Evaluation of a contour integral considered in lecture 6

We saw that
\[
\frac{\sqrt{\pi}(1 + i)}{2\sqrt{2}} = \int_0^\infty e^{ix^2} \, dx = \int_0^\infty e^{iR^2e^{i\pi/4}}e^{i\pi/8} \, dR = I
\]
when we integrated from \((0, 0)\) to \((0, R)\), then along \(|z| = R\) to the ray \(\theta = \pi/8\), then along that ray to \((0, 0)\), after letting the arc move off to infinity.

Now \(e^{i\pi/8} = 2^{-1/4}\sqrt{1+i}\). Thus we have
\[
I = \sqrt{1+i} \int_0^\infty e^{-t^2}e^{it^2} \, dt.
\]

Thus
\[
\int_0^\infty e^{-t^2} \cos(t^2) \, dt = \sqrt{\pi/8} \Re \sqrt{1+i}.
\]

But if \(\sqrt{1+i} = a + ib\), the \(1 + i = a^2 - b^2 + 2iab\), \(a^2 - b^2 = 1\), \(2ab = 1\). Solving for \(a > 0\) we get \(a = \sqrt{\frac{1+\sqrt{2}}{\sqrt{2}}}\). Thus
\[
\int_0^\infty e^{-t^2} \cos(t^2) \, dt = \frac{1}{4\sqrt{\pi}}\sqrt{1 + \sqrt{2}}.
\]