Assignment 3.

Given September 16, due September 23. Last revised, September 16.
Objective: Markov chains, II and lattices.

1. We have a three state Markov chain wih transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} .$$

(Some of the transition probabilities are $P(1 \to 1) = \frac{1}{2}$, $P(3 \to 1) = \frac{1}{3}$, and $P(1 \to 2) = \frac{1}{4}$. Let $\tau = \min(t \mid X_t = 3)$.) Even though this τ is not bounded (it could be arbitrarily large), we will see that $P(\tau > t) \leq Ca^t$ for some $a < 1$ so that the probability of large τ is very small. This is enough to prevent the stopping time paradox (take my word for it). Suppose that at time $t = 1$, all states are equally likely.

a. Consider the quantities $u_t(j) = P(X_t = j \text{ and } \tau > t)$. Find a matrix evolution equation for a two component vector made from the $u_t(j)$ and a submatrix, $\widetilde{P}$, of P.

b. Solve this equation using the the eigenvectors and eigenvalues of $\widetilde{P}$ to find a formula for $m_t = P(\tau = t)$.

c. Use the answer of part b to find $E[\tau]$. It might be helpful to use the formula

$$\sum_{t=1}^{\infty} tP(\tau = t) = \sum_{t=1}^{\infty} P(\tau \geq t) .$$

Verify the formula if you want to use it.

d. Consider the quantities $f_t(j) = P(\tau \geq t \mid X_1 = j)$. Find a matrix recurrence for them.

e. Use the method of question 1 to solve this and find the f_t.

2. Let P be the transition matrix for an n state Markov chain. Let $v(k)$ be a function of the state $k \in \mathcal{S}$. For this problem, suppose that paths in the Markov chain start at time $t = 0$ rather than $t = 1$, so that $X = (X_0, X_1, \ldots)$. For any complex number, z, with $|z| < 1$, consider the sum

$$E\left[\sum_{t=0}^{\infty} z^t v(X_t) \mid \mathcal{F}_0\right] . \tag{1}$$

Of course, this is a function of $X_0 = k$, which we call $f(k)$. Find a linear matrix equation for the quantities f that involves P, z, and v. Hint: the sum

$$E\left[\sum_{t=1}^{\infty} z^t v(X_t) \mid \mathcal{F}_1\right] .$$

may be related to (1) if we take out the common factor, z.

3. (Change of measure) Let P be the probability measure corresponding to ordinary random walk with step probabilities $a = b = c = 1/3$. Let Q be the measure for the random walk where the up, stay, and down probabilities from site x depend on x as

$$(c(x), b(x), a(x)) = \frac{1}{3} e^{-\beta(x)} \left(e^{-\alpha(x)}, 1, e^{\alpha(x)} \right) .$$

We may choose $\alpha(x)$ arbitrarily and then choose $\beta(x)$ so that $a(x) + b(x) + c(x) = 1$. Taking $\alpha \neq 0$ introduces drift into the random walk. The state space for walks of length T is the set of paths $x = x(0), \cdots, x(T)$ through the lattice. Assume that the probability distribution for $x(0)$ is the same for P and Q. Find a formula for $Q(x)/P(x)$. The answer is a discrete version of Girsanov's formula.

4. In the urn process, suppose balls are either *stale* or *fresh.* Assume that the process starts with all n stale balls and that all replacement balls are fresh. Let τ be the first time all balls are fresh. Let $X(t)$ be the number of stale balls at time t. Show that $X(t)$ is a Markov chain and write the transition probabilities. Use the backward or forward equation aproach to calculate the quantities $a(x) = E_x(\tau)$. This is a two term recurrence relation ($a(x+1) = \text{something} \cdot a(x)$) that is easy to solve. Show that at time τ, the colors of the balls are iid red with probability p. Use this ti explain the binomial formula for the invariant distribution of colors.