Homework 7, due Monday, June 18. Hand in the starred (*) problems. This is the last assignment of the term.

P. 191/2, 5, 11, *12, 18, *20

Answer P191/12. If \( n = \prod p_i^{a_i} \), the number of divisors is \( \prod (a_i + 1) \). This is odd if and only if each \( (a_i + 1) \) is odd, hence each \( a_i \) is even. But this is precisely the condition that \( n \) is a perfect square. As for \( \sigma_k(n) \), each odd prime \( p \) such that \( p^a||n \) contributes a factor \( 1 + p + \ldots + p^k \) consists of the sum of \( k+1 \) odd numbers. This is odd if and only if \( k \) is even. However, if \( p = 2 \), this is always odd. So the necessary and sufficient condition that \( \sigma_k(n) \) is odd is \( n = 2^{\alpha}m^2 \) with \( m \) odd. But depending on whether \( \alpha \) is even or odd, the condition is that \( n = m^2 \) or \( n = 2m^2 \), where \( m \) is arbitrary.

Answer P191/20. Clearly, \( \Omega(n) \) satisfies the condition \( \Omega(mn) = \Omega(m) + \Omega(n) \) if \( (m,n) = 1 \). Therefore, if \( (m,n) = 1 \), then

\[
\lambda(mn) = (-1)^{\Omega(mn)} = (-1)^{\Omega(m)+\Omega(n)} = (-1)^{\Omega(m)}(-1)^{\Omega(n)} = \lambda(m)\lambda(n)
\]

So \( \lambda(n) \) is multiplicative. Thus, \( \sum d|n \lambda(d) \) is multiplicative. To compute it, we find

\[
F(n) = \sum_{d|n} \lambda(d) = \lambda(1) + \lambda(p) + \ldots + \lambda(p^k) = 1 - 1 + \ldots + (-1)^k
\]

This is 1 if \( k \) is even, 0 if \( k \) is odd. Thus, if \( n = \prod p_i^{a_i} \), \( F(n) = 0 \) if one of the \( a_i \) is odd, so that \( n \) is not a square, and \( F(n) = 1 \) if all of the \( a_i \) are even - that is, \( n \) is a square. This is the result.

P. 195/2, 3, *5, 18, *19

P195/5. It is a simple matter to show that \( |\mu(n)| \) is multiplicative, Therefore, so is \( F(n) = \sum_{d|n} |\mu(n)| \). If \( a \geq 1 \), \( F(p^a) = |\mu(1)| + |\mu(p)| = 2 \) since \( \mu(p^k) = 0 \) for \( k \geq 2 \). Therefore if the prime decomposition of \( n = \prod p_i^{a_i} \), we have \( F(n) = \prod_{i=1}^{k} 2 = 2^k \), where \( k \) is the number of distinct prime factors of \( n \). This is the result. A direct proof using the binomial theorem is also possible.

P195/19. Rewriting, we must show that

\[
n/\phi(n) = \sum_{d|n} \mu(d)^2/\phi(d)
\]

Since \( \mu(n)^2/\phi(n) \) is multiplicative, it is enough to verify this for prime powers. We have

\[
\sum_{d|p^k} \mu(d)^2/\phi(d) = 1 + 1/(p - 1) = p/(p - 1)
\]
since terms involving $\mu(p^a)$ for $a \geq 2$ vanish. But
\[
\frac{n}{\phi(n)} = \frac{p^k}{\phi(p^k)} = \frac{p^k}{p^{k-1}(p-1)} = \frac{p}{p-1}
\]
This proves the result because the multiplicative functions agree on prime powers.

P. 136/7, 8

P. 140/1, 2, 4, 6, *8, 10
Answer P140/8.

\[
\left(\frac{10}{p}\right) = \left(\frac{2}{p}\right) \left(\frac{5}{p}\right) = \left(\frac{2}{5}\right)
\]
The answer depends on $p \mod 5$ and 8. We separate into cases according as the first factor is 1 or $-1$.

Case 1. $\left(\frac{2}{p}\right) = 1$. Here $p \equiv \pm 1 \mod 8$, and $p \equiv \pm 1 \mod 5$. Solving simultaneously, this gives $p \equiv \pm 1 \mod 40$ and $p \equiv \pm 9 \mod 40$.

Case 2. $\left(\frac{2}{p}\right) = -1$. Here $p \equiv 3 \mod 8$ or $p \equiv 5 \mod 8$, and $p \equiv \pm 2 \mod 5$. Writing $p = 5 + 8t$, we must have $5 + 8t \equiv \pm 2 \mod 5$, so $-2t \equiv \pm 2 \mod 5$, so $t = 5s \pm 1$ and $p = 5 + 8(5s \pm 1)$ or $p \equiv 13$ or $-3 \mod 40$.

Writing $p = 3 + 8t$, we get $3 + 8t \equiv \pm 2 \mod 5$, so $-2t \equiv 2 \pm 2 \mod 5$, so $t = 5s$ or $t = -2 + 5s$. This gives $p = 3 + 40s$ or $p = -13 + 40s$, so $p \equiv 3$ or 27 mod 40.

Summarizing, the possibilities are $p \equiv 1, 39, 9, 31, 13, 37, 3, 27$.

*Find all solutions of the congruence $x^2 \equiv 58 \mod 77$.
Answer, This is equivalent to the system $x^2 \equiv 58 \equiv 3 \mod 11$ and $x^2 \equiv 58 \equiv 2 \mod 7$. The mod 11 congruence has the solutions $x \equiv \pm 5 \mod 11$, so $x = \pm 5 + 11t$. The mod 7 congruence has the solution $x \equiv \pm 3 \mod 7$. Using the + sign in each case, we substitute the mod 11 solution to get $5 + 11t \equiv 3 \mod 7$ or $4t \equiv -2 \mod 7$. This gives $t \equiv 3 \mod 7$, so we can write $t = 3 + 7s$. Substituting in the equation $x = 5 + 11t$, we get $x = 5 + 11(3 + 7s) = 38 + 77s$ or $x \equiv 38 \mod 77$. Using the negative sign in each congruence leads to $x \equiv -38 \mod 77$. This gives two solutions $x \equiv \pm 38 \mod 77$. Using $5 + 11t \equiv -3 \mod 7$, we get $4t \equiv -8 \mod 7$, so $t \equiv -2 \mod 7$, and $t = -2 + 7s$. Substituting in $x = 5 + 11t$, we get $x = 5 + 11(-2 + 7s) = -17 + 77s$. This leads to two more solutions $x \equiv \pm 17 \mod 77$. So there are four solutions in all: $x \equiv \pm 17 \mod 77$ and $x \equiv \pm 38 \mod 77$. 