Applied Mathematics II **PROBLEM SET 5** **Due March 27, 2001**

1.(15pts) Using the similarity method, derive the fundamental solution $U(x,y,t)$ of the heat equation in 2D, satisfying

$$U_t - k(U_{xx} + U_{yy}) = 0,\ t > 0,$$

$$\lim_{t\to 0+} U = \delta(x)\delta(y).$$

Note , you may assume $U = U(r,t), r^2 = x^2+y^2$, and $U_{xx}+U_{yy} = U_{rr}+r^{-1}U_r$, so that the integral condition is

$$2\pi \int_0^\infty U r dr = 1,\ t > 0.$$

Also, we may assume U and its derivatives vanish at infinity. Show that

$$U = \frac{1}{4\pi kt} e^{-r^2/(4kt}.$$

Use the solution to write the solution to th IVP in the Cauchy form for 2D.

2.(15pts) Express the solution of the following inhomogeneous problem as the sum of a Fourier series (giving explicitly the coefficients) and a function of the form $Ax + B$:

$$T_t - kT_{xx} = 0, 0 < x < 2\pi,$$

$$T(0,t) = 4, T(2\pi,t) = 3, T(x,0) = 0.$$

3.(10pts) Suppose that the IVP for the 1D heat equation on $-\infty < x < +\infty$ is solved with $u(x,0) = f(x), f(-x) = -f(x)$. Show that the solution may be written in the form

$$u(x,t) = \int_0^{+\infty} f(\xi)[U(x-\xi,t) - U(x+\xi,t)]d\xi,$$

where U is the fundamental solution. From this deduce that u is the solution to the IVP for the half interval $x > 0$, with initial values $f(x)$ on this interval, subject to the boundary condition $u(0,t) = 0$. This is an example of *the reflection method.*

4.(20pts) Consider the following problem with mixed boundary condition:

$$T_t - kT_{xx} = 1, 0 < x < L,$$

$$T(0,t) = 0, T_x(L,t) = \alpha T(L,t),$$

where α is a positive constant. Find an equation for the eigenvalues λ_n, giving the eigensolutions

$$\phi_n(x,t) = e^{-\lambda_n kt}\psi_n(x)$$

and by means of a sketch show there are an infinite number of them. (You do not have to determine the eigenvalues explicitly.) Find the asymptotic form of the positive eigenvalues for large n. Be sure to consider all possible values of α. Under what conditions on αL will the solution of the IBVP invariably decay to zero?

5.(20pts) Use the Fourier transform to find the fundamental solution of the partial differential equation

$$\frac{\partial u}{\partial t} + k\frac{\partial^4 u}{\partial x^4} = 0.$$

Express you answer as an inverse Fourier transform. Show that the answer has the form $u = t^{-1/4} f(x/t^{1/4})$. Derive a differential equation for f from this form and show that it reduces to $\eta f + 4f\eta\eta\eta = 0$. Show that the Fourier integral solution satisfies this differential equation.

6.(20pts) The ocean may considered for this problem a 2D domain where contaminants such as oil from an oil spill diffuse with difusivity k. Suppose that a spill of volume Q occurs at $t = 0$ at a point $(0, y_0)$ where $y_0 > 0$ and the line $y = 0$ is a coastline. The spill is cleaned as it arrives at the coast, so that the flux of oil onto the beach is given by

$$F = \int_{\infty}^{+\infty} -k\frac{\partial u}{\partial y}(x, 0, t)dx,$$

where $u(x, y, t)$ is the oil density, solving the diffusion equation in 2D with diffusivity k. Give an expresssion for F as aa function of time. At what time, in terms of y_0, k, does the flux reach a maximum?. Use the fundamental solution from problem 1.