Assignment 2.

Given January 26, due February 9.

We want to study the function

$$f(x) = \int_0^1 \cos\left(xt^2\right) dt \quad . \tag{1}$$

Step 1: Write a program to estimate $f(x)$ using a panel integration method. The program should be modular, well documented, robust, and clean. At the lowest level should be a procedure that takes as input x and $n = 1/\Delta t$ and returns the approximate integral with that x and Δt value. This routine should be written so that another person could easily substitute a different panel method by changing a few lines of code.

Step 2: Verify the correctness of the program by checking that it gives the right answer for small x. We can estimate $f(x)$ for small x using a few terms of its Taylor series. This series can be computed by integrating the Taylor series for $\cos(xt^2)$ term by term.

Step 3: With $x = 1$, do a convergence study to verify the second order accuracy of the trapezoid rule and the fourth order accuracy of Simpson's rule.

Step 4: Write a routine that uses the routine from Step 1 and Richardson estimation to find an n that gives $f(x)$ to within a specified error tolerance. The input should be x and the desired error bound. The output should be the estimated value of f and the number of points used. This routine should be robust enough to quit and report failure if it is unable to achieve the requested accuracy.

Step 5: For large x we have the approximation

$$f(x) \sim \sqrt{\frac{\pi}{8x}} + \frac{1}{2x}\sin(x) - \frac{1}{16x^2}\cos(x) + \cdots \quad . \tag{2}$$

Make a few plots showing f and its approximations using one, two and all three terms on the right side of (2) for x in the range $1 \leq x \leq 1000$. In all cases we want to evaluate f so accurately that the error in our f value is much less than the error of the approximation (2). Note that even for a fixed level of accuracy, more points are needed for large x. Why?