The following problems are intended to fill in some material outlined at the end of lecture 3.

Recall that we have calculated the fundamental solution of the wave equation in 3D. $R(\mathbf{x},t) = R_3(\mathbf{x},t)$, in the form

$$R_3(\mathbf{x},t) = \frac{1}{4\pi c^2}\frac{\delta(t-r/c)}{r}.$$

Recall $r = (x^2+y^2+z^2)^{1/2}$ and R satisfies

$$R_t t - c^2\nabla^2 R = \delta(x)\delta(y)\delta(z)\delta(t) \equiv \delta(\mathbf{x})\delta(t).$$

Also, R solves the homogeneous wave equation for $t>0$ with initial conditions

$$R(\mathbf{x},0) = 0, R_t(\mathbf{x},0) = \delta(\mathbf{x}).$$

Here $\mathbf{x} = (x,y,z)$ According to the superposition principle given in class, the function $u(\mathbf{x},t)$ satisfying the wave equation for $t>0$ and initial conditions

$$u(\mathbf{x},0) = 0, u_t(\mathbf{x},0) = \psi(\mathbf{x})$$

is given by

$$u = \frac{1}{4\pi c^2}\int\int\int \frac{\delta(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})}{|\mathbf{x}-\mathbf{x}'|}\psi(\mathbf{x}')dV'.$$

Here dV' indicates the volume element attached to $\mathbf{x}'$, the dummy variable of integration.

Problem 1: Show that the last equation may be written as

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2}\frac{1}{t}\int\int_{\sigma(\mathbf{x},t)}\psi(\mathbf{x}')dS',$$

where dS' is the area element attached to $\mathbf{x}'$ and $\sigma(\mathbf{x},t)$ denotes the sphere of radius ct and center at $\mathbf{x}$.

Hints: First set $\mathbf{x}' = \mathbf{x}+\mathbf{y}$ so that in the integral $|\mathbf{y}| = ct$. Then use a spherical coordinate system centered at $\mathbf{x}$ to make $dS' = c^2t^2d\Omega$ where $d\Omega$ is the element of solid angle attached to $\mathbf{y}$. Then do the integration with respect to the radius $|\mathbf{y}| = \rho$ say. You will then have

$$u = \frac{1}{4\pi c^2}\int\int\int \delta(t-\rho/c)\psi(\mathbf{x}+\mathbf{y})\rho d\Omega d\rho.$$

Now use the following two properties of the one dimensional delta function:

$$\delta(-x) = \delta(x), \delta(cx) = |c|^{-1}\delta(x),$$

where c is any nonzero real number. With these you should obtain the indicated result.

Recall now that the Kirchoff solution of $u_{tt} - c^2\nabla^2 u = 0, t>0$ with initial values $u(\mathbf{x},0), u_t(\mathbf{x},0)$ prescribed is

$$u = \frac{1}{4\pi c^2}\frac{1}{t}\int\int_{\sigma(\mathbf{x},t)} u_t(\mathbf{x}',0)dS' + \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\frac{1}{t}\int\int_{\sigma(\mathbf{x},t)} u(\mathbf{x}',0)dS'.$$

Consider then again the problem of the bursting balloon. The pressure p of the sound field can them be taken as the variable u of the Kirchoff solution, with $p(\mathbf{x},0) = p_b$ for $0 \le r \le r_b$ and zero elsewhere. Also $p_t(\mathbf{x},0) = 0$.

Problem 2: Show that the Kirchoff solution gives the same answer in the bursting balloon problem as we obtained from the D'Alembert solution.

Hints: Let an observer be located at a point A a distance $R > r_b$ from the center of the balloon. Consider the intersection of a spherical surface of radius ct, centered at A, with the balloon. The part of the surface within the balloon is bounded by a circle, and any line drawn from A to this circle will make an angle α with the line from the center of the balloon to A (draw a side view). From geometry we know that the area of the spherical surface within the balloon is then $2\pi(1-\cos\alpha)c^2t^2$. Also from the geometry of the triangle we have that $r_b^2 = R^2 + c^2t^2 - 2Rct\cos\alpha$. Using this information and the Kirchoff formula, find the pressure field as ct varies through the balloon and obtain the "N" wave we obtained before.

Consider now the fundamental solution of the wave equation in 2D, $R_2(x,y,t)$ say, satisfying

$$u_{tt} + c^2\nabla^2 u = \delta(x)\delta(y)\delta(t).$$

As we indicated in class, this solution can be obtained by the "method of descent". Here the descent is from 3 to 2 dimensions. If $R(\mathbf{x},t)$ is the fundamental solution in 3D, the idea is to apply the principle of superposition, and write

$$R_2(x,y,t) = \int_{-\infty}^{+\infty} R_3(\mathbf{x},t)dz.$$

Problem 3: Formally applying the 3D wave operator to the integrand of the last equation, verify that this gives us the 2D solution.

Problem 4: Show that, if $r_2 = (x^2+y^2)^{1/2}$,

$$R_2(x,y,t) = \begin{cases} 0, & \text{if } ct < r_2, \\ \frac{1}{2\pi c}\frac{1}{\sqrt{c^2t^2 - r_2^2}}, & \text{if } 0 < r_2 < ct. \cdot \end{cases}$$

Hints: You sill need to make use of the following result for 1D delta functions: Let $f(x)$ be a twice differentiable function with exactly one 1 simple zero x_0 in the interval $[a,b]$. Then

$$\int_a^b \delta(f(x))\phi(x)dx = \frac{1}{|f'(x_0)|^{-1}}\phi(0).$$

To prove this, write $f(x) = f'(x_0)(x-x_0) + f''(k(x,x_0))(x-x_0)^2/2$. Then use the earlier rule for $\delta(cx)$, then the basic definition of the delta function.