Quotient Spaces and Direct Sums.
In what follows, we take V as a finite dimensional vector space over a field F. Let $W \subseteq V$ be a subspace. For x and y in V, define

$$x \equiv y \text{ mod } W \text{ if and only if } y - x \in W$$

It follows that the relation $x \equiv y$ mod W is an equivalence relation on V. Namely
(i) $x \equiv x$ mod W for any $x \in V$. (reflexivity)
(ii) If $x \equiv y$ mod W then $y \equiv x$ mod W (symmetry)
(iii) If $x \equiv y$ mod W and $y \equiv z$ mod W then $x \equiv z$ mod W (transitivity)

General theory shows that if we define the equivalence class $\overline{x} = \{y | y \equiv x\}$, then the following properties hold.
(iv) The equivalence classes partition V. Namely, distinct equivalent classes are disjoint, and their union is V.
(v) $x \equiv y$ mod W if and only if $\overline{x} = \overline{y}$.
It is easily shown that $\overline{x} = x + W$.

Item (v) shows that equivalence can be converted to equality, using equivalence classes. Thus, $\overline{x} = \overline{y}$ if and only if $x + W = y + W$ if and only if $x \equiv y$ if and only if $y - x \in W$.

The following algebraic properties of equivalence can easily be shown:
(vi) If $x \equiv y$ mod W and $c \in F$ then $cx \equiv cy$ mod W.
(vii) If $x \equiv y$ mod W and $u \equiv v$ mod W then $x + u \equiv y + v$ mod W.

Properties (v1) and (vii) can be stated as "Equivalence is compatible with scalar multiplication and addition." It allows us to define addition and scalar multiplication of the equivalence classes in the following natural way.

(viii) $$c\overline{x} = \overline{cx}; \; \overline{x} + \overline{y} = \overline{x + y}$$

It is easy to verify that the equivalent classes of V mod W form a vector space using this definition. This vector space is called the quotient space V/W. Paraphrasing,

V/W is the space of all cosets $x + W$, with $x \in V$, with

$$(x + W) + (y + W) = (x + y) + W \text{ and } c(x + W) = cx + W.$$

Note: The zero element of V/W is $0 + W$ or W.

Definition. Let $W \subseteq V$ as before. Define the map $p{:}V \to V/W$ by the equation $p(x) = x + W$. Then p is called the canonical map of V into the quotient space V/W.

It is an easy matter to show that
(1) p is linear.
(2) p is onto.
(3) The $\ker(p) = W$.

Corollary. $\dim(V/W) = \dim(V) - \dim(W)$.
In fact, since the image of p is V/W and the kernel is W, we have $\dim(V/W) + \dim(W) = \dim(V)$, by the theorem on dimensions of range and kernel.

Theorem. Let $L{:}V \to U$, and let $W = \ker(L)$. Then $L(V)$ is isomorphic with V/W.
For the proof, consider the map $T{:}V/W \to U$ according to the formula $T(x + W) = L(x)$. It is well defined, since if $x + W = y + W$, we have $x - y \in W$, so $L(x - y) = 0$ (since W is the kernel of L) and so $L(x) = L(y)$. The image of T is clearly $L(V)$, and the kernel of T is clearly W, since $T(x + W) = 0$ if and only if $L(x) = 0$ if and only if $x \in W$ or $x + W$ is the zero element of V/W. So T Is the required isomorphism of V/W onto $L(V)$.

Definition. Let $W \subseteq V$ as before. We say the $x_1, x_2, \ldots, x_k$ are linearly independent mod W provided $\sum_{i=1}^{k} a_k v_k \equiv 0 \bmod W$ implies $a_i = 0$ for $1 \leq i \leq k$.

It is an easy matter to show that $x_1, x_2, \ldots, x_k$ are linearly independent mod W if and only if $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_k}$ are linearly independent vectors of V/W.

Theorem: Let $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_r}$ be a basis of V/W, and let $w_1, w_2, \ldots, w_s$ be a basis for W. Then $x_1, \ldots, x_r, w_1, \ldots, w_s$ is a basis for V.
Proof: We already know that $r + s = \dim(V)$ by the theorem of dimensions. So it is enough to show that these vectors are linearly independent. If $\sum a_i x_i + \sum b_j w_j = 0$, we have $\sum a_i x_i \equiv 0 \bmod W$, so $\sum a_i \overline{x_i} = 0$ in V/W. Thus, each $a_i = 0$. This implies $\sum b_j w_j = 0$ and so each $b_i = 0$, proving the result.

Direct Sums. Let V_1 and V_2 be vector spaces over the same field F. The direct sum $V = V_1 \oplus V_2$ is the vector space defined as follows.
1. The elements of V are the ordered couples (v_1, v_2) where $v_i \in V_i$ for $i = 1, 2$.
2. Addition and scalar multiplication are defined component-wise:
$(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2)$.
$a(v_1, v_2) = (av_1, av_2)$

It is a simple matter to verify that $V_1 \oplus V_2$ is a vector space using these definitions. Both V_1 and V_2 are naturally isomorphic to subspaces of $V = V_1 \oplus V_2$. The subspace of all elements $\overline{V_1} = \{(v_1, 0) | v_1 \in V_1\}$ is clearly isomorphic to V_1 using the isomorphism $v_1 \mapsto (v_1, 0)$. Similarly, V_2 is naturally isomorphic to the subspace $\overline{V_2}$ of all elements $(0, v_2)$ with $v_2 \in V_2$.

The process naturally extends to the direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_k$ of k vector spaces

$V_i,\ 1 \le i \le k$.

The above is an external construction of a direct sum. More typically direct sums can sometimes be found internally. Namely, suppose V is a vector space, and $V_i \subseteq V$ are subspaces for $1 \le i \le k$. Suppose further that

$$V = V_1 + \cdots + V_k$$

and

$$0 = v_1 + v_2 + \cdots + v_k \text{ for } v_i \in V_i \text{ implies } v_i = 0, \text{for } 1 \le i \le k$$

Then V is isomorphic to $V_1 \oplus \cdots \oplus V_k$.

For a proof, define $V' = V_1 \oplus \cdots \oplus V_k$, and consider the map $T{:}V' \to V$ defined by

$$T(v_1, \ldots, v_k) = v_1 + \cdots + v_k$$

These two conditions show that this map is onto, and 1-1. Linearity is straight-forward. Following the text, we shall write $V = V_1 \oplus \cdots \oplus V_k$ in this situation. Note that the two conditions generalize the notion of a basis. In fact, if $v_1, \ldots, v_n$ are n vectors in a space V, and we define $V_i = \mathrm{span}(\{v_i\})$, it is clear that V is the direct sum of the V_i if and only if the v_i form a basis of V.

When $V = V_1 \oplus \cdots \oplus V_k$, and each V_i is finite dimensional, it can be easily verified that $\dim(V) = \sum\limits_i \dim(V_i)$ and a basis of V is the union of bases of V_i.

A reason for considering direct sums is that the analysis of a structure is often simplified by analyzing substructures that are used a building blocks of that structure. This becomes clear when we discuss linear transformations on a vector space V.

Invariant subspaces.
Let $T{:}V \to V$. A subspace $U \subseteq V$ is called T-invariant, provided $T(U) \subseteq U$. Analysis of T will be simplified if a basis γ of U is chosen, and then expanded to a full basis β of V. The matrix $[T]_\beta$ is then seen to have the simpler form

$$\begin{pmatrix} A & C \\ O & B \end{pmatrix}$$

where A is and $m \times m$ matrix (m is the dimension of U) and B is an $(n-m)\times(n-m)$ matrix (n is the dimension of V.) Here $A = [T|U]_\gamma$. ($T|U$ is the map T restricted to U and regarded a transformation of U.) We used this idea in the discussion of the Cayley Hamilton theorem, and in the analysis of eigenspaces. It is easy to show that the characteristic polynomial of $T|U$ divides the characteristic polynomial of T.

Now suppose, for $T{:}V \to V$, that there are subspaces U and W, such that $V = U \oplus W$ and T is U-invariant and W-invariant. Then we say that T is (U, W)-invariant. In this case, if a basis δ of W is adjoined to a basis γ of U to form a basis β of V, the matrix $[T]_\beta$ has the form

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

Here $A = [T|U]_\gamma$ and $B = [T|W]_\delta$. We have broken up the transformation into simpler parts - simpler because the dimensions of U and W are smaller than the dimension of V. This process can continue until there are no longer any pairs of subspaces which are invariant under T. This will lead to subspaces $U_1, \ldots, U_k$ whose direct sum is V and which are all T invariant. It leads to a basis of V and a matrix of T given by

$$\begin{pmatrix} A_1 & O & \ldots & O \\ O & A_2 & \ldots & O \\ \ldots & \ldots & \ddots & \ldots \\ O & O & \ldots & A_k \end{pmatrix} \tag{1}$$

The extreme case is a diagonalizable transformation. Here, each U_i is 1 dimensional and each matrix A_i is a scalar (a 1×1 matrix).

This decomposition of a transformation can be done externally. Let $V_1, \ldots, V_k$ be vector spaces, and let T_i be a transformation of V_i for $1 \leq i \leq k$. Let $V = V_1 \oplus \cdots \oplus V_k$. We define the direct sum $T = T_1 \oplus \cdots \oplus T_k$ of these transformation by defining $T(v_1, \ldots, v_k) = (Tv_1, \ldots, Tv_k)$. The matrix of T with respect to the obvious basis for V will then be of the form (1).