# Mechanics - Lecture 8, 3/28/2018

[Note: there was no class 3/14 (spr brk) or 3/21 (bad weather). We'll finish the "Lecture 7" notes before starting these.]

New topic: three lectures on "classical mechanics," emphasizing links to pde and the calculus of variations.

My syllabus lists some books – they're good places to read more – but I won't really follow any of them directly.

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Some basic examples involve motion of one or more bodies in $\mathbb{R}^3$ (or $\mathbb{R}^n$):

$$(*) \quad m_i \ddot{x}_i = f_i$$

$m_i$ = mass of $i^{th}$ body
$\ddot{x}_i$ = accel of $i^{th}$ body
$f_i$ = force on $i^{th}$ body

(here $x_i(t) \in \mathbb{R}^n$). We'll usually focus on conservative forces

$$(**) \quad f_i = -\frac{\partial U}{\partial x_i}$$

where $U = U(x_1, \dots, x_N)$ is fn of positions of the bodies

U is often called the "potential energy").
Note (from calculus) that $(f_1, \cdots, f_N)$ is
conservative iff the assoc "work"

$$\int_{\substack{\text{initial}\\\text{config}}}^{\substack{\text{final}\\\text{config}}} \sum{}' f_i \cdot dx_i$$

s path-independent (it equals U(initial config) − U(final config)). Key feature of Newton's eqns *) with conservative force field: the "total energy"

$$H = \underbrace{\tfrac{1}{2}\sum{}' m_i |\dot{x}_i|^2}_{\text{kinetic}} + \underbrace{U}_{\text{potential}}$$

s conserved, since

$$\frac{dH}{dt} = \sum{}' m_i \langle \dot{x}_i, \ddot{x}_i \rangle + \langle \frac{\partial U}{\partial x_i}, \dot{x}_i \rangle = 0$$

hen (*) and (**) hold.

xample 1: a single particle in a central force field,

$$m\ddot{x} = -\nabla\varphi(|x|) \quad ;$$

reduces to a scalar ODE for $r = |x|$, namely

$$m\ddot{r} = -\partial\psi/\partial r$$

with $\psi(r) = \varphi(r) + \frac{\text{const}}{r^2}$ (this is not trivial; see the discn later concerning 2-body pbm, esp. pg 8.12). Gravitation is a special case: $\varphi = -k/r$, so $\psi = -k/r + \frac{\text{const}}{r^2}$.

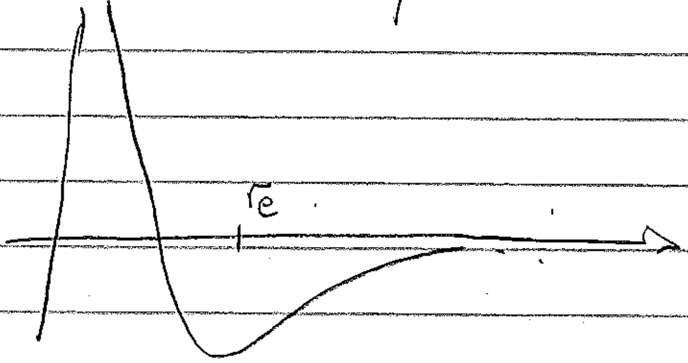
Example 2: many particles interacting by pairwise attraction/repulsion

$$m\ddot{x}_i = -\sum_{j \neq i} \frac{\partial}{\partial x_i} V(|x_i - x_j|)$$

Widely used example (eg for modeling fluids): The Lennard-Jones potential

$$V(r) = c\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

where $\sigma$, $c$ are constants. Graph looks like



where $r_e$ solves $(\sigma/r_e)^6 = \frac{1}{2}$. Particles repel when $|X_i - X_j| < r_e$ & attract when $|X_i - X_j| > r_e$, but attraction is negligible as $|X_i - X_j|/r_e \to \infty$ while repulsion $\to \infty$ as $|X_i - X_j|/r_e \to 0$

Example 2.5 (Similar to ex 2 but not a special case): 3 particles interacting by gravity

$$m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i}$$

$$U = \frac{-m_1 m_2}{|X_1 - X_2|^2} - \frac{m_1 m_3}{|X_1 - X_3|^2} - \frac{m_2 m_3}{|X_2 - X_3|^2}$$

(Different from EX 2 since potential assoc particles i+j is proportional to $m_i m_j$).

xample 3: 1D particle attached to a 1D spring

$$m\ddot{X} = -\alpha x \qquad x \in \mathbb{R}$$

(Exactly solvable of course: $x(t) = A\sin(\omega t + \varphi)$, $\omega^2 = \alpha^2/m$.)

Example 4: nodal values of wave eqn, discretized in space (but not time).

$$\ddot{W}_i = \frac{W_{i+1} + W_{i-1} - 2W_i}{(\Delta X)^2}$$

(here the potential is $U = \sum' \left|\frac{W_i - W_{i-1}}{\Delta X}\right|^2$, and we must pay attn to bdry conds; if the wave eqn has $W(X,t)$ defined for $0<X<1$ with Dir bc $W \equiv 0$ at $X=0$ + $X=1$ then discrete version involves $W_1, \dots, W_{N-1}$ with $\Delta X = \frac{1}{N}$ and law for $\ddot{W}_1$ and $\ddot{W}_{N-1}$ takes into acct that $W_0 = 0$, $W_N = 0$.)

Explicit solns are useful

a) to gain intuition
b) to perturb around them (eg the solar system is a perturbation of 2-body problems involving each planet + the sun)

But: explicit solns are rare! Aside from linear cases (Examples 3 + 4) the most accessible examples of explicit solns are

① two body problems

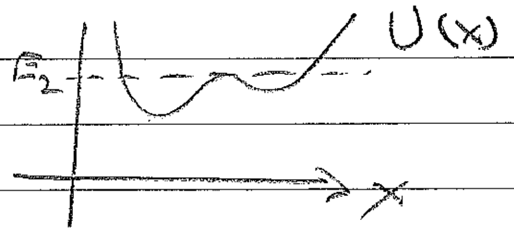
② systems with one spatial degree of freedom (eg "planar pendulum" $\ddot{\theta} = -\sin\theta$)

Brief descn of ②: key tool is phase plane analysis. Evolution eqn defines a flow in 2D phase plane. If force is conservative (always true in 1D if force depends on position but not velocity) then

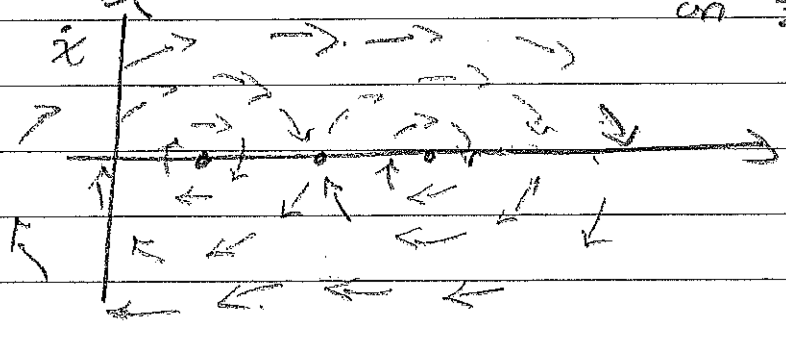
$$H = \text{kinetic} + \text{potential energy}$$

is conserved, so system moves on 1D curve $H = \text{const}$ in 2D space $(x, \dot{x})$. This makes visualization easy even if formulas are not readily available.

Example: $\ddot{x} = -U'(x)$ where



Phase plane picture is filled out by trajectories on $\frac{1}{2}\dot{x}^2 + U(x) = E$.



Local max of U is unstable;
Local min of U are stable (but not asymptotically stable); for $E > E_2$, trajectory surrounds all 3 crit pts.

Cons of angular + linear momentum: an interacting particle system is *closed* if the only forces present are those assoc to pairwise interactions (via pairwise potentials $U = \sum U_{ij}(|x_i - x_j|)$. In such a system

$$m_i \ddot{x}_i = \sum_{j \neq i}' F_{ij} \qquad \text{where } F_{ij} \parallel x_i - x_j \text{ and } F_{ij} = -F_{ji}$$

Key property of such systems: *linear momentum is conserved*. Here

$$\text{lin momentum} = \sum_{i=1}^{N}{}' m_i \dot{x}_i \qquad (\text{a vector})$$

and proof of cons is elementary:

$$\frac{d}{dt} \sum_i' m_i \dot{x}_i = \sum_i' m_i \ddot{x}_i$$

$$= \sum_i \sum_{j \neq i}' F_{ij} = 0$$

since $F_{ij} = -F_{ji}$.

*Impt corollary*: center of mass has accel 0 in such a system. (Since law of motion is invariant wrt to subtracting a linear motion, we can suppose wlog that center of mass stays fixed.)

Another key property of closed system: angular momentum is also conserved. Here

$$\text{angular momentum} = \sum_i x_i \wedge m_i \dot{x}_i$$

(vector cross-product in $\mathbb{R}^3$!). Pf is again elementary:

$$\frac{d}{dt} \sum_i x_i \wedge m_i \dot{x}_i = \sum_i x_i \wedge m_i \ddot{x}_i$$

$$= \sum_i \sum_{j \neq i} x_i \wedge F_{ij}$$

For any pair $i \neq j$, $F_{ij} = \lambda(x_j - x_i)$ + $F_{ji} = -\lambda(x_j - x_i)$

so

$$x_i \wedge F_{ij} + x_j \wedge F_{ji} = \lambda\, x_i \wedge (x_j - x_i) - \lambda\, x_j \wedge (x_j - x_i)$$
$$= \lambda x_i \wedge x_j + \lambda x_j \wedge x_i = 0.$$

Thus entire sum $= 0$.

Corollary: If a single particle in $\mathbb{R}^3$ moves in a central force field, then it remains in the plane determined by its initial posn and velocity.

__Pf:__ Repeat the proof of cons of angular momentum (this time $\ddot{x}_i \parallel x_i$) or else recognize that this is like the case of 2 particles with $m_2 = \infty$. Since $M = x \wedge m\dot{x}$ is constant, $x(t)$ and $\dot{x}(t)$ remain in the plane $\perp M$.

I promised to discuss the 2-body plm. Let's do that now:

$$m_1 \ddot{x}_1 = -\frac{\partial V}{\partial x_1} \qquad m_2 \ddot{x}_2 = -\frac{\partial V}{\partial x_2}$$

where $U = U(|x_1 - x_2|)$

__Claim:__ ① $z = x_1 - x_2$ satisfies

$$\frac{m_1 m_2}{m_1 + m_2} \ddot{z} = -\nabla_z U(|z|)$$

ie it evolves like motion of a single particle of mass $\frac{m_1 m_2}{m_1+m_2}$ in the central force field assoc to $U$. (Set $\bar{m} = \frac{m_1 m_2}{m_1 + m_2}$)

8,10

② $z$ remains in a plane

③ in polar coords $M = \dot{\varphi}(t)\, r^2$ is constant (this is the out-of-plane component of angular momentum) and $r(t)$ behaves like a 1D particle with potential energy

$$V(r) = U(r) + \frac{M^2 \overline{m}}{2r^2}$$

Since we understand 1D systems well (using phase plane analysis) clearly ③ lets us analyze the system more or less completely. [For special case of gravitational interaction, a little extra work ⇒ Kepler's laws, cf Arnold Chap 2.8 or Jose + Saletan Chap 2.3]

<u>Explain the claims:</u>

About ①: we expect reduction to a one-particle system, since motion of center of mass is trivial. Actually, pf is elementary & does <u>not</u> use cons of momentum: mult $x_1$-eqn by $m_2$, $x_2$-eqn by $m_1$, & subtract to get

$$m_2 m_1 \ddot{x}_1 = -m_2 \partial_z U \quad , \quad m_1 m_2 \ddot{x}_2 = m_1 \partial_z U$$

so $$m_1 m_2 \ddot{z} = -(m_1 + m_2)\, \partial_z U(|z|)$$

as asserted.

About ②: Cons of momentum (combined with ①) shows that $z$ stays in a plane.

About ③: if $r, \varphi$ are polar coords in plane of motion, let

$$\frac{z}{|z|} = (\cos\varphi, \sin\varphi) = e_r$$ be radial unit vector

and

$$e_\varphi = (-\sin\varphi, \cos\varphi) = e_r^{\perp}$$ be orthog to it.

By calculus,

$$\dot{z} = \dot{r}\, e_r + r\dot{\varphi}\, e_\varphi$$

Cons of momentum says $z \wedge \dot{z}$ is constant. But this is $r e_r \wedge (\dot{r} e_r + r\dot{\varphi} e_\varphi) = r^2 \dot{\varphi}\, e_r \times e_\varphi$. So $r^2 \dot{\varphi}$ is constant.

Differentiating further, + using $\dot{e}_r = \dot{\varphi}\, e_\varphi$
$\dot{e}_\varphi = -\dot{\varphi}\, e_r$

we get

$$\ddot{z} = \ddot{r} e_r + \dot{r} \dot{\varphi} e_\varphi + \dot{r} \dot{\varphi} e_\varphi + r \ddot{\varphi} e_\varphi - r \dot{\varphi} \dot{\varphi} e_r$$

ie $\ddot{z} = (\ddot{r} - r\dot{\varphi}^2) e_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi}) e_\varphi$

Since the force field is central,

$$\bar{m}(\ddot{r} - r\dot{\varphi}^2) = -U'(r), \quad 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$$

But $\dot{\varphi} = \frac{M}{r^2}$ (recall: $M$ = const determined by initial conds). So

$$\bar{m}\ddot{r} = -U'(r) + \bar{m} r \frac{M^2}{r^4}$$

$$= -U'(r) - V'(r) \quad \text{with } V = \frac{\bar{m} M^2}{2 r^2}$$

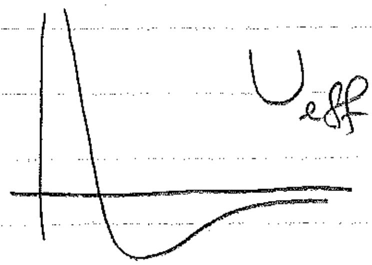
* * *

You probably know that for the gravitational law $U(r) = -k/r$ the orbits are all ellipses (if bdd) or hyperbolas (if unbounded). But that's very special to the inverse-square grav force law. How can ③ be used to get qualitative information in a robust way? Use phase plane analysis! Choosing several constants =1 for simplicity, we're left to consider

$$U_{eff}(r) = \frac{1}{2r^2} - \frac{1}{r}$$

extra term $V(r)$ with $\bar{m} = M = 1$

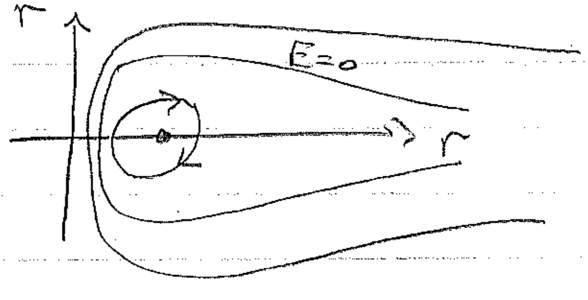
grav potential with $k=1$



Trajectories in phase plane are assoc

$$\frac{1}{2}\dot{r}^2 + U_{eff}(r) = E \quad (\text{constant})$$

These curves look like



closed orbits for $E<0$; unbounded orbits for $E \geq 0$; there's one orbit at which $r$ is constant and $\dot{r}=0$ — it corresp to $E =$ min value of $U_{eff}$.