Derivative Securities, Fall 2012 – Homework 4. Distributed 10/24, due 11/7.
Corrective footnote added to Problem 7 is new as of 11/7. Problem 5a corrected 11/7. Typo in comment after Problem 4 corrected 11/11.

Problems 1–5 reinforce our discussion of SDE's and the Ito calculus. Problem 6 makes use of the Black-Scholes PDE. Problem 7 reinforces the material in the Section 7 notes on pricing path-dependent options.

(1) Let s solve the SDE $ds = \mu s\, dt + \sigma s\, dw$, where μ and σ are constant. Find the SDE solved by

  (a) $y = As$, where A is constant
  (b) $y = \sqrt{s}$
  (c) $y = \cos(s)$
  (d) $y = s^3 t^2$.

(2) We continue to assume that s solves $ds = \mu s\, dt + \sigma s\, dw$, with μ and σ constant. Find a function f such that $f(s(t))$ is a martingale (that is, such that the SDE describing $f(s(t))$ has no dt term).

(3) This problem should help you understand Ito's formula. If w is Brownian motion, then Ito's formula tells us that $z = w^2$ satisfies the stochastic differential equation $dz = 2w dw + dt$. Let's see this directly:

  (a) Suppose $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$. Show that $w^2(t_{i+1}) - w^2(t_i) = 2w(t_i)(w(t_{i+1}) - w(t_i)) + (w(t_{i+1}) - w(t_i))^2$, whence

$$w^2(b) - w^2(a) = 2\sum_{i=0}^{N-1} w(t_i)(w(t_{i+1}) - w(t_i)) + \sum_{i=0}^{N-1} (w(t_{i+1}) - w(t_i))^2$$

  (b) Let's assume for simplicity that $t_{i+1} - t_i = (b-a)/N$. Find the mean and variance of $S = \sum_{i=0}^{N-1}(w(t_{i+1}) - w(t_i))^2$.
  (c) Conclude by taking $N \to \infty$ that

$$w^2(b) - w^2(a) = 2\int_a^b w\, dw + (b-a).$$

(4) Here's a cute application of the Ito calculus. Let $w(t)$ be Brownian motion (with $w(0) = 0$), and consider

$$\beta_k(t) = E[w^k(t)].$$

Show using Ito's formula that for $k = 2, 3, \ldots$,

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\, ds.$$

Deduce that $E[w^4(t)] = 3t^2$. What is $E[w^6(t)]$?

[Comment: the moments of w can also be calculated from its distribution function, since $w(t)$ is Gaussian with mean 0 and variance[1] t. But the method in this problem is easier, and good practice with Ito's lemma.]

(5) Consider the solution of

$$ds = r(t)s\,dt + \sigma(t)s\,dw, \quad s(0) = s_0. \tag{1}$$

where $r(t)$ and $\sigma(t)$ are deterministic functions of time.

(a) Show that[2] $\log s(t)$ is a Gaussian random variable, with mean $\log s(0) + \int_0^t [r(s) - \frac{1}{2}\sigma^2(s)]\,ds$ and variance $\int_0^t \sigma^2(s)\,ds$.

(b) Show that $s(T) = s_0 \exp\left([\overline{r} - \frac{1}{2}\overline{\sigma}^2]T + \overline{\sigma}\sqrt{T}Z\right)$ where Z is a standard Gaussian,

$$\overline{r} = \frac{1}{T}\int_0^T r(s)\,ds \quad \text{and} \quad \overline{\sigma}^2 = \frac{1}{T}\int_0^T \sigma^2(s)\,ds.$$

[Comment: The price of a non-dividend-paying stock has the form (1) under the risk-neutral measure. A forward price also satisfies an equation of this form, with $r(t) \equiv 0$. According to this problem, when $r(t)$ and $\sigma(t)$ are nonconstant but depend deterministically on time, we can value options by using the standard Black-Scholes formulas with r and σ replaced by $\overline{r}$ and $\overline{\sigma}$.]

(6) In HW3 we considered a derivative whose payoff was $s^n(T)$ at maturity, where $s(t)$ has lognormal dynamics with constant volatility σ, and where the risk-free rate is r (also constant). We showed there that the derivative has value

$$s^n(t)\exp\left([\tfrac{1}{2}\sigma^2 n(n-1) + r(n-1)](T-t)\right)$$

at time t. Let's give a different derivation of the same result, using the Black-Scholes PDE.

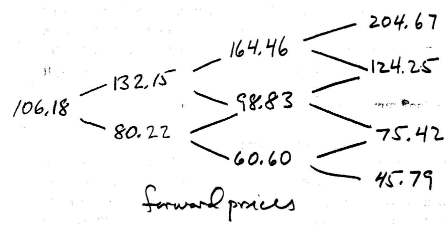
(a) Substitute $V(s,t) = h(t)s^n$ into the Black-Scholes PDE. What ODE must $h(t)$ solve? What is the appropriate final-time condition?

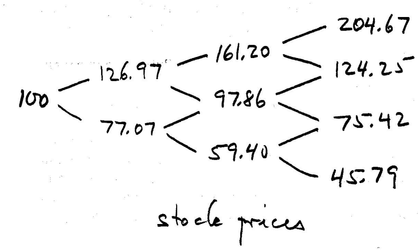
(b) Verify that $h(t) = \exp\left([\frac{1}{2}\sigma^2 n(n-1) + r(n-1)](T-t)\right)$ solves the ODE you found in (a), with the appropriate final-time condition.

(7) This problem asks you to value some path-dependent options using one of the methods discussed in the Section 7 notes. Use the 3-period forward and stock price trees shown in the figure, with branching probabilities equal to 1/2. [The trees were established as follows: assume each period is a year; the current stock price is 100; there is a continuous dividend rate of $q = 5\%$; the risk-free rate is $r = 7\%$; and the volatility is $\sigma = 25\%$. This suggests a multiplicative forward tree with "up factor" $u = \exp(-\frac{1}{2}(.25)^2 + .25) = 1.2445$. Since we have set the branching probabilities to

[1] typo here corrected 11/11

[2] The original version said "log s(t) is a Gaussian random variable. Show that its mean is $\int_0^t [r(s) - \frac{1}{2}\sigma^2(s)]\,ds$." That expression for the mean is wrong, for lack of the additive term $\log s(0)$.





1/2, the "down factor" must be $d = 2 - u = .7555$. The stock tree is determined by the forward tree, $s_t = F_t e^{-(r-q)(T-t)}$.][3]

(a) Value an American call with strike 90 and maturity 3 years.

(b) Value a call with strike 95 on the average year-end stock price. (This option's payoff at time $T = 3$ is $\max\{(s_1 + s_2 + s_3)/3 - 95, 0\}$.)

(c) Value a down and out barrier call with barrier 85 and strike 95. (This option is worthless if s_1, s_2, or s_3 is below 85; otherwise its payoff at time $T = 3$ is $(s_3 - 95)_+$.)

[3] Unfortunately, I made a copying error in reporting the forward price tree, which makes two trees in the figure inconsistent. In period 2, where my forward price tree says 98.83, the number produced by the procedure just described is 99.83. The stock price tree is correct (ie it is what the procedure gives), since $99.83 * e^{-.02} = 97.86$. By the way, the crucial structural property for a stock price tree when the dividend rate q is nonzero and the branching probabilities are 1/2 is $s_{\text{now}} = e^{-(r-q)\delta t}[\frac{1}{2}s_{\text{up}} + \frac{1}{2}s_{\text{down}}]$; this assures that the stock is correctly priced by the tree. Our procedure for getting the stock price tree from the forward price tree guarantees that the stock price tree has this property. A previous "correction" – asserting that the stock price tree should say 96.87 where the figure says 97.86 – was simply wrong [with this change, the tree would not price the stock correctly]. But solutions that use a correct method to price the options will be counted correct regardless of which stock price tree you used.