Applied Mathematics II   **PROBLEM SET 2**   **Due February 6, 2001**

1. (30 pts) Problem 11 on page 101 of G&L. Take $L = 1$ for numerical calculations.

2. (15 pts) Problem 8 on page 106 of G&L. You may use the following differential inequality: If $y(x) \geq 0$ is differentiable with $dy/dx \leq -cy$, $c > 0$, for all $x \geq 0$, then $y(x) \leq y(0)e^{-cx}$ for all $x \geq 0$.

3. (20 pts) (a) Show that the sequence of integrable functions $f_n(x) = n^{-1}|x|^{\frac{1}{n}-1} \to 2\delta(x)$ as $n \to \infty$, as follows: Show that for every test function ϕ and $\epsilon = n^{-\alpha}$, $\alpha > 0$,

$$\int_{-\infty}^{+\infty} \phi(x)f_n(x)dx = \int_{-\infty}^{-\epsilon} \phi(x)f_n(x) + \int_{-\epsilon}^{+\epsilon} \phi(x)f_n(x)dx + \int_{+\epsilon}^{\infty} \phi(x)f_n(x)dx$$

$$\to 0 + 2\phi(0) + 0, \ n \to \infty.$$

Recall any test function has continuous derivatives of all orders for all x and vanishes outside a finite interval.

(b) Show that $x\delta'(x) = -\delta(x)$ where δ' is the derivative of $\delta(x)$.

4. (20 points) Show that sound waves may be treated using a potential ϕ satisfying the wave equation $\phi_{tt} - c^2\nabla^2\phi = 0$ with velocity $\mathbf{u} = \nabla\phi$ and pressure $p = -\phi_t/\rho_0$. Here $c^2 = dp/d\rho(\rho_0)$. Find the potential of the three-dimensional, spherically symmetric sound field emitted by an oscillating sphere. Assume oscillations are of amplitude a about a radius r_0, where $a << r_0$, so the boundary condition to be satisfied is

$$\frac{\partial\phi}{\partial r}(r_0, 0) = a\sin\omega t$$

approximately. (Why is the last expression approximately correct?) Assume also that only outgoing waves appear in the solution. From your expression for ϕ compute the radial velocity field $u_r = \frac{\partial\phi}{\partial r} = \phi_r$. Your expression will involve the dimensionless parameter $\Omega = \omega r_0/c$. (The frequency of the standard note A above middle C is 440 Hertz, or cycles per second, in which case $\omega = 880\pi$. Taking $r_0 = 1cm$ and $c = 350m/sec$ we get $\omega r_0/c \approx .08$. We may take the parameter to be of order unity.) The length scale $R = c/\omega \approx 12cm$ separates the "near" and "far" fields of the sound pattern. Discuss the r-variation of ϕ_r in the two cases $r_0 + a < r << R$ cm and $r >> R$.

5. (15 points) Establish uniqueness of the solution of the IVP for the wave equation in N dimensions using the energy method, in the following case:

(a) Show that if u is twice continuously differentiable and $u_{tt} - c^2\nabla^2u = 0$, for $t > 0$, all $\mathbf{x}$, with $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_N^2}$, and also either u or u_t is identically zero outside a sphere of some large radius R centered at the origin, then the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty}(u_t^2 + c^2(\nabla u)^2)dx_1\ldots dx_N$$

is constant in time. Use the fact that then the range of influence of the initial data lies inside the sphere or radius $R + ct$ at time t.

(b) Then obtain uniqueness by considering the null initial conditions $u_t = 0, u = 0$.