1. (a) \[
\frac{3-2i}{1-4i} = \frac{3-2i}{1-4i} \cdot \frac{1+4i}{1+4i} = \frac{3-2i+12i+8}{17} = \frac{11+6i}{17}
\]

(b) \((1+i)^{\frac{1}{3}} = (\sqrt[12]{2} e^{i\pi/4})^{\frac{1}{3}} = 2^{\frac{1}{6}} e^{i\pi/12} (1, e^{2\pi i/3}, e^{4\pi i/3})\]

\[
= 2^{\frac{1}{6}} e^{i\pi/12}, 2^{\frac{1}{6}} e^{i\pi/12} 2^{\frac{1}{6}} e^{(2\pi i/3) + \frac{1}{3}}, 2^{\frac{1}{6}} e^{(4\pi i/3) + \frac{1}{3}}
\]

(c) \[\left|\frac{1}{1-2z}\right| = 2 \Rightarrow |1-2z| < \frac{1}{2} \text{ the interior of the circle of radius } \frac{1}{2} \text{ centered at } z = 1.
\]

(d) \(z = 1\) is an essential singularity of \(e^{\frac{1}{1-z}}\)

\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \left(1-\frac{z}{m!}\right)^m \text{ and hence of } \frac{e^{\frac{1}{1-z}}}{(1-\frac{z}{m!})^m}
\]

\(z = \pm i\) are poles of order 3 since \((1+2i)^3 = (-1)^3 (1+2i)^3\), and \(e^{\frac{1}{1-z}}\) is analytic on \(\not=0\ \text{ at } z = \pm i\).

(E)(i) \(\Delta^2 (2xy) = 0, \text{ } u_x = 2y = v_y, \text{ } u_y = 2x = -v_x, \text{ } u_{yy} = 2x = -v_x, \text{ } u_{xx} = \frac{y^2 - x^2}{v} + C, \text{ } u_{yx}\)

(ii) \(\Delta^2 x^2 y = 2y \not= 0 \text{ so } f(x) \text{ is not analytic.}\)
(2) By the Cauchy integral formula,

\[ f'(0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^2} \, dz \]

So,

\[ |f'(0)| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f| \, d\theta \leq \frac{2\pi \cdot 1}{2\pi} = 1 \]

(3) (a) \[ \frac{1}{(z^2+1)(z-1)^2} = \frac{1}{(2z-1)(z-1)^2} \]

\[ = \frac{1}{z^3} \left[ 1 + \frac{2}{z} (z-1)^2 \right] (z-1)^2 \]

\[ = \frac{1}{3} \sum_{m=0}^{\infty} c_1^m \left( \frac{2}{3} \right)^m (z-1)^{-m-2} \]

(b) The series converges absolutely if \( |z-1| < \frac{3}{2} = R_{max} \)

(c) \[ \frac{1}{(z^2+1)(z-1)^2} = \frac{1}{(2z-1)(z-1)^2} \]

\[ = \frac{1}{2(z-1)} \left( 1 + \frac{3}{2(z-1)} \right) (z-1)^2 \]

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{3}{2} \right)^n (-1)^n \frac{1}{(z-1)^{2+n}} \]
\[ \oint \frac{1 + z^2}{(1 - z)^3} \, dz \]

\[ = 2\pi i \ \text{Res} \left( \frac{1 + z^2}{(z - 1)^3} \right) \]

\[ = 2\pi i \left( \frac{1}{2} \cdot 2 \right) = 2\pi i \quad z = (z - 1 + 1) \]

On note: \[ 1 + z^2 = 1 + (z - 1)^2 + 2(z - 1) + 1 \]

\[ \therefore \text{Res} = 1 \quad z = 1 \]

or use: \[ \text{Res} = \frac{1}{2} \left( 1 + z^2 \right)^{\prime} \bigg|_{z=1} = 1 \]
5. Within $|z|=2$ we see that we can choose
\[ z^5 = f \quad \text{and} \quad 5z^2 - 1 = g \quad \text{so that} \]
\[ |f| = 2^5 = 32, \quad |g| \leq 13. \quad \text{Hence there are 5 roots in} \quad |z| < 2. \]

On $|z|=1$, we may take $f = 3z^7$ if $|z|=1$
\[ g = 5z^2 - 1, \quad |g| \leq 2, \quad \text{to conclude that} \]
there are 2 roots in $|z| < 1$. Thus there
are 3 roots in $1 \leq |z| < 2$, since
there are no roots in $|z|=1$.

6. \[ I = \int_0^\infty \frac{\sin x}{x(1+x^2)} \, dx = \frac{1}{i} \lim_{R \to \infty} \int_{-R}^R \frac{e^{ix}}{x(1+x^2)} \, dx \]
   \[ c_1 \quad c_2 \quad c_3 \quad c_4 \]
   \[ 2I = \lim_{R \to \infty} \lim_{\epsilon \to 0} \int_{C_R} f(z) \, dz, \quad f(z) = \frac{e^{iz}}{z(1+z^2)} \]
   \[ \int_{C_R} f(z) \, dz = 2\pi i \left[ \frac{e^{iz}}{z(1+z^2)} \right]_{z=-i}^{z=i} \]
   \[ = 2\pi i \left( \frac{e^{-1}}{i(2i)} - \frac{e^{i}}{-i(2i)} \right) = \frac{\pi - i\pi}{4} \]
   \[ \lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) \, dz = +\pi i \quad \text{since} \quad f(z) \text{has a} \]
   \[ \text{simple pole at} \quad z=0 \]
   \[ \text{with residue 1.} \]

\[ \lim_{R \to \infty} \left| \int_{C_\epsilon} f(z) \, dz \right| \leq \frac{\pi R}{2} \int_0^{\pi/2} e^{-\theta^2/R} \, d\theta \to 0 \quad \text{as} \quad R \to \infty \]

Thus \[ 2I = \lim_{R \to \infty} \left( \pi i - \pi i \epsilon^1 \right) = \pi \left( 1 - \frac{\pi i}{i} \right) \]