Notes on the red light-green light problems

For the case of light traffic, $\rho_0 < 1/2$, we asked for the value of ρ_0 that ill make the shock cross the intersection at the moment the light again turns red, assuming this is at time $t_{R_2} - 2t_{G_1}$. Note that if this is the case then the shock again forms from scratch and the next red-green sequence is the same as the one already studied, so we have established the periodic solution with period $2t_{G_1}$.

Now the shock path has the equation

$$\dot{\xi(t)} = 1 - \rho_0 - \frac{1}{2}(1 - \frac{\xi}{(t - t_{G_1})}),$$

with initial condition $\xi(t_{G_1}/(1-\rho_0)) = -\rho t_{G_1}/(1-\rho_0)$. Thus

$$\xi(t) = (1 - 2\rho_0)(t - t_{G_1}) - A\sqrt{t_{G_1}}\sqrt{t - t_{G_1}},$$

with $A = 2\sqrt{\rho_0 - \rho_0^2}$. If $\xi(2t_{G_1}) = 0$, we must have $1 - 2\rho_0 = 2\sqrt{\rho - \rho^2}$ or $1 - 8\rho_0 + 8\rho_0^2 = 0$. Since $\rho_0 < 1/2$, the only acceptable root is

$$\rho_0 = \frac{2 - \sqrt{2}}{4} = .14645.$$

This value is independent of the duration of the red (green) light. It is suprisingly small, indicating how relatively light traffic can be choked at a light.

In thinking about the case of heavy traffic, $\rho > 1/2$, is is perhaps useful take $\rho_0 = 3/4$ and carefully draw the $t - x$ diagram starting at $t = 0$ with $\rho = 1, x < 0, = 0, x > 0$ and a green light. Thus you start with a fan. Making the period of the light = 1 minute, the light turns read at $t = 1$, and a curved hock path develops with the fan to the left of the shock and $\rho = 1$ to the right. Carry your diagram to the next fan and shock and you will get an idea of the situatiion.

Another good exercise: determine the path of the vehicle initially at $x = -1/2$ in the $t - x$ diagram for heavy traffic.