k) = \sum_{i=0}^{k-1} e_i \tau_{Y_i}, \tag{2.1}$$

where the e_i's are mean-one i.i.d. exponential random variables. We always suppose that e_i's are coupled with X and Y in this way.

Let $n \in \mathbb{N}$ large. We will consider the processes Y and X before the first exit from the large ball $\mathbb{D}(n) = \mathbb{D}(mr(n))$, where[1]

$$r(n) = \begin{cases} \pi^{-1/2} 2^{n/2} n^{(1-\alpha)/2}, & \text{if } d = 2, \\ 2^{n/2}, & \text{if } d \geq 3. \end{cases} \tag{2.2}$$

Let ζ_n be the exit time of Y from $\mathbb{D}(n)$,

$$\zeta_n = \inf\{k \in \mathbb{N} : Y(k) \notin \mathbb{D}(n)\}. \tag{2.3}$$

In $\mathbb{D}(n)$, a principal contribution to the clock process comes from traps with depth of order $g(n)$ where

$$g(n) = \begin{cases} n^{-1} 2^{n/\alpha}, & \text{if } d = 2, \\ 2^{n/\alpha}, & \text{if } d \geq 3. \end{cases} \tag{2.4}$$

We define, as in [BČM06],

$$T_\varepsilon^M(n) = \{x \in \mathbb{D}(n) : \varepsilon g(n) \leq \tau_x < M g(n)\}. \tag{2.5}$$

If M or ε are omitted, it is understood $M = \infty$, resp. $\varepsilon = 0$. We always suppose that $\varepsilon < 1 < M$. We further introduce two d-dependent constants κ, γ. For $d = 2$ we choose

$$\gamma < 1 - \alpha \qquad \text{and} \qquad \kappa = \frac{5}{1-\alpha}, \tag{2.6}$$

for $d \geq 3$

$$\gamma = 1 - \frac{1}{3d} \qquad \text{and} \qquad \kappa = \frac{1}{d}. \tag{2.7}$$

We then define the coarse-graining scale $\rho(n)$ as

$$\rho(n) = \begin{cases} \pi^{-1/2} 2^{n/2} n^{\gamma/2}, & \text{if } d = 2, \\ 2^{\gamma n/2}, & \text{if } d \geq 3. \end{cases} \tag{2.8}$$

[1]The scales for $d = 2$ are chosen to agree with [BČM06]

We will often abbreviate

$$h(n) = r(n)/\rho(n). \tag{2.9}$$

The last scale we need is the "proximity" scale

$$\nu(n) = \begin{cases} \pi^{-1/2} 2^{n/2} n^{-\kappa/2}, & \text{if } d = 2, \\ 2^{\kappa n/2}, & \text{if } d \geq 3. \end{cases} \tag{2.10}$$

Observe that $\nu \ll \rho$. We use $\mathcal{E}(n)$, $\mathcal{B}(n)$ to denote the sets

$$\mathcal{E}(n) = \big\{ x \in \mathbb{D}(n) : \operatorname{dist}(x, T_\varepsilon^M(n)) > \nu(n) \big\}, \tag{2.11}$$

$$\mathcal{B}(n) = \begin{cases} \emptyset, & \text{if } d = 2, \\ \big\{ x \in T_\varepsilon^M(n) : (\exists y \neq x : y \in T_\varepsilon^M(n), \operatorname{dist}(x,y) \leq \nu(n) \big\}, & \text{if } d \geq 3. \end{cases} \tag{2.12}$$

For all objects defined above we will often skip the dependence on n in the notation.

We now introduce the coarse-graining procedure. Let j_i^n be a sequence of stopping times for Y given by $j_0^n = 0$ and

$$j_i^n = \min \big\{ k > j_{i-1}^n : \operatorname{dist}(Y(k), Y(j_{i-1}^n)) > \rho(n) \big\}, \qquad i \in \mathbb{N}. \tag{2.13}$$

For every $i \in \mathbb{N}_0$ we define the *score* of the part of the trajectory between j_i^n and j_{i+1}^n as follows. Let

$$\lambda_{i,1}^n = \min\{ k \geq j_i^n : Y(k) \in T_\varepsilon^M \} \tag{2.14}$$

and $y_i^n = Y(\lambda_{i,1}^n)$. Let further

$$\begin{aligned} \lambda_{i,2}^n &= \min\{ k \geq \lambda_{i,1}^n : \operatorname{dist}(Y(k), y_i^n) > \nu(n) \}, \\ \lambda_{i,3}^n &= \min \big[\{ k \geq \lambda_{i,1}^n : Y(k) \in T_\varepsilon^M \setminus y_i^n \} \cup \{ k \geq \lambda_{i,2}^n : Y(k) \in T_\varepsilon^M \} \big]. \end{aligned} \tag{2.15}$$

If the part of the trajectory between j_i^n and j_{i+1}^n satisfies

$$\operatorname{dist}(Y(j_i^n), \partial \mathbb{D}(n)) > \rho(n), \qquad Y(j_i^n), Y(j_{i+1}^n) \in \mathcal{E}(n) \tag{2.16}$$

and

$$\lambda_{i,1}^n < \lambda_{i,2}^n < j_{i+1}^n \leq \lambda_{i,3}^n, \qquad \operatorname{dist}\big(y_i^n, \partial D_{Y(j_i^n)}(\rho(n)) \big) > \nu(n), \qquad y_i^n \notin \mathcal{B}(n), \tag{2.17}$$

then we define the score of this part as

$$s_i^n = \sum_{k=\lambda_{i,1}^n}^{\lambda_{i,2}^n} e_k \tau_k \mathbb{1}\{ Y(k) = y_i^n \}. \tag{2.18}$$

If (2.16) is satisfied and $\lambda_{i,1}^n \geq j_{i+1}^n$, we set $s_i^n = 0$. In both these cases s_i^n records the time spend by X in T_ε^M during the i^{th} part of the trajectory. In all other cases we set $s_i^n = \infty$. This value marks the part of trajectory where something "bad" happens. We use $J(n)$ to denote the index of the first bad part,

$$J(n) = \min\{ i : s_i^n = \infty \}. \tag{2.19}$$

We finally introduce two families of random variables, $s_x^n \in [0, \infty)$ and $r_x^n \in \mathbb{Z}^d$, indexed by $x \in \mathbb{D}(n)$. By definition, the law of s_x^n is the same as the law of s_i^n conditionally on $Y(j_i^n) = x$ (and on $\boldsymbol{\tau}$). Similarly, the law of r_x^n is the same as the law of $Y(j_{i+1}^n) - Y(j_i^n)$ conditionally on the same event.

We will need these properties of the random variables s_x^n.

Lemma 2.1. *Let*

$$\mathcal{E}_0(n) = \{x \in \mathcal{E}(n) : \mathrm{dist}(x, \partial\mathbb{D}(n)) > \rho(n)\}. \tag{2.20}$$

Then, for every ε, M *and for* $\mathbb{P}$*-a.e. random environment* $\boldsymbol{\tau}$

(i)

$$\max_{x\in\mathcal{E}_0(n)} \mathbb{P}[s_x^n = \infty|\boldsymbol{\tau}] = o\big(h(n)^{-2}\big). \tag{2.21}$$

(ii)

$$\lim_{n\to\infty} \max_{x\in\mathcal{E}_0(n)} \Big| h(n)^2 \big\{1 - \mathbb{E}\big[e^{-\lambda s_x^n/2^{n/\alpha}} \big| s_x^n < \infty, \boldsymbol{\tau}\big]\big\} - F_d(\lambda)\Big| = 0. \tag{2.22}$$

Here

$$F_d(\lambda) = \mathcal{K}_d \Big\{ p_\varepsilon^M - \int_\varepsilon^M \frac{\alpha}{1 + \mathcal{K}'_d \lambda z} \cdot \frac{1}{z^{\alpha+1}} \mathrm{d}z \Big\}, \tag{2.23}$$

$p_\varepsilon^M = \varepsilon^{-\alpha} - M^{-\alpha}$ *and*

$$\mathcal{K}_d = \begin{cases} (\log 2)^{-1}, \\ 1, \end{cases} \qquad \mathcal{K}'_d = \begin{cases} \pi^{-1}\log 2, & \text{if } d = 2, \\ G_d(0), & \text{if } d \geq 3, \end{cases} \tag{2.24}$$

(iii)

$$\lim_{n\to\infty} \max_{x\in\mathcal{E}_0(x)} \Big| h(n)^2 \mathbb{P}[s_x^n \neq 0|\boldsymbol{\tau}] - \mathcal{K}_d p_\varepsilon^M \Big| = 0. \tag{2.25}$$

For $d = 2$ (i) follows from Section 5 , (ii) from Lemma 6.4, and (iii) from Lemma 5.7 of [BČM06]. We give in Section 4 a proof for $d \geq 3$ taken from [Čer03].

It is worth noting that (i) of the previous lemma implies that (ii) holds also when conditioning on $s_x < \infty$ is removed. As a corollary of (i) we also get

Corollary 2.2. *For every* $\delta, T > 0$ *there exists* m *independent of* ε *and* M*, such that* $\boldsymbol{\tau}$*-a.s. for* n *large*

$$\mathbb{P}[J(n)/h(n)^2 \geq T|\boldsymbol{\tau}] \geq 1 - \delta. \tag{2.26}$$

Proof. By (1.2)

$$\mathbb{P}[0 \notin \mathcal{E}_0(n)] \leq \sum_{x\in D(\nu(n))} \mathbb{P}[x \in T_\varepsilon^M] \leq C\nu(n)^d g(n)^{-\alpha}. \tag{2.27}$$

This is $O(n^{-\kappa+\alpha})$ for $d = 2$ and $O(2^{-n/2})$ for $d \geq 3$. In both cases the Borel-Cantelli lemma implies that $\boldsymbol{\tau}$-a.s. $0 \in \mathcal{E}_0(n)$ for n large. Therefore, by Lemma 2.1, $\mathbb{P}[s_0^n = \infty|\boldsymbol{\tau}] = o(h^{-2})$. Moreover, by the second condition in (2.16), if $s_0^n < \infty$, then the first part of the trajectory ends in $\mathcal{E}$. Actually, it ends in $\mathcal{E}_0$ since the set $\mathcal{E} \setminus \mathcal{E}_0$ is at distance $r - \rho \gg \rho$ from the origin. Therefore, the second part of the trajectory starts in $\mathcal{E}_0$ and thus $s_1^n = \infty$ with probability $o(h^{-2})$. This remains true for all parts of the trajectory before the walk approaches the boundary of $\mathbb{D}$. However, since $r/\rho = h$, the expected number of parts needed to approach $\partial\mathbb{D}$ scales as m^2h^2. Therefore, it is possible to choose m large enough such that Th^2 parts stay in $D(mr - \rho)$ with probability larger than $1 - \delta/2$. The probability that at least one of these parts is bad is $Th^2 o(h^{-2}) = o(1)$. This finishes the proof. □

The behaviour of the random variables r_x^n is easy to control.

Lemma 2.3. *For every $\xi \in \mathbb{R}^d$ and for all $x \in \mathcal{E}(n)$*

$$\lim_{n\to\infty} h(n)^{-2}\big\{1 - \mathbb{E}\big(e^{-\xi\cdot r_x^n/r(n)}\big)\big\} = -\frac{|\xi|^2}{2d}. \tag{2.28}$$

Proof. By definition $|r_x| = \rho(1+o(1)) = r/h \ll r$. Using the Taylor expansion and the symmetry of the distribution of r_x we get

$$\mathbb{E}\big[e^{-\xi\cdot r_x/r}\big] = 1 + \mathbb{E}\Big[\frac{1}{2}h^{-2}\Big(\xi\cdot\frac{r_x}{\rho}\Big)^2\Big] + O(h^{-4}). \tag{2.29}$$

It follows, e.g., from Lemma 1.7.4 of [Law91] that the distribution of r_x/ρ converges to the uniform distribution on the sphere of radius one. The result then follows by an easy integration. □

The reason why the scores s_i^n were introduced in [BČM06] is that the sum of scores is a good approximation for the clock process.

Lemma 2.4. *For any $\delta > 0$ and $T > 0$ one can choose ε, M and m such that $\boldsymbol{\tau}$-a.s. for all n large enough,*

$$\mathbb{P}\Big[\frac{1}{2^{n/\alpha}}\max\Big\{\Big|S(j_k^n) - \sum_{j=0}^{k-1} s_j^n\Big| : k\in\{1,\dots,h^2T\}\Big\} \geq \delta\Big|\boldsymbol{\tau}\Big] < \delta. \tag{2.30}$$

The proof of this lemma for $d \geq 2$ can be found on pages 30-31 of [BČM06]. For $d \geq 3$ it is proved in Section 4.

3. Proofs of Theorems 1.3 and 1.4

We prove Theorem 1.4 first. The next lemma gives the convergence of fixed-time marginals.

Lemma 3.1. *The finite-dimensional distributions of the pair (S_N, Y_N) converge to those of (V_α, B_d).*

In order to prove Lemma 3.1 we will need an important lemma describing the asymptotic behaviour of the *joint* Laplace transform of r_x^n and s_x^n.

Lemma 3.2. *For $\mathbb{P}$-a.e. random environment $\boldsymbol{\tau}$ and for all $\lambda > 0$, $\xi \in \mathbb{R}^d$*

$$\lim_{n\to\infty} h(n)^2\Big\{1 - \mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x^n}{2^{n/\alpha}} - \frac{\xi\cdot r_x^n}{r(n)}\Big)\Big|s_x<\infty,\boldsymbol{\tau}\Big]\Big\} = F_d(\lambda) - \frac{|\xi|^2}{2d} \tag{3.1}$$

uniformly in $x \in \mathcal{E}_0(n)$.

Proof. Note first that by Lemma 2.1(i) $\mathbb{P}[s_x = \infty|\boldsymbol{\tau}] = o(h(n)^{-2})$. Therefore, we can remove the conditioning on $s_x < \infty$. To shorten the expressions we do not explicitly write conditioning on $\boldsymbol{\tau}$ in this proof. By a trivial decomposition

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}} - \frac{\xi\cdot r_x}{r(n)}\Big)\Big] \\
&= \mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}} - \frac{\xi\cdot r_x}{r(n)}\Big)\mathbb{1}\{s_x=0\}\Big] + \mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}} - \frac{\xi\cdot r_x}{r(n)}\Big)\mathbb{1}\{s_x\neq 0\}\Big] \quad (3.2)\\
&= \mathbb{E}\Big[\exp\Big(-\frac{\xi\cdot r_x}{r(n)}\Big)\mathbb{1}\{s_x=0\}\Big] + \mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\mathbb{1}\{s_x\neq 0\}\Big]\cdot\mathcal{R}(n),
\end{aligned}$$

where, since $|r_x| = \rho(1+o(1))$,

$$e^{-\rho(n)|\xi|/r(n)} \le \mathcal{R}(n) \le e^{\rho(n)|\xi|/r(n)} \tag{3.3}$$

and therefore $\mathcal{R}(n) = 1 + o(1)$. The first expectation on the right-hand side of (3.2) can be rewritten using Lemma 2.3,

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big(-\frac{\xi\cdot r_x}{r(n)}\Big)\mathbb{1}\{s_x = 0\}\Big] \\ &\quad = \mathbb{E}\Big[\exp\Big(-\frac{\xi\cdot r_x}{r(n)}\Big)\Big] - \mathbb{E}\Big[\exp\Big(-\frac{\xi\cdot r_x}{r(n)}\Big)\mathbb{1}\{s_x \neq 0\}\Big] \\ &\quad = 1 + \frac{|\xi|}{2dh(n)^2} + o\big(h(n)^{-2}\big) - \mathcal{R}(n)\mathbb{P}[s_x \neq 0], \end{aligned} \tag{3.4}$$

where $\mathcal{R}(n)$ satisfies again (3.3). We rewrite the second expectation of (3.2) using Lemma 2.1(ii),

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\mathbb{1}\{s_x \neq 0\}\Big] \\ &\quad = \mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big] - \mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\mathbb{1}\{s_x = 0\}\Big] \\ &\quad = 1 - h(n)^{-2}F_d(\lambda) + o\big(h(n)^{-2}\big) - \mathbb{P}[s_x = 0] \\ &\quad = -h(n)^{-2}F_d(\lambda) + o\big(h(n)^{-2}\big) + \mathbb{P}[s_x \neq 0]. \end{aligned} \tag{3.5}$$

Putting everything together we get

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}} - \frac{\xi\cdot r_x}{\rho(n)h(n)}\Big)\Big] \\ &\quad = 1 + \frac{|\xi|}{2dh(n)^2} - \frac{F_d(\lambda)}{h(n)^2} + o\big(h(n)^{-2}\big) + \{1-\mathcal{R}(n)\}\mathbb{P}[s_x \neq 0]. \end{aligned} \tag{3.6}$$

Since $1 - \mathcal{R}(n) = o(1)$ and by Lemma 2.1(iii) $\mathbb{P}[s_x \neq 0] = O\big(h(n)^{-2}\big)$, the proof is finished. ☐

Proof of Lemma 3.1. To check the convergence of the finite-dimensional distributions of (S_N, Y_N) we choose $n = n(N) \in \mathbb{N}$ and $t = t(N) \in [1, 2^{1/\alpha})$ such that

$$N = 2^{n(N)/\alpha} t(N). \tag{3.7}$$

It is easy to see from definitions of n, t and $r(n)$ that

$$f(N) = c_1 r(n(N)) t(N)^{\alpha/2}, \tag{3.8}$$

where

$$c_1 = c_1(d,\alpha) = \begin{cases} \pi^{1/2}(\alpha^{-1}\log 2)^{(1-\alpha)/2}, & \text{if } d = 2, \\ 1, & \text{if } d \ge 3. \end{cases} \tag{3.9}$$

We further set $c_2 = c_2(d,\alpha) = 1/\big(C_d(\alpha)c_1(d,\alpha)\big)$.

Later we will take the limit $n \to \infty$ for a fixed value of $t \in [1, 2^{1/\alpha})$ instead of taking the limit $N \to \infty$. We will show that this limit exists and does not depend on t. Moreover, as can be seen from the proof, the convergence is uniform in t. Therefore also the limit as $N \to \infty$ exists. We will not comment on the issue of uniformity during the proof. Hence, instead of the convergence of (S_N, Y_N) we show that (in the sense of the finite-dimensional distributions)

$$\Big(\frac{1}{t2^{n/\alpha}}S(c_2^{-2}r(n)^2t^\alpha\cdot), \frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}Y(c_2^{-2}r(n)^2t^\alpha\cdot)\Big) \xrightarrow{n\to\infty} (V_\alpha(\cdot), B_d(\cdot)) \qquad \forall t \in [1, 2^{1/\alpha}). \tag{3.10}$$

Let $r_k^n = Y(j_{k+1}^n) - Y(j_k^n)$. We will approximate the processes on the left-hand side of the last display by sum of scores s_j^n and of displacement r_j^n. It follows from the properties of the simple random walk that the exit time j_1^n from the ball $D(\rho(n))$ satisfies $\mathbb{E}[j_1^n] = \rho(n)^2(1 + o(1))$. Therefore, by the law of large numbers, a.s. for any $\delta' > 0$, $u \le T$ and n large enough

$$j^n_{\lfloor(1-\delta')c_2^{-2}h(n)^2t^\alpha u\rfloor} \le c_2^{-2}r(n)^2t^\alpha u \le j^n_{\lfloor(1+\delta')c_2^{-2}h(n)^2t^\alpha u\rfloor}. \tag{3.11}$$

Since $S(\cdot)$ is increasing, $S(c_2^{-2}r(n)^2t^\alpha u)$ can be approximated from above and below by $S(j^n_{\lfloor(1\pm\delta')c_2^{-2}h(n)^2t^\alpha u\rfloor})$. Lemma 2.4 then yields that for ε small and M, m, n large

$$\mathbb{P}\Big[\Big|\frac{1}{t2^{n/a}}S(j^n_{\lfloor(1\pm\delta')c_2^{-2}h(n)^2t^\alpha u\rfloor}) - \sum_{i=0}^{\lfloor(1\pm\delta')c_2^{-2}h(n)^2t^\alpha u\rfloor-1} s_i^n\Big| \ge \delta \Big| \boldsymbol{\tau}\Big] \le \delta. \tag{3.12}$$

Similarly, it follows from the properties of the simple random walk that for any $\delta > 0$ it is possible to choose δ' such that

$$\mathbb{P}\Big[\Big|\frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}Y(c_2^{-2}r(n)^2t^\alpha u) - \frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}\sum_{i=0}^{\lfloor(1\pm\delta')c_2^{-2}h(n)^2t^\alpha u\rfloor-1} r_i^n\Big| \ge \delta\Big] \le \delta. \tag{3.13}$$

Let $0 = u_0 < u_1 < \cdots < u_q \le T$, $\lambda_i > 0$ and $\xi_i \in \mathbb{R}^d$, where $i \in \{1, \dots, q\}$. To prove the convergence of the finite-dimensional distributions we will prove the $\boldsymbol{\tau}$-a.s. convergence of the Laplace transform

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^q \frac{\lambda_i}{t2^{n/\alpha}}\big\{S(c_2^{-2}r(n)^2t^\alpha u_i) - S(c_2^{-2}r(n)^2t^\alpha u_{i-1})\big\} \\ + \frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}\xi_i \cdot \big\{Y(c_2^{-2}r(n)^2t^\alpha u_i) - Y(c_2^{-2}r(n)^2t^\alpha u_{i-1})\big\}\Big)\Big|\boldsymbol{\tau}\Big].\end{aligned} \tag{3.14}$$

The discussion of the last paragraph implies that it suffices to show the convergence of

$$\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^q \sum_{k\in B_v(n,i)} \frac{\lambda_i}{t2^{n/\alpha}}s_k^n + \frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}\xi_i \cdot r_k^n\Big)\Big|\boldsymbol{\tau}\Big] \tag{3.15}$$

for $v = \pm\delta'$, where

$$B_v(n,i) = \{\lfloor(1+v)c_2^{-2}h(n)^2t^\alpha u_{i-1}\rfloor, \dots, \lfloor(1+v)c_2^{-2}h(n)^2t^\alpha u_i\rfloor - 1\}, \tag{3.16}$$

and to show that as $\delta' \to 0$ both limits coincide.

Let $\mathcal{Q}_n$ be the set of all finite sequences $\{x_\ell \in \mathbb{Z}^d : \ell \in 0, \dots, \lfloor c_2^{-2} h(n)^2 t^\alpha u_q \rfloor - 1\}$. Expression (3.15) can be written as

$$\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{q}\sum_{k\in B_v(n,i)} \frac{\lambda_i}{t2^{n/\alpha}} s_k^n + \frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}\xi_i \cdot r_k^n\Big)\Big|\boldsymbol{\tau}\Big] = \sum_{\{x_\ell\}\in Q_n} \mathbb{P}[Y(j_\ell^n) = x_\ell \forall \ell] \\ \times \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{q}\sum_{k\in B_v(n,i)} \frac{\lambda_i s_k^n}{t2^{n/\alpha}} + \frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}\xi_i \cdot r_k^n\Big)\Big|\boldsymbol{\tau}, Y(j_\ell^n) = x_\ell \forall \ell\Big]. \tag{3.17}$$

The last summation can be further divided into two parts. We first consider sequences such that $\{x_\ell\} \not\subset \mathcal{E}_0$. It follows from Lemma 2.1 and Corollary 2.2 that the sum of probabilities of such sequences can be made arbitrarily small by choosing ε, M and m. We can therefore ignore them. The contribution of the remaining sequences $\{x_\ell\}$ can be evaluated using Lemma 3.2. Indeed, let $\omega = \lfloor c_2^{-2} h(n)^2 t^\alpha u_q \rfloor - 1$. Observe that given $\boldsymbol{\tau}$ and $Y(j_\omega^n) = x_\omega$, the distribution of (s_ω^n, r_ω^n) is independent of the past of the walk and is the same as the distribution of $(s_{x_\omega}, r_{x_\omega})$. Therefore,

$$\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{q}\sum_{k\in B_v(n,i)} \frac{\lambda_i}{t2^{n/\alpha}} s_k^n + \frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}\xi_i \cdot r_k^n\Big)\Big|\boldsymbol{\tau}, Y(j_\ell^n) = x_\ell \forall \ell \le \omega\Big] \\ = \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{q}\sum_{\substack{k\in B_v(n,i)\\ k\le \omega-1}} \frac{\lambda_i}{t2^{n/\alpha}} s_k^n + \frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}\xi_i \cdot r_k^n\Big)\Big|\boldsymbol{\tau}, Y(j_\ell^n) = x_\ell \forall \ell \le \omega - 1\Big] \\ \times \mathbb{E}\Big[\exp\Big(-\frac{\lambda_r}{t2^{n/\alpha}} s_{x_\omega} - \frac{c_2\sqrt{d}}{r(n)t^{\alpha/2}}\xi_i \cdot r_{x_\omega}\Big)\Big|\boldsymbol{\tau}\Big]. \tag{3.18}$$

The last expectation is bounded uniformly in x_ω by

$$1 - (1\pm\delta)h(n)^{-2}\Big(F_d\Big(\frac{\lambda_q}{t}\Big) - \frac{|\xi_q|^2}{2d}\frac{dc_2^2}{t^\alpha}\Big). \tag{3.19}$$

Therefore, we can sum over x_ω and repeat the same manipulation for $x_{\omega-1}$. Iterating, we find that the sum over $\{x_\ell\} \subset \mathcal{E}_0$ is bounded from above by

$$\mathbb{P}[Y(j_\ell^n) \in \mathcal{E}_0 \forall \ell \le \omega] \prod_{i=1}^{q} \Big\{1 - \frac{1}{h(n)^2}\Big((1-\delta)F_d\Big(\frac{\lambda_i}{t}\Big) - (1+\delta)\frac{|\xi_i|^2}{2d}\frac{dc_2^2}{t^\alpha}\Big)\Big\}^{|B_v(n,i)|} \\ = \prod_{i=1}^{q} \exp\Big\{\Big(-(1-\delta)\frac{t^\alpha}{c_2^2}F_d\Big(\frac{\lambda_i}{t}\Big) + (1+\delta)\frac{|\xi_i|^2}{2}\Big)(u_i - u_{i-1})(1+v)\Big\}. \tag{3.20}$$

A lower bound can be constructed analogously. Obviously, as $\delta, \delta' \to 0$ and for $v = \pm\delta'$ the upper and lower bound coincide.

Finally, taking $\varepsilon = 0$ and $M = \infty$ in the definition (2.23) of $F_d(\lambda)$ we find by an easy integration that

$$F_d(\lambda) \xrightarrow[\varepsilon\to 0]{M\to\infty} \mathcal{K}_d(\mathcal{K}'_d\lambda)^\alpha \Gamma(1+\alpha)\Gamma(1-\alpha). \tag{3.21}$$

Therefore, using definitions (3.9) and (1.6),

$$\frac{t^\alpha}{c_2^2} F_d\Big(\frac{\lambda}{t}\Big) \xrightarrow[\varepsilon\to 0]{M\to\infty} \lambda^\alpha. \tag{3.22}$$

This finishes the proof of Lemma 3.1. □

To finish the proof of Theorem 1.4 we need to show the tightness.

Lemma 3.3. *The sequence of the distributions of* (S_N, Y_N) *is tight in* $D([0,T], M_1) \times D^d([0,T], U)$

Proof. To check the tightness for S_N in $D([0,T], M_1)$ we use Theorem 12.12.3 of [Whi02]. Since S_N are increasing, it is easy to see that condition (i) of this theorem is equivalent to the tightness of $S_N(T)$ which can be easily checked from Lemma 3.1. In order to check condition (ii) of the theorem remark that for increasing functions the oscillation function w_s used in [Whi02] is equal to zero. So checking (ii) boils down to controling the boundary oscilations $\bar v(x, 0, \delta)$ and $\bar v(x, T, \delta)$. For the first quantity (using again the monotonicity of S_N) this amounts to check that for any $\varepsilon, \eta > 0$ there is δ such that $\mathbb{P}[S_N(\delta) \geq \eta] < \varepsilon$ which follows again from Lemma 3.1. The reasoning for $\bar v(x, T, \delta)$ is analogous.

For Y_n the proof of the tightness is analogous to the same proof for Donsker's Invariance Principle. The tightness of both components implies the tightness of (S_N, Y_N) in the product topology on $D([0,T], M_1) \times D^d([0,T], U)$. □

Obviously, Lemmas 3.1 and 3.3 imply Theorem 1.4. We can now easily derive Theorem 1.3.

Proof of Theorem 1.3. It is easy to check from definitions (1.4) and (1.7) that $X_N(\cdot) = Y_N(S_N^{-1}(\cdot))$. Let $D_{u,\uparrow}$ denote the subset of $D([0,T])$ consisting of unbounded increasing functions. By Corollary 13.6.4 of [Whi02] the inverse map from $D_{u,\uparrow}(M_1)$ to $D_{u,\uparrow}(U)$ is continuous at strictly increasing functions. Since the Lévy process V_α (the limit of S_N in $(D_{u,\uparrow}, M_1)$) is a.s. strictly increasing, the distribution of S_N^{-1} converges to the distribution of V_α^{-1} weakly on $D_{u,\uparrow}(U)$ and the limit is a.s. continuous. The composition $(f, g) \mapsto f \circ g$ as the mapping from $D^d([0,T], U) \times D_{u,\uparrow}(U)$ to $D^d([0,T], U)$ is continuous at $C^d \times C$ (here C is the space of continuous function) as is easy to check. The weak convergence of X_N on $D^d([0,T], U)$ then follows. □

4. Proofs of the coarse graining estimates for $d \geq 3$

We give here the proofs of Lemmas 2.1 and 2.4 for $d \geq 3$. These proofs are adapted from [Čer03] and use similar techniques as in [BČM06] for $d = 2$. In general, the proofs become slightly simpler because the random walk is transient if $d \geq 3$, and all important quantities (like Green's function, hitting probabilities, etc.) depends only polynomially on the radius of the ball, logarithmic corrections are not required.

4.1. **Proof of Lemma 2.1 for** $d \geq 3$**.** Lemma 2.1 controls the distribution of the random scores s_x for $x \in \mathcal{E}_0$. Typically, s_x equals to the time that X started at x spends in T_ε^M before exiting $D_x(\rho)$. In some exceptional cases $s_x = \infty$. We first show that the probability that this happens is $o(h(n)^{-2})$, that is we prove (i) of Lemma 2.1.

As follows from the definition of s_x the exceptional cases that we need to control are:

(a) the exit point from $D_x(\rho(n))$ is not in $\mathcal{E}(n)$,
(b) Y hits a trap in T_ε^M that is at distance smaller than $\nu(n)$ from $\partial D_x(\rho(n))$,
(c) Y hits two different traps in T_ε^M before the exit of $D_x(\rho(n))$,
(d) Y hits a trap from $\mathcal{B}(n)$ (see (2.12) for definition),
(e) Y hits a trap y in T_ε^M, exits $D_y(\nu(n))$ and then returns to y before exiting $D_x(\rho(n))$.

We now bound the probability of all these events. For the event (a) we have

Lemma 4.1. *Let $P_1(n,x)$ be the probability that the simple random walk started at x exits $D_x(\rho(n))$ at some site that is not in $\mathcal{E}$. Then $\boldsymbol{\tau}$-a.s. for every $x \in \mathcal{E}_0$, $P_1(n,x) \le Cg(n)^{-\alpha}\nu(n)^d = o(h(n)^{-2})$.*

Proof. Let $A_x = A_x(n)$ denote the annulus

$$A_x(n) = D_x(\rho(n)+\nu(n)) \setminus D_x(\rho(n)-\nu(n)). \tag{4.1}$$

We first show that there exists K such that $\boldsymbol{\tau}$-a.s. for n large enough

$$|A_x \cap T_\varepsilon^M| \le K\rho^{d-1}\nu g^{-\alpha} \qquad \text{for all } x \in \mathbb{D}. \tag{4.2}$$

The number of the sites in A_x is bounded by $|A_x| \le c'\rho^{d-1}\nu$. Hence, for x fixed

$$\begin{aligned} &\mathbb{P}\big[|A_x \cap T_\varepsilon^M| \ge K\rho^{d-1}\nu g^{-\alpha}\big] \\ &\quad \le \exp\big(-\lambda K\rho^{d-1}\nu g^{-\alpha}\big)\big\{1 + cg^{-\alpha}\varepsilon^{-\alpha}(e^\lambda - 1)\big\}^{c'\rho^{d-1}\nu} \\ &\quad \le \exp\big\{\rho^{d-1}g^{-\alpha}\nu[-\lambda K + c(e^\lambda - 1)]\big\}. \end{aligned} \tag{4.3}$$

Summing over $x \in \mathbb{D}$ we bound the probability that (4.2) is violated by

$$cr(n)^2 \exp\big\{\rho^{d-1}g^{-\alpha}\nu[-\lambda K + c(e^\lambda - 1)]\big\}. \tag{4.4}$$

Since $\rho^{d-1}g^{-\alpha}\nu = 2^{(d-1)\gamma/2+n\kappa/2-1}$ and $(d-1)\gamma/2+\kappa/2-1 > 0$ for our choice of constants, the fact (4.2) follows by choosing K large and using Borel-Cantelli lemma.

If (4.2) is true, then there is at most $cK\rho^{d-1}\nu g^{-\alpha}\nu^{d-1}$ points on the boundary of $D_x(\rho)$ that are not in $\mathcal{E}$. The probability that Y exits $D_x(\rho)$ in any of such points is $O(\rho^{1-d})$ (see [Law91] Lemma 1.7.4). Hence,

$$\mathbb{P}_x\big[Y \text{ exits } D_x(\rho) \text{ in } \mathcal{E}^c\big] \le cK\rho^{d-1}g^{-\alpha}\nu^d\rho^{1-d} = Cg^{-\alpha}\nu^d = o(h(n)^{-2}). \tag{4.5}$$

This finishes the proof. ☐

Next, we bound the probability that (b) happens.

Lemma 4.2. *Let $P_2(n,x)$ be the probability that the simple random walk started at x hits a trap in $T_\varepsilon^M(n) \cap A_x(n)$ before exiting $D_x(\rho(n))$. Then $\boldsymbol{\tau}$-a.s. for all n large, $P_2(n,x) \le C\rho(n)\nu(n)g(n)^{-\alpha} = o(h(n)^{-2})$ for all $x \in \mathbb{D}$.*

Proof. According to (4.2) there is $\boldsymbol{\tau}$-a.s. at most $K\rho^{d-1}\nu g^{-\alpha}$ traps in $A_x \cap T_\varepsilon^M$. The probability that the walk hits one such trap y is by (A.4) bounded from above by $c|x-y|^{2-d}$. There exists constant C such that for all $y \in A_x$, $|x-y|^{2-d} \le C\rho^{2-d}$. The required probability is thus smaller than $C\rho^{d-1}\nu g^{-\alpha}\rho^{2-d} \le C\rho\nu g^{-\alpha}$. ☐

To proof (c) we need two technical lemmas first.

Lemma 4.3. *Let $\mathcal{P}_x(n)$ denote the probability that the simple random walk started at x hits the set T_ε^M before exiting $D_x(\rho(n))$. Then for any $\delta > 0$ and $\boldsymbol{\tau}$-a.s. there is n_0 such that for all $n \geq n_0$ and for all $x \in \mathcal{E}_0(n)$*

$$\mathcal{P}_x(n) \in \big(\mathcal{K}_d(1-\delta)h(n)^{-2}p_\varepsilon^M, \mathcal{K}_d(1+\delta)h(n)^{-2}p_\varepsilon^M\big). \tag{4.6}$$

Proof. Let

$$V_x(n) = \sum_{y \in T_\varepsilon^M} \mathbb{P}_x\big[Y \text{ hits } y \text{ before exiting } D_x(\rho(n)) | \boldsymbol{\tau}\big]. \tag{4.7}$$

We first show that for any δ and $\boldsymbol{\tau}$-a.s. there exists n_0 such that for $n \geq n_0$ and for all $x \in \mathcal{E}_0$

$$(1-\delta)\mathcal{K}_d p_\varepsilon^M h(n)^{-2} \leq V_x(n) \leq (1+\delta)\mathcal{K}_d p_\varepsilon^M h(n)^{-2}. \tag{4.8}$$

Let $\mu = 1 - 2/(3d)$ and $\iota(n) = 2^{\mu n/2}$, therefore $\kappa < \mu < \gamma$ and $\nu(n) \ll \iota(n) \ll \rho(n)$. Recall that $B_x(r)$ denotes the cube centred at x with side r. Let $\mathcal{D}_n = D_x(\rho - 2\iota) \setminus B_x(\iota)$. We divide the sum in (4.7) into three parts. We use Σ_1 to denote the sum over $y \in T_\varepsilon^M \cap \mathcal{D}_n$, Σ_2 to denote the sum over $y \in T_\varepsilon^M \cap B_x(\iota)$, and Σ_3 to denote the sum over $y \in T_\varepsilon^M \cap (D_x(\rho) \setminus D_x(\rho - 2\iota))$. The reason why we introduce the third sum is the error term in (A.5) which is too large for the traps that are too close to the border of $D(\rho)$.

The main contribution comes from Σ_1, so we treat it first. We cover $\mathcal{D}_n$ by cubes with side ι. It is not difficult to show that $\boldsymbol{\tau}$-a.s. for n large

$$\big|B_x(\iota) \cap T_\varepsilon^M\big| \in \big((1-\delta)p_\varepsilon^M \iota^d g^{-\alpha}, (1+\delta)p_\varepsilon^M \iota^d g^{-\alpha}\big) \tag{4.9}$$

for all $x \in \mathbb{D}$ such that $\operatorname{dist}(x, \partial\mathbb{D}) \geq \iota\sqrt{2}$. Indeed, let

$$F_x = \big\{|B_x(\iota) \cap T_\varepsilon^M| \geq (1+\delta)p_\varepsilon^M \iota^d g^{-\alpha}\big\}. \tag{4.10}$$

Then for any small η and n large enough $\mathbb{P}[x \in T_\varepsilon^M] \leq (1+\eta)p_\varepsilon^M g^{-\alpha}$. Hence, for $\lambda > 0$

$$\begin{aligned} \mathbb{P}[F_x] &\leq \exp\big(-\lambda(1+\delta)p_\varepsilon^M \iota^d g^{-\alpha}\big)\big\{1 + (e^\lambda - 1)(1+\eta)p_\varepsilon^M g^{-\alpha}\big\}^{\iota^d} \\ &\leq \exp\big\{p_\varepsilon^M \iota^d g^{-\alpha}[-\lambda(1+\delta) + (e^\lambda - 1)(1+\eta)]\big\}. \end{aligned} \tag{4.11}$$

For any δ one can choose λ and η small enough such that the exponent in the last expression is negative. Hence,

$$\mathbb{P}[F_x] \leq \exp(-c\iota^d g^{-\alpha}) \tag{4.12}$$

for n large enough. Summing over all x and using the definitions of ι and g we get

$$\mathbb{P}\Big[\bigcup_x F_x\Big] \leq Cr^d \exp\big(-c2^{n(d\mu-2)/2}\big). \tag{4.13}$$

Since $d\mu - 2 > 0$, the upper bound for (4.9) is finished. The proof of the lower bound is completely analogous.

We can now actually estimate Σ_1. Without loss of generality we set $x = 0$. Let

$$H_n = \big\{z \in \iota(n)\mathbb{Z}^d \setminus \{0\} : B_z(\iota) \cap \mathcal{D}_n \neq \emptyset\big\}. \tag{4.14}$$

Using the bound (A.5) we get

$$\begin{aligned}\Sigma_1 &\le \sum_{y\in T_\varepsilon^M\cap\mathcal{D}_n} a_d\big\{|y|^{2-d}-\rho^{2-d}+O(|y|^{1-d})\big\}\big(1+O(\rho-|y|)^{2-d}\big)\\ &\le \sum_{z\in H_n}\sum_{\substack{y\in T_\varepsilon^M\\ y\in B_z(\iota)}} a_d\big\{|y|^{2-d}-\rho^{2-d}+O(|y|^{1-d})\big\}\big(1+O(\rho-|y|)^{2-d}\big),\end{aligned}\tag{4.15}$$

where $a_d=\frac{d}{2}\Gamma(\frac{d}{2}-1)\pi^{-d/2}$. Obviously, for any $y\in B_z(\iota)$, $\big||y|^{2-d}-|z|^{2-d}\big|\le c\iota|z|^{1-d}$. This together with (4.9) yields the bound

$$\Sigma_1\le\sum_{z\in H_n}(1+\delta)p_\varepsilon^M\iota^d g^{-\alpha}a_d\big\{|z|^{2-d}-\rho^{2-d}+O(\iota|z|^{1-d})\big\}+\mathcal{R},\tag{4.16}$$

where

$$\mathcal{R}=\sum_{z\in H_n}\sum_{\substack{y\in T_\varepsilon^M\\ y\in B_z(\iota)}} a_d\big\{|y|^{2-d}-\rho^{2-d}+O(|y|^{1-d})\big\}O(\rho-|y|)^{2-d}.\tag{4.17}$$

Every site y from the last summation satisfies $|y|\le\rho-\iota$. Therefore, $O(\rho-|y|)^{2-d}=O(\iota^{2-d})$. The error term $\mathcal{R}$ is thus much smaller than the sum in (4.16) which we now estimate. Replacing the summation by integration (making again an error of order $\iota|z|^{1-d}$) we get

$$\begin{aligned}\Sigma_1 &\le (1+\delta)p_\varepsilon^M g^{-\alpha}\int_{\mathcal{D}} a_d\big\{|z|^{2-d}-\rho^{2-d}+O(\iota|z|^{1-d})\big\}\mathrm{d}z+\mathcal{R}\\ &\le (1+\delta)p_\varepsilon^M g^{-\alpha}\rho^2 a_d\omega_d\Big(\frac12-\frac1d\Big)(1+o(1))\le(1+2\delta)\mathcal{K}_d p_\varepsilon^M h(n)^{-2},\end{aligned}\tag{4.18}$$

where ω_d denotes as before the surface of the d-dimensional unit sphere. The lower bound for Σ_1 can be obtained in the same way. It is actually much simpler, because the lower bound (A.3) on the hitting probability is less complicated than the upper bound (A.5). Hence,

$$\Sigma_1\ge(1-2\delta)\mathcal{K}_d p_\varepsilon^M h(n)^{-2}.\tag{4.19}$$

It remains to show that Σ_2 and Σ_3 are $o(h(n)^{-2})$. To estimate Σ_2 we need a finer description of the homogeneity of the environment than (4.9). Let $i_{\max}$ be the smallest integer satisfying $2^i\nu(n)\ge\iota(n)$, i.e. $i_{\max}\sim(\mu-\kappa)n/2$. Then $\boldsymbol{\tau}$-a.s. for n large, all $i\in\{0,\dots,i_{\max}\}$, and all $x\in\mathbb{D}$

$$\big|B_x(2^i\nu)\cap T_\varepsilon^M\big|\le n\big(1\vee 2^{id}\nu^d g^{-\alpha}\big).\tag{4.20}$$

Indeed, fix $i\in\{-1,\dots,i_{\max}\}$ first. Then for any $x\in\mathbb{D}$ we have

$$\begin{aligned}&\mathbb{P}\big[|B_x(2^{n\gamma+i})\cap T_\varepsilon^M|\ge n(1\vee 2^{id}\nu^d g^{-\alpha})\big]\\ &\quad\le\exp\big(-\lambda n(1\vee 2^{id}\nu^d g^{-\alpha})\big)\big\{1+c(e^\lambda-1)\varepsilon^{-\alpha}g^{-\alpha}\big\}^{2^{id}\nu^d}\\ &\quad\le C\exp(-c\lambda n).\end{aligned}\tag{4.21}$$

Summing over $x\in\mathbb{D}$ and $i\in\{-1,\dots,i_{\max}\}$ we get an upper bound for the probability of the complement of (4.20) which is of order $nr(n)^d e^{-\lambda n}$. Therefore, choosing λ large enough, (4.20) is true $\mathbb{P}$-a.s. for n large enough.

Let $E = \{-1, 0, 1\}^d \setminus \{0, 0, 0\}$. Let $\mathcal{O}_i$ be the union of $3^d - 1$ cubes of size $2^i\nu$ centred at $2^i\nu E$,

$$\mathcal{O}_i = \bigcup_{x \in E} B_{x2^i\nu}(2^i\nu). \tag{4.22}$$

To bound Σ_2 we cover the cube $B(\iota)$ (we suppose again that $x = 0$) by $\bigcup_{i=0}^{i_{\max}} \mathcal{O}_i$. Observe that our covering does not contain $B(\nu)$. However, $B(\nu) \subset D(\nu)$ and $0 \in \mathcal{E}_0$, so that $B(\nu) \cap T_\varepsilon^M = \emptyset$.

By (A.4) and (4.20) we get

$$\Sigma_2 \le C \sum_{i=0}^{i_{\max}} n\big(1 \vee 2^{id}\nu^d g^{-\alpha}\big)(2^i\nu)^{(2-d)} \le C \sum_{i=0}^{(\mu-\kappa)n/2} n\big\{(2^i\nu)^{2-d} \vee 2^{2i}\nu^2 g^{-\alpha}\big\}. \tag{4.23}$$

The fist term in the braces is decreasing in i and the second one is increasing. The sum is thus bounded by $Cn^2(\nu^{2-d} \vee \iota^2 g^{-\alpha})$. However, both terms, $n^2\nu^{2-d}$ and $n^2\iota^2 g^{-\alpha}$, are much smaller than $h(n)^{-2}$ for our choice of constants. This means that $\Sigma_2 \ll \Sigma_1$.

The sum Σ_3, that is the sum over $y \in T_\varepsilon^M \cap (D(\rho) \setminus D(\rho - 2\iota))$ can be bounded in the same way as the probability of hitting a trap in the annulus $A_x \cap T_\varepsilon^M$ was bounded in Lemma 4.2. Following the same reasoning (with $\nu(n)$ replaced by $\iota(n)$) we get $\Sigma_3 \le \rho\iota g^{-\alpha} \ll h(n)^{-2}$. This completes the proof. □

The second technical lemma that we need to bound the event (c) also provides the required bound for the event (d).

Lemma 4.4. *Let $W_x(n) = \sum_{y \in \mathcal{B}(n)} \mathbb{P}_x[Y$ hits y before exiting $D_x(\rho(n))|\boldsymbol{\tau}]$. Then $\boldsymbol{\tau}$-a.s., for and for all $x \in \mathcal{E}(n)$ and n large enough $W_x(n) = o(h(n)^{-2})$.*

Proof. The proof is very similar to the previous one. We divide the sum into three parts in the same way as before. We keep the notation Σ_1, Σ_3, Σ_3 for these parts. Since $\mathcal{B} \subset T_\varepsilon^M$, it follows from the previous proof that Σ_2 and Σ_3 are $o(h(n)^{-2})$. Hence, it remains to bound Σ_1 from above. This can be achieved by the same calculation as before if we show that

$$\big|B_x(\iota) \cap \mathcal{B}\big| = o\big(\big|B_x(\iota) \cap T_\varepsilon^M\big|\big) = o\big(\iota^d g^{-\alpha}\big) \tag{4.24}$$

for all $x \in \mathbb{D}$ (compare this with (4.9)). We will show that $\boldsymbol{\tau}$-a.s. for n-large and for all $x \in \mathbb{D}$

$$\big|B_x(\iota) \cap \mathcal{B}(n)\big| \le n^2\nu^d g^{-2\alpha} r^d =: \phi(n). \tag{4.25}$$

This bound is not optimal but sufficient for our purposes. Indeed, using the definitions of g, ν and r we find that $\phi(n) = O(n^2 2^{(d-3)n/2})$ which is much smaller than $\iota^d g^{-\alpha} = O(2^{(d-8/3)n/2})$.

Let $\mathcal{L}_n$ denote the grid $\iota(n)\mathbb{Z}^d$. Then, $|\mathcal{L}_n \cap \mathbb{D}| \le c(r/\iota)^d$. We use A to denote the event that there exists a cube of side ι containing more than $\phi(n)$ bad sites. If A is true, then there is also a cube of side 2ι centred on $\mathcal{L}_n$ that contains more than $\phi(n)$ bad sites. Therefore,

$$\mathbb{P}[A] \le \sum_{x \in \mathcal{L}_n \cap \mathbb{D}} \mathbb{P}\big[|B_x(2\iota) \cap \mathcal{B}| \ge \phi(n)\big] \le C(r/\iota)^d \mathbb{P}\big[|B(2\iota) \cap \mathcal{B}| \ge \phi(n)\big]. \tag{4.26}$$

Using the definition of $\mathcal{B}$ and the union bound we get that $\mathbb{P}[x \in \mathcal{B}] \le c\varepsilon^{-2\alpha}\nu^d g^{-2\alpha}$. Therefore, by Markov inequality,

$$P\big[|B(2\iota) \cap \mathcal{B}| \ge \phi(n)\big] \le \phi(n)^{-1}\mathbb{E}\Big[\sum_{x\in B(2\iota)} \mathbb{1}\{x \in \mathcal{B}\}\Big] \le C\varepsilon^{-2\alpha}n^{-2}(r/\iota)^{-d}. \tag{4.27}$$

Putting this into (4.26) we obtain $\mathbb{P}[A] \le Cn^{-2}$. Therefore, (4.25) follows by Borel-Cantelli lemma and the proof is completed. □

We now use the last two lemmas to bound the probability of the event (c).

Lemma 4.5. *Let $P_3(n,x)$ denote the probability that the simple random walk started at x hits two traps from $T_\varepsilon^M(n)$ before exiting $D_x(\rho(n))$. Then $\boldsymbol{\tau}$-a.s. for every $x \in \mathcal{E}_0$, $P_3(n,x) = o(h(n)^{-2})$.*

Proof. For $A \subset \mathbb{Z}^d$ we use $\mathcal{Y}(x,\rho,A)$ to denote the number of different traps from A visited by the simple random walk Y before the exit from $D_x(\rho)$. Then,

$$\begin{aligned} P_3(n,x) &= \mathbb{P}\big[\mathcal{Y}(x,\rho,T_\varepsilon^M) \ge 2|\boldsymbol{\tau}\big] \\ &\le \mathbb{P}\big[\mathcal{Y}(x,\rho,T_\varepsilon^M) \ge 2\big|\mathcal{Y}(x,\rho,T_\varepsilon^M\setminus\mathcal{B}) \ge 1,\boldsymbol{\tau}\big]\mathbb{P}\big[\mathcal{Y}(x,\rho,T_\varepsilon^M\setminus\mathcal{B}) \ge 1\big|\boldsymbol{\tau}\big] \\ &\quad + \mathbb{P}\big[\mathcal{Y}(x,\rho,T_\varepsilon^M) \ge 2\big|\mathcal{Y}(x,\rho,\mathcal{B}) \ge 1,\boldsymbol{\tau}\big]\mathbb{P}\big[\mathcal{Y}(x,\rho,\mathcal{B}) \ge 1\big|\boldsymbol{\tau}\big]. \end{aligned} \tag{4.28}$$

By Lemma 4.3,

$$\mathbb{P}\big[\mathcal{Y}(x,\rho,T_\varepsilon^M\setminus\mathcal{B}) \ge 1\big|\boldsymbol{\tau}\big] = O\big(h(n)^{-2}\big), \tag{4.29}$$

and, by Lemma 4.4,

$$\mathbb{P}\big[\mathcal{Y}(x,\rho,\mathcal{B})\big|\boldsymbol{\tau}\big] = o\big(h(n)^{-2}\big). \tag{4.30}$$

If we show that

$$\mathbb{P}\big[\mathcal{Y}(x,\rho,T_\varepsilon^M) \ge 2\big|\mathcal{Y}(x,\rho,T_\varepsilon^M\setminus\mathcal{B}) \ge 1,\boldsymbol{\tau}\big] = O(h(n)^{-2}) = o(1), \tag{4.31}$$

then the lemma follows from (4.28)–(4.31). To prove (4.31) we denote by y the first visited trap from T_ε^M. Then from the strong Markov property and from $D(x,\rho) \subset D(y,2\rho)$ if follows that

$$\mathbb{P}\big[\mathcal{Y}(x,\rho,T_\varepsilon^M) \ge 2\big|\mathcal{Y}(x,\rho,T_\varepsilon^M\setminus\mathcal{B}) \ge 1,\boldsymbol{\tau}\big] \le \mathbb{P}\big[\mathcal{Y}(y,2\rho,T_\varepsilon^M\setminus\{y\}) \ge 1\big|\boldsymbol{\tau}\big]. \tag{4.32}$$

The fact that the right-hand side of the last formula is $O(h(n)^{-2})$ follows from the same argument as Lemma 4.3. This argument works because $y \in T_\varepsilon^M \setminus \mathcal{B}$ and therefore $(T_\varepsilon^M \cap D_y(\nu)) \setminus \{y\} = \emptyset$. The fact that the ball considered in (4.32) is two-times larger than in Lemma 4.3 does not change the asymptotic behaviour, it only changes the prefactor. □

It remains to exclude the event (e).

Lemma 4.6. *Let $P_5(n,x)$ denote the probability that the simple random walk started at x hits a trap $y \in T_\varepsilon^M(n)$ exits $D_y(\nu(n))$ and returns to y before exiting $D_x(\rho(n))$. Then $\boldsymbol{\tau}$-a.s. for every $x \in \mathcal{E}_0(n)$, $P_5(n,x) = o(h(n)^{-2})$.*

Proof. Due to Lemma 4.2 we can suppose that $\mathrm{dist}(y,\partial D_x(\rho)) \ge \nu$. Let $p_{\mathrm{return}}(x,y)$ denote the probability that the simple random started at y that have exited $D_y(\nu)$ returns to y before exiting $D_x(\rho)$. Obviously $P_5(n,x) \le \max\{p_{\mathrm{return}}(x,y) : y \in D_x(\rho)\cap T_\varepsilon^M\}$. Let

$G_A(x,y)$ denote the Green's function of Y killed on the first exit from the set $A \subset \mathbb{Z}^d$. By the decomposition on the first exit from $D_y(\nu)$,

$$G_{D_x(\rho)}(y,y) = G_{D_y(\nu)}(y,y) + p_{\mathrm{return}}(x,y) G_{D_x(\rho)}(y,y). \tag{4.33}$$

Hence, by (A.1), uniformly for $y \in D_x(\rho) \cap T_\varepsilon^M$ and $\mathrm{dist}(y, \partial D_x(\rho)) \ge \nu$,

$$p_{\mathrm{return}}(x,y) = 1 - \frac{G_{D_y(\nu)}(y,y)}{G_{D_x(\rho)}(y,y)} \le 1 - \frac{G_{D(\nu)}(0,0)}{G_{D(2\rho)}(0,0)} = O(\nu^{2-d}) = o(h(n)^{-2}). \tag{4.34}$$

This finishes the proof of (e) and therefore also of Lemma 2.1(i) for $d \ge 3$. □

We now show Lemma 2.1(ii). Since s_x records the time that X spends in T_ε^M, we should first control the distribution of the depth of the first hit trap in T_ε^M. To this end we choose a function $\sigma(n)$ such that

$$\sigma(n) \ge 1/n, \qquad \lim_{n\to\infty} \sigma(n) = 0, \tag{4.35}$$

and, with L defined in (1.2),

$$|L(g(n)x) - 1| = o(\sigma(n)) \quad \text{for all } x \ge \varepsilon. \tag{4.36}$$

Further, let $z_n(i)$ be a sequence satisfying $\varepsilon = z_n(0) < z_n(1) < \dots < z_n(R_n) = M$, and $z_n(i+1) - z_n(i) \in (\sigma(n), 2\sigma(n))$ for all $i \in \{0, \dots, R_n - 1\}$. Let p_i^n denote the factor

$$p_i^n = \frac{1}{z_n(i)^\alpha} - \frac{1}{z_n(i+1)^\alpha}. \tag{4.37}$$

Lemma 4.7. *Let $\mathcal{P}_x(n,i)$ denote the probability that the simple random walk started at x hits the set $T_{z_n(i)}^{z_n(i+1)}(n)$ before exiting $D_x(\rho(n))$. Then for any δ and a.e. $\boldsymbol{\tau}$ there is n_0 such that for all $n \ge n_0$, for all $x \in \mathcal{E}_0(n)$ and for all $i \in \{0, \dots, R_n\}$*

$$\mathcal{P}_x(n,i) \in \big(\mathcal{K}_d(1-\delta)h(n)^{-2}p_i^n, \mathcal{K}_d(1+\delta)h(n)^{-2}p_i^n\big). \tag{4.38}$$

Proof. We will need the following technical claims.

Lemma 4.8. *For any $\delta > 0$, $\boldsymbol{\tau}$-a.s. for all n large*
(i) $\mathbb{P}\big[0 \in T_{z_n(i)}^{z_n(i+1)}\big] \in \big((1-\delta)g^{-\alpha}p_i^n, (1+\delta)g^{-\alpha}p_i^n\big)$,
(ii) for all $x \in \mathbb{D}$ and $i \in \{0, \dots, R-1\}$

$$\big|B_x(\iota) \cap T_{z_n(i)}^{z_n(i+1)}\big| \in \big((1-\delta)\iota^d g^{-\alpha} p_i^n, (1+\delta)\iota^d g^{-\alpha} p_i^n\big). \tag{4.39}$$

Proof. By (1.2) we have

$$p_i^n = g^{-\alpha}\Big[\Big(\frac{1}{z_n(i)^\alpha} - \frac{1}{z_n(i+1)^\alpha}\Big) + \frac{L\big(gz_n(i)\big)}{z_n(i)^\alpha} - \frac{L\big(gz_n(i+1)\big)}{z_n(i+1)^\alpha}\Big]. \tag{4.40}$$

To prove (i) we should thus show that

$$\frac{L\big(gz_n(i)\big)}{z_n(i)^\alpha} - \frac{L\big(gz_n(i+1)\big)}{z_n(i+1)^\alpha} = o\Big(\frac{1}{z_n(i)^\alpha} - \frac{1}{z_n(i+1)^\alpha}\Big). \tag{4.41}$$

However, this is true since $z_n(i)^{-\alpha} - z_n(i+1)^{-\alpha} \asymp \sigma(n)$ and, as follows from (4.36), $L\big(gz_n(i)\big) = o(\sigma(n))$.

The claim (ii) can be proved exactly in the same way as (4.9) was proved; the estimate on $\mathbb{P}[x \in T_\varepsilon^M]$ should be replaced by the first claim of the lemma. The easy proof is left to the reader. □

We can now finish the proof of Lemma 4.7. We use $V_{x,i}(n)$ to denote

$$V_{x,i}(n) = \sum_{y\in T^{z_n(i+1)}_{z_n(i)}} \mathbb{P}_x\big[Y \text{ hits } y \text{ before exiting } D_x(\rho)\big|\boldsymbol{\tau}\big]. \tag{4.42}$$

Lemma 4.8(ii) and the procedure used to show Lemma 4.3 (or more precisely (4.7)) give

$$(1-\delta)\mathcal{K}_d p_i^n h(n)^{-2} \le V_{x,i}(n) \le (1+\delta)\mathcal{K}_d p_i^n h(n)^{-2}. \tag{4.43}$$

For $\mathcal{P}_x(n,i)$ we have then

$$\mathcal{P}_x(n,i) \le V_{x,i}(n) \le (1+\delta)\mathcal{K}_d p_i^n h(n)^{-2}. \tag{4.44}$$

The corresponding lower bound can be obtained using Lemma 4.5 and Bonferroni's inequality. Indeed, using the notation introduced before (4.28),

$$\begin{aligned}\mathcal{P}_x(n,i) &\ge V_{x,i}(n) - \mathbb{P}\big[\mathcal{Y}(x,\rho,T^{z_n(i+1)}_{z_n(i)}) \ge 2\big]\\ &\ge V_{x,i}(n) - \mathbb{P}\big[\mathcal{Y}(x,\rho,T^M_\varepsilon) \ge 2\big] \ge (1-2\delta)\mathcal{K}_d p_i^n h(n)^{-2}.\end{aligned} \tag{4.45}$$

This completes the proof. □

We can now show Lemma 2.1(ii), i.e. to show that $\boldsymbol{\tau}$-a.s.

$$\lim_{n\to\infty} \max_{x\in\mathcal{E}_0(n)} \Big|h(n)^2\big\{1-\mathbb{E}\big[e^{-\lambda s_x/2^{n/\alpha}}\big|s_x<\infty,\boldsymbol{\tau}\big]\big\} - F_d(\lambda)\Big| = 0. \tag{4.46}$$

When the simple random walk Y hits a deep trap y before exiting $D_x(\rho(n))$ and $s_x < \infty$, then s_x is simply the time spent in y before the exit from $D_y(\nu(n))$. The process Y hits y a geometrical number times. The mean of this geometrical variable is $G_{D(\nu(n))}(0,0)$. Each visit takes an exponential time with mean τ_y. Using the expression (A.1) from Appendix we get the following formula for the conditional Laplace transform of s_x.

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|\tau_y, s_x<\infty\Big] = \frac{1}{1+\lambda\tau_y 2^{-n/\alpha}G_d(0)\big(1+o(1)\big)}. \tag{4.47}$$

The probability that $s_x = \infty$ is $o(h(n)^{-2})$. Therefore,

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|s_x<\infty,\boldsymbol{\tau}\Big] = \mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|\boldsymbol{\tau}\Big]\big(1+o(h(n)^{-2})\big). \tag{4.48}$$

The last expectation can be estimated using Lemma 4.7 and (4.47),

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|\boldsymbol{\tau}\Big] \ge\\ &\qquad\big(1-(1+\delta)\mathcal{K}_d p^M_\varepsilon h(n)^{-2}\big) + \mathcal{K}_d h(n)^{-2}\sum_{i=1}^{R_n}\frac{p_i^n(1-\delta)}{1+\lambda z_n(i)G_d(0)(1+o(1))}.\end{aligned} \tag{4.49}$$

For n large the last expression is bounded from below by

$$1-\mathcal{K}_d h(n)^{-2}\Big(p^M_\varepsilon - \int_\varepsilon^M \frac{\alpha}{1+\lambda G_d(0)z}\cdot\frac{1}{z^{\alpha+1}}\,\mathrm{d}z\Big) - \delta C h(n)^{-2}p^M_\varepsilon = 1-h(n)^{-2}(F_d(\lambda)+O(\delta)). \tag{4.50}$$

This and (4.48) give an upper bound for $1-\mathbb{E}\big[e^{-\lambda s_x/2^{n/\alpha}}\big|s_x<\infty,\boldsymbol{\tau}\big]$. A corresponding lower bound can be constructed analogously. This finishes the proof of Lemma 2.1(ii).

Lemma 2.1(iii) follows trivially from Lemma 4.3 and Lemma 2.1(i).

4.2. **Proof of Lemma 2.4 for $d \geq 3$.** We want to show that for any $\delta > 0$ and $T > 0$ it is possible to choose ε, M and m such that for $\boldsymbol{\tau}$-a.s. and n large enough

$$\mathbb{P}\Big[\frac{1}{2^{n/\alpha}} \max\Big\{\Big|S(j_k^n) - \sum_{j=0}^{k-1} s_j^n\Big| : k \in \{1, \dots, h(n)^2 T\}\Big\} \geq \delta\Big] < \delta. \tag{4.51}$$

The sum of scores records (if s_i^n stay finite) only the time spent in T_ε^M. Let $\mathcal{G}_n$ be the event $\{s_j^n < \infty : j \leq Th(n)^2\}$. As follows from Corollary 2.2, probability of $\mathcal{G}_n^c$ can be made smaller than $\delta/2$ by choosing m large. Conditionally on $\mathcal{G}_n$, the difference in (4.51) is positive and it increases with k. It is therefore bounded by

$$S(j_{Th(n)^2}^n) - \sum_{j=0}^{Th(n)^2-1} s_j^n, \tag{4.52}$$

which is simply the time spent in T^ε and T_M during the first $j_{Th(n)^2}^n$ parts.

We first show that the time spent in T^ε is small.

Lemma 4.9. *For any $\delta > 0$ there exists ε such that for a.e. $\boldsymbol{\tau}$ and n large enough*

$$\mathbb{P}\Big[\Big\{\sum_{i=0}^{j_{Th(n)^2}^n} e_i \tau_{Y(i)} \mathbb{1}\{Y(i) \in T^\varepsilon\} \geq 2^{n/\alpha}\delta\Big\} \cap \mathcal{G}_n \Big| \boldsymbol{\tau}\Big] \leq \delta. \tag{4.53}$$

Proof. On $\mathcal{G}_n$ the first $Th(n)^2$ parts of the trajectory stays in $\mathbb{D}(n)$. The probability in (4.53) is thus bounded from above by

$$\mathbb{P}\Big[\sum_{i=0}^{\zeta_n} e_i \tau_{Y(i)} \mathbb{1}\{Y(i) \in T^\varepsilon\} \geq 2^{n/\alpha}\delta \Big| \boldsymbol{\tau}\Big], \tag{4.54}$$

where ζ_n is the exit time of Y from $\mathbb{D}(n)$ (see (2.3)). We show that there exists a constant K_1 independent of ε such that for a.e. $\boldsymbol{\tau}$ and n large enough

$$\mathbb{E}\Big[\sum_{i=0}^{\zeta_n} e_i \tau_{Y(i)} \mathbb{1}\{Y(i) \in T^\varepsilon\} \Big| \boldsymbol{\tau}\Big] \leq K_1 \varepsilon^{1-\alpha} 2^{n/\alpha}. \tag{4.55}$$

The claim of the lemma then follows by the Markov inequality.

To prove (4.55) we bound the expected time spent in traps with $\tau_x \leq 1$ first,

$$\begin{aligned}\mathbb{E}\Big[\sum_{i=0}^{\zeta_n} e_i \tau_{Y(i)} \mathbb{1}\{\tau_{Y(i)} \leq 1\} \Big| \boldsymbol{\tau}\Big] &= \sum_{x \in \mathbb{D}} \tau_x G_{\mathbb{D}}(0, x) \mathbb{1}\{\tau_x \leq 1\} \\ &\leq \sum_{x \in \mathbb{D}} G_{\mathbb{D}}(0, x) = \mathbb{E}(\zeta_n) = O(2^n) \ll 2^{n/\alpha}.\end{aligned} \tag{4.56}$$

We divide the remaining part of T^ε into disjoint sets $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$, $i \in \{1, \dots, i_{\max}\}$, where $i_{\max}$ is an integer satisfying

$$1/2 \leq 2^{-i_{\max}} \varepsilon g(n) < 1. \tag{4.57}$$

From the condition (1.2) it can be showed easily that the probability that a fixed site x is in $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$ is bounded by

$$p_{n,i} := \mathbb{P}\big[x \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big] \leq \mathbb{P}[\tau_x \geq 2^{-i} \varepsilon g(n)] \leq c \varepsilon^{-\alpha} g^{-\alpha} 2^{i\alpha}. \tag{4.58}$$

For any fixed $i \in \{1, \dots, i_{\max}\}$ and K' large we can write

$$
\begin{aligned}
&\mathbb{P}\Big[\mathbb{E}\Big[\sum_{i=0}^{\zeta_n - 1} e_i \tau_{Y(i)} \mathbb{1}\big\{Y(i) \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big\}\Big|\boldsymbol{\tau}\Big] \geq K' \varepsilon^{1-\alpha} 2^{i(\alpha-1)} 2^{n/\alpha}\Big] \\
&\quad = \mathbb{P}\Big[\sum_{x\in\mathbb{D}} G_{\mathbb{D}}(0,x)\tau_x \mathbb{1}\big\{x \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big\} \geq K' \varepsilon^{1-\alpha} 2^{i(\alpha-1)} 2^{n/\alpha}\Big] \\
&\quad \leq \mathbb{P}\Big[\sum_{x\in\mathbb{D}} G_{\mathbb{D}}(0,x) \mathbb{1}\big\{x \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big\} \geq K' \varepsilon^{-\alpha} 2^{i\alpha-1}\Big].
\end{aligned}
\tag{4.59}
$$

Using the Markov inequality (with $\lambda_n > 0$) this can be bounded by

$$
\begin{aligned}
&\leq \exp(-\lambda_n K' \varepsilon^{-\alpha} 2^{i\alpha-1}) \prod_{x\in\mathbb{D}} \big[(1-p_{n,i}) + p_{n,i} e^{\lambda_n G_{\mathbb{D}}(0,x)}\big] \\
&\leq \exp(-\lambda_n K' \varepsilon^{-\alpha} 2^{i\alpha-1}) \prod_{x\in\mathbb{D}} \big[1 + c 2^{i\alpha} g^{-\alpha} \varepsilon^{-\alpha}\big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big)\big].
\end{aligned}
\tag{4.60}
$$

Since $x \geq \log(1+x)$, we have

$$
\log \prod_{x\in\mathbb{D}} \big[1 + c 2^{i\alpha} g^{-\alpha} \varepsilon^{-\alpha}\big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big)\big] \leq \sum_{x\in\mathbb{D}} c 2^{i\alpha} g^{-\alpha} \varepsilon^{-\alpha}\big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big). \tag{4.61}
$$

Let $\lambda_n = n/2G_{\mathbb{D}}(0,0)$. We divide the last sum into two parts. First, we sum over the sites that are close to the origin, $|x| \leq n^{2/(d-2)}$. Since $G_{\mathbb{D}}(0,x) \leq G_{\mathbb{D}}(0,0)$, we have

$$
\begin{aligned}
&\sum_{x\in D(n^{2/(d-2)})} c 2^{i\alpha} g^{-\alpha} \varepsilon^{-\alpha}\big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big) \\
&\qquad \leq C n^{2d/(d-2)} 2^{i\alpha-n} \varepsilon^{-\alpha} e^{\lambda_n G_{\mathbb{D}}(0,0)} \leq C n^{2d/(d-2)} 2^{i\alpha} 2^{-n} \varepsilon^{-\alpha} e^{n/2}.
\end{aligned}
\tag{4.62}
$$

The last expression tends to 0 as $n \to \infty$.

By (A.2), $G_{\mathbb{D}}(0,x) \leq cn^{-2}$ for $x \in \mathbb{D}(n) \setminus D(n^{2/(d-2)})$. Therefore, the argument of the exponential in (4.61) is smaller than $c'n^{-1}$. Using the fact that $e^x - 1 \leq 2x$ for x sufficiently close to 0 we get $e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1 \leq cnG_{\mathbb{D}}(0,x)$ and thus

$$
\begin{aligned}
&\sum_{x\in\mathbb{D}\setminus D(n^{2/(d-2)})} c 2^{i\alpha} g^{-\alpha} \varepsilon^{-\alpha}\big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big) \\
&\qquad \leq \sum_{x\in\mathbb{D}\setminus D(n^{2/(d-2)})} C n 2^{i\alpha} \varepsilon^{-\alpha} g^{-\alpha} G_{\mathbb{D}}(0,x) \leq C 2^{i\alpha} \varepsilon^{-\alpha} n.
\end{aligned}
\tag{4.63}
$$

Here we used $\sum_{x\in\mathbb{D}} G_{\mathbb{D}}(0,x) = O(r(n)^2) = O(g^\alpha)$. From (4.62) and (4.63) it follows that the expression in (4.60) can be bounded from above by

$$
\exp(-K' c n \varepsilon^{-\alpha} 2^{i\alpha}) \exp(C n \varepsilon^{-\alpha} 2^{i\alpha}). \tag{4.64}
$$

Therefore, it is possible to choose K' large enough such that this bound decreases exponentially with n for all $i \in \{0, \dots, i_{\max}\}$.

Summing over all possible values of i gives

$$
\mathbb{P}\Big[\bigcup_{i=0}^{i_{\max}} \Big(\mathbb{E}\Big[\sum_{i=0}^{\zeta_n - 1} e_i \tau_{Y(i)} \mathbb{1}(Y(i) \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}})\Big|\boldsymbol{\tau}\Big] \geq K' \varepsilon^{1-\alpha} 2^{i(\alpha-1)} 2^{n/\alpha}\Big)\Big] \leq cne^{-c'n}. \tag{4.65}
$$

The Borel-Cantelli lemma then yields

$$\mathbb{E}\Big[\sum_{i=0}^{\zeta_n-1} e_i\tau_{Y(i)}\mathbb{1}(Y(i)\in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}})\Big|\boldsymbol{\tau}\Big] \le K'\varepsilon^{1-\alpha}2^{i(\alpha-1)}2^{n/\alpha} \tag{4.66}$$

$\boldsymbol{\tau}$-a.s. for all i and for n large enough. Combining (4.56) and (4.66) we get easily (4.55). This finishes the proof of Lemma 4.9. ☐

We show now that the set T^M can be safely ignored.

Lemma 4.10. *For every δ there exist m and M such that for a.e. $\boldsymbol{\tau}$ and n large enough*

$$\mathbb{P}\big[\{Y \textit{ hits } T_M \textit{ before } j^n_{Th(n)^2}\}\cap\mathcal{G}_n|\boldsymbol{\tau}\big] \le \delta. \tag{4.67}$$

Proof. As in the proof of Lemma 4.9 we can replace $j^n_{Th(n)^2}$ by ζ_n. We use again Borel-Cantelli lemma,

$$\mathbb{P}\big[\mathbb{P}[Y \text{ hits } T_M(n) \text{ before } \zeta_n|\boldsymbol{\tau}] \ge \delta\big] \le e^{-\lambda_n\delta}\mathbb{E}\big[\exp\big\{\lambda_n\mathbb{P}[Y \text{ hits } T_M(n) \text{ before } \zeta_n|\boldsymbol{\tau}]\big\}\big], \tag{4.68}$$

However,

$$\begin{aligned}&\log\mathbb{E}\big[\exp\big\{\lambda_n\mathbb{P}[Y \text{ hits } T_M(n) \text{ before } \zeta_n|\boldsymbol{\tau}]\big\}\big]\\ &\qquad\le \log\mathbb{E}\big[\exp\big\{\lambda_n\sum_{x\in\mathbb{D}}\mathbb{P}[Y \text{ hits } x \text{ before } \zeta_n]\mathbb{1}(x\in T_M)\big\}\big].\end{aligned} \tag{4.69}$$

Since $\mathbb{P}[x\in T_M]\le cM^{-\alpha}g^{-\alpha}$, we get

$$\begin{aligned}&\le \sum_{x\in\mathbb{D}}\log\big\{1+cM^{-\alpha}g^{-\alpha}\big(\exp\{\lambda_n\mathbb{P}[Y \text{ hits } x \text{ before } \zeta_n]\}-1\big)\big\}\\ &\le \sum_{x\in\mathbb{D}} cM^{-\alpha}\gamma^{-\alpha}\big\{\exp(\lambda_n\mathbb{P}[Y \text{ hits } x \text{ before } \zeta_n])-1\big\}.\end{aligned} \tag{4.70}$$

We choose $\lambda_n = n/2$ and divide the sum into two parts. For $|x|\le n^{2/(d-2)}$ we use $\mathbb{P}[Y \text{ hits } x \text{ before } \zeta_n]\le 1$. Hence,

$$\sum_{x\in D(n^{2/(d-2)})} cM^{-\alpha}\gamma^{-\alpha}\big\{\exp(\lambda_n\mathbb{P}[Y \text{ hits } x \text{ before } \zeta_n])-1\big\} \le cn^{2d/(d-2)}2^{-n}e^{n/2}, \tag{4.71}$$

which becomes negligible as $n\to\infty$.

By (A.4), for $|x|\ge n^{2/(d-2)}$ the argument of the exponential in (4.69) is smaller than cn^{-1} and thus

$$\exp\big(\lambda_n\mathbb{P}[Y \text{ hits } x \text{ before } \zeta_n]\big)-1\le cn|x|^{2-d} \tag{4.72}$$

for some large c. We have thus

$$\begin{aligned}&\sum_{x\in\mathbb{D}\setminus D(n^{2/(d-2)})} cM^{-\alpha}g^{-\alpha}\big\{\exp(\lambda_n\mathbb{P}[Y \text{ hits } x \text{ before } \zeta_n])-1\big\}\\ &\qquad\le cM^{-\alpha}g^{-\alpha}n\sum_{x\in\mathbb{D}\setminus D(n^{2/(d-2)})}|y|^{2-d}\le cM^{-\alpha}n.\end{aligned} \tag{4.73}$$

Inserting (4.71) and (4.73) into (4.68) we get

$$\mathbb{P}\big[\mathbb{P}[Y \text{ hits } T_M(n) \text{ before } \zeta_n|\boldsymbol{\tau}]\ge\delta\big]\le c\exp(-n\delta+c'M^{-\alpha}n). \tag{4.74}$$

The proof is finished by taking M large enough. ☐

Appendix A. Properties of the d-dimensional random walk

We summarise here some useful facts about the Green's function and hitting probabilities of the simple random walk in the large ball $D(r) \subset \mathbb{Z}^d$, $d \geq 3$. The following lemma is taken from [Law91], Proposition 1.5.9.

Lemma A.1. *The Green's function $G_{D(r)}(\cdot,\cdot)$ of the simple random walk killed on exit from the ball $D(r)$ satisfies*

$$G_{D(r)}(0,0) = G_d(0) - O\big(r^{2-d}\big). \tag{A.1}$$

and

$$G_{D(r)}(0,x) = a_d\big(|x|^{2-d} - r^{2-d}\big) + O\big(|x|^{1-d}\big), \tag{A.2}$$

where $a_d = \frac{d}{2}\Gamma(\frac{d}{2}-1)\pi^{-d/2}$.

The hitting probabilities are controlled by the following lemma.

Lemma A.2. *Let $p_r(0,x)$ denote the probability that the simple random walk started at 0 hits x before exiting $D(r)$. The function $p_r(0,x)$ satisfies*

$$p_r(0,x) \geq \frac{a_d}{G_d(0)}\big(|x|^{2-d} - r^{2-d}\big) + O\big(|x|^{1-d}\big), \tag{A.3}$$

$$p_r(0,x) \leq a_d\big(|x|^{2-d} - r^{2-d}\big) + O\big(|x|^{1-d}\big). \tag{A.4}$$

More precisely $p_n(0,x)$ can be bounded from above by

$$p_r(0,x) \leq \frac{a_d}{G_d(0)}\big(|x|^{2-d} - r^{2-d} + O\big(|x|^{1-d}\big)\big)\big(1 + O\big((n-|x|)^{2-d}\big)\big). \tag{A.5}$$

Proof. The first two claims follows from equation (A.2),

$$G_{D(r)}(0,x) = p_r(0,x)G_{D(r)}(x,x), \tag{A.6}$$

and from $1 \leq G_{D(r)}(x,x) \leq G_d(0)$. The third fact is a consequence of (A.6) and

$$\begin{aligned} G_{D(r)}(x,x)^{-1} &\leq G_{D_x(r-|x|)}(x,x)^{-1} = G_{D(r-|x|)}(0,0)^{-1} \\ &= G_d(0)^{-1} + O\big((r-|x|)^{2-d}\big), \end{aligned} \tag{A.7}$$

which is a consequence of (A.1). ☐

Aknowledgement. We thank Takashi Kumagai for suggesting this problem of the scaling limit in $d \geq 2$.

References

[BČ06a] G. Ben Arous and J. Černý, *The arcsine law as a universal aging scheme for trap models*, to appear in Comm. Pure Appl. Math. (2006).

[BČ06b] G. Ben Arous and J. Černý, *Dynamics of trap models*, Mathematical Statistical Physics, Les Houches summer school proceedings, vol. LXXXIII, Elsevier, 2006, to be published.

[BČM06] G. Ben Arous, J. Černý, and T. Mountford, *Aging for Bouchaud's model in dimension two*, Probab. Theory Related Fields **134** (2006), no. 1, 1–43.

[Ber96] J. Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996.

[Čer03] J. Černý, *On two properties of strongly disordered systems, aging and critical path analysis*, Ph.D. thesis, EPF Lausanne, 2003.

[Fel71] W. Feller, *An introduction to probability theory and its applications. Vol. II.*, Second edition, John Wiley & Sons Inc., New York, 1971.

[FIN02] L. R. G. Fontes, M. Isopi, and C. M. Newman, *Random walks with strongly inhomogeneous rates and singular diffusions: convergence, localization and aging in one dimension*, Ann. Probab. **30** (2002), no. 2, 579–604.

[FM06] L. R. Fontes and P. Mathieu, *K-processes, scaling limit and aging for the REM-like trap model*, preprint, math.PR/0603198, 2006.

[GM03] R. Gorenflo and F. Mainardi, *Fractional diffusion processes: Probability distributions and continuous time random walk*, Processes with Long Range Correlations (G. Rangarajan and M. Ding, eds.), Lecture Notes in Physics, no. 621, Springer-Verlag, Berlin, 2003, pp. 148–166.

[Hil00] R. Hilfer (ed.), *Applications of fractional calculus in physics*, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

[Law91] G. F. Lawler, *Intersections of random walks*, Probability and its Applications, Birkhäuser Boston Inc., Boston, MA, 1991.

[MS04] M. M. Meerschaert and H.-P. Scheffler, *Limit theorems for continuous-time random walks with infinite mean waiting times*, J. Appl. Probab. **41** (2004), no. 3, 623–638.

[MK00] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 77.

[MS84] E. W. Montroll and M. F. Shlesinger, *On the wonderful world of random walks*, Nonequilibrium phenomena, II, Stud. Statist. Mech., XI, North-Holland, Amsterdam, 1984, pp. 1–121.

[MW65] E. W. Montroll and G. H. Weiss, *Random walks on lattices. II*, Journal of Mathematical Physics **6** (1965), no. 2, 167–181.

[SZ97] A. I. Saichev and G. M. Zaslavsky, *Fractional kinetic equations: solutions and applications*, Chaos **7** (1997), no. 4, 753–764.

[SZK93] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, *Strange kinetics*, Nature **363** (1993), 31–37.

[Sko56] A. V. Skorohod, *Limit theorems for stochastic processes*, Teor. Veroyatnost. i Primenen. **1** (1956), 289–319.

[Whi02] W. Whitt, *Stochastic-process limits*, Springer Series in Operations Research, Springer-Verlag, New York, 2002, An introduction to stochastic-process limits and their application to queues.

[Zas05] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, 2005.

[Zas02] G. Zaslavsky, *Chaos, fractional kinetics, and anomalous transport*, Physics Reports **371** (2002), 461–580.

Gérard Ben Arous, Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, N.Y. 10012-1185, and École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland
E-mail address: gerard.benarous@epfl.ch

Jiří Černý, École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland
E-mail address: jiri.cerny@epfl.ch