Applied Mathematics II **PROBLEM SET 4** **Due March 6, 2001**

1.(20pts) Evaluate the following integrals asymptotically using the method of stationary phase:

$$(a)\quad \int_0^{\pi/2} e^{ix\cos t}dt\ , x \to +\infty, \quad (b)\quad \int_0^{\infty} e^{i(xk^2-kt)}dk,\ x,t\to\infty,\ x/t\ fixed \neq 0.$$

2.(25 pts) Consider a dispersive medium governed by the wave equation

$$u_{tt} - u_{xx} + 4\epsilon^2 u_{xxxx} = 0.$$

Taking ϵ small, convert to characteristic coordinates $\alpha = x - t, \beta = x + t$, let $u(\alpha,\beta) = f(\alpha,\tau), \tau = \epsilon^2\beta$, and derive an approximate equation, given that f tends to zero for large α,

$$f_\tau - f_{\alpha\alpha\alpha} = 0,$$

for the long-time evolution of a wave moving to the right. Solve this approximate equation in $\beta \geq 0$ given an initial condition

$$f(\alpha,0) = \int_a^b \phi(k)e^{ik\alpha}dk, 0 < a < b < \infty$$

with given $\phi(k)$. Determine the form of the wave for large $\tau, \alpha\ > 0,\ \alpha/\tau\ \ fixed$ in terms of $\phi(k)$, using the method of stationary phase.

3.(20pts) Using the two modes $\omega = \pm\sqrt{|k|g}$ for linear waves in deep water give the solution of the initial value problem in two dimensions with

$$\eta_t(x,0) = 0, \eta(x,0) = \eta_0(x) = \int_{-\infty}^{+\infty} e^{ikx}e^{-(ak)^2}dk$$

where a is a constant. Use the method of stationary phase to estimate the wave height $\eta(x,t)$ for large x, t.

4.(35pts) A linear vibrating string is modified by attaching it to a bed of springs which exerts a force proportional to the displacement of a point of the string from a stretched-straight position $u(x,t) = 0$. Verify that the equation for the displacement $u(x,t)$ is then

$$u_{tt} - c^2u_{xx} + \lambda^2 u = 0,$$

where $c^2 = T/\rho, \lambda$ are constants. Solve the initial value problem for this equation given that

$$u(x,0) = \int_0^\infty e^{-k^2}e^{ikx}dk,\ \ u_t(x,0) = 0.$$

Study the structure of the solution for large x, t using the method of stationary phase. Verify that the group velocity lies between $-c$ and $+c$. Focus on the waves moving to the right $(x, t > 0)$ and give an explicit expression for $\sqrt{t}u(x,t)$ as a function of x/t for $x, t \to \infty, 0 < x/t < 1$.