Assignment 8.

Given November 11, due November 18. Last revised, November 11.
Objective: Diffusions and diffusion equations.

1. An Ornstein Uhlenbeck process is a stochastic process that satisfies the stochastic differential equation

$$dX(t) = -\gamma X(t)dt + \sigma dW(t) \; . \tag{1}$$

a. Write the backward equation for $f(x,t) = E_{x,t}[V(X(T)]$.

b. Show that the backward equation has (Gaussian) solutions of the form $f(x,t) = A(t)\exp(-s(t)(x-\xi(t))^2/2)$. Find the differential equations for A, ξ, and s that make this work.

c. Show that $f(x,t)$ does not represent a probability distribution, possibly by showing that $\int_{-\infty}^{\infty} f(x,t)dt$ is not a constant.

d. What is the large time behavior of $A(t)$ and $s(t)$? What does this say about the nature of an Ornstein Uhlenbeck reward that is paid long in the future as a function of starting position?

2. The forward equation:

a. Write the forward equation for $u(x,t)$ which is the probability density for $X(t)$.

b. Show that the forward equation has Gaussian solutions of the form

$$u(x,t) = \frac{1}{\sqrt{2\pi\sigma(t)^2}} e^{-(x-\mu(t))^2/2\sigma^2(t)} \; .$$

Find the appropriate differential equations for μ and σ.

c. Use the explicit solution formula for (1) from assignment 7 to calculate $\mu(t) = E[X(t)]$ and $\sigma(t) = \text{var}[X(t)]$. These should satisfy the equations you wrote for part b.

d. Use the approximation from (1): $\Delta X \approx -\gamma X \Delta t + \sigma \Delta W$ (and the independent increments property) to express $\Delta\mu$ and $\Delta(\sigma^2)$ in terms of μ and σ and get yet another derivation of the answer in part b. Use the definitions of μ and σ from part c.

e. Differentiate $\int_{-\infty}^{\infty} xu(x,t)dx$ with respect to t using the forward equation to find a formula for $d\mu/dt$. Find the formula for $d\sigma/dt$ in a similar way from the forward equation.

f. Give an abstract argument that $X(t)$ should be a Gaussian random variable for each t (something is a linear function of something), so that knowing $\mu(t)$ and $\sigma(t)$ determines $u(x,t)$.

g. Find the solutions corresponding to $\sigma(0) = 0$ and $\mu(0) = y$ and use them to get a formula for the transition probability density (Green's function) $G(y,x,t)$. This is the probability density for $X(t)$ given that $X(0) = y$.

h. The transition density for Brownian motion is $G_B(y,x,t) = \frac{1}{\sqrt{2\pi t}} \exp(-(x-y)^2/2t)$. Derive the transition density for the Ornstein Uhlenbeck process from this using the Cameron Martin Girsanov formula (warning: I have not been able to do this yet, but it must be easy since there is a simple formula for the answer. Check the bboard.).

i. Find the large time behavior of $\mu(t)$ and $\sigma(t)$. What does this say about the distribution of $X(t)$ for large t as a function of the starting point?

3. Duality:

a. Show that the Green's function from part 2 satisfies the backward equation as a function of y and t.

b. Suppose the initial density is $u(x,0) = \delta(x-y)$ and that the reward is $V(x) = \delta(x-z)$. Use your expressions for the corresponding forward solution $u(x,t)$ and backward solution $f(x,t)$ to show by explicit integration that $\int_{-\infty}^{\infty} u(x,t)f(x,t)dx$ is independent of t.