Mechanics - Lecture 3, 2/7/2018

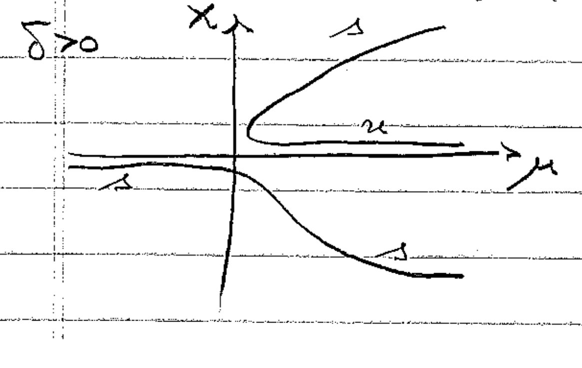
[We stopped last wk at pg 2.11 of "Lecture 2 notes," so on 2/7 we'll start with buckling of the elastica, understood via formal expansion + (briefly) by Liapunov-Schmidt reduction.]

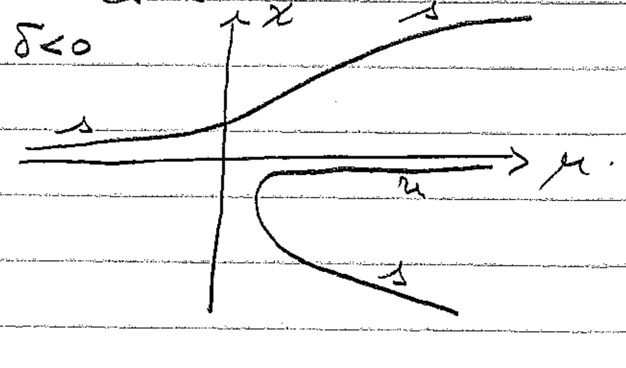
A little more on bifurcation: a small imperfection will change the bifurcation diagram in an essential way. (This is important, since real systems usually have imperfections.)

Simplest example: let's perturb eqn $x^3 = \mu x$ (where $x, \mu \in \mathbb{R}$; note that it describes the crit pts of $\frac{1}{4}x^4 - \frac{\mu}{2}x^2$, which is convex for $\mu < 0$ [$x=0$ is the only crit pt] but not for $\mu > 0$ [crit pts are $x = \pm\sqrt{\mu}$ and $x=0$]) by considering

$$x^3 - \mu x + \delta = 0$$

(which describes crit pts of $\frac{1}{4}x^4 - \frac{\mu}{2}x^2 + \delta x$). The presence of $\delta \neq 0$ "breaks" bif diagram into two disconnected components





(The labels "s" + "u" corresp to local min or local max of assoc "energy" $\frac{1}{4}x^4 - \frac{\mu}{2}x^2 + \delta x$, with $\mu + \delta$ held fixed.)

<u>Note</u>: effect of $\delta \neq 0$ involves fractional powers of δ (thus: not so small, even if δ is small) since

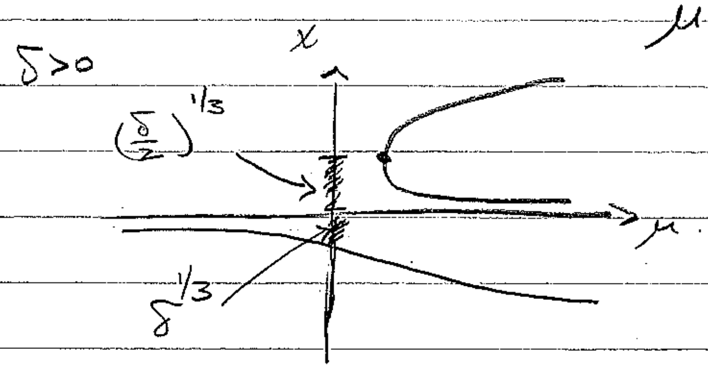
$$\mu = 0 \implies x^3 = -\delta \implies x = -(\delta^{1/3})$$

vertical tangent in bif diagram

$$\iff \frac{d\mu}{dx} = 0$$

$$\iff 3x^2 = \mu \iff x = (\mu/3)^{1/2}$$

$$\mu^{3/2} = \frac{3\sqrt{3}}{2}\delta$$



A problem on HW2 asks you to show that the buckling of an elastica with a bit of intrinsic curvature is essentially the same as the preceding example. (Gravity provides another physically natural imperfection; see

Howell-Kozyreff-Ockendon Chap 4, prob 4.17.)

<u>Digression</u>: in the late 70's people asked whether one can "classify all possible ways an imperfection can break the bifurcation diagram." Answer was yes in many cases; see Golubitsky + Schaeffer, "A theory for imperfect bifurcation via singularity theory", CPAM 32 (1979) 21-98.

<u>New topic</u>: start discn of <u>3D (nonlinear) elasticity</u>. Recommended reading:

- 1^{st} 20 pages of Marsden + Hughes
- chapter 5 of Howell-Kozyreff-Ockendon

As usual we must address

a) strain (ie description of the stretching + shrinking assoc to a given deformation)

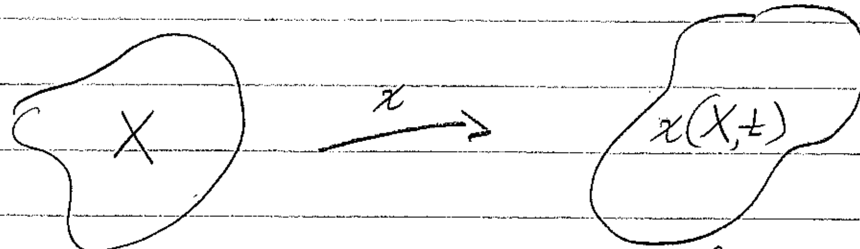
b) stress (ie description of forces by which a body acts on itself)

c) balance laws (always assoc to conservation of linear + angular momentum)

d) constitutive laws (relation btwn stress + strain, which characterizes the material under consideration)

<u>Start with strain</u> (analysis of deformation; "kinematics"; closely linked to differential geometry)

Basic picture:



$\Omega \subset \mathbb{R}^3$
reference ("Lagrangian") coordinates

deformed ("Eulerian") coordinates

$X = \{X_\alpha\}$, $x = \{x_i\}$ conventions for ref + deformed coords

$F_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}$ "deformation gradient"

<u>note:</u> $X \to x(X)$ should be orientation preserving ($\det F > 0$) and injective (no interpenetration)

<u>Polar decomposition</u>: if $\det F > 0$ then F can be written uniquely as

$$F = RU$$

where $U = (F^T F)^{1/2}$ is pos def symmetric and R is an orientation-preserving rotn

<u>Eigenvalues</u> of U are "principal <u>stretches</u>"

Nonlinear strain is $U - I$ or $F^T F - I$ (depends on author).

Main pt: locally, any deformation is like a pos def symm lin map composed with a rotn.

<u>Comments</u> (several long digressions):

1) Which matrix-valued fns $F_{i\alpha}(X)$ are def gradients?

Ans is standard: in a simply-connected domain, $F_{i\alpha} = \partial x_i / \partial X_\alpha$ iff each row is curl-free.

(<u>Note</u>: when studying crystals with dislocations, req't that $F = \partial x / \partial X$ is relaxed; presence of nonzero curl reflects presence of dislocations in the lattice.)

2) Which matrix-valued fns $g_{\alpha\beta}(X)$ arise as $F^T F = \left(\sum_{i=1}^{3} F_{\alpha i} F_{i\beta}\right)_{\alpha\beta}$ for some $F_{i\alpha} = \partial x_i / \partial X_\alpha$?

Ans is subtle, but classical: $F^T F$ is a <u>Riemannian metric</u> (namely, the standard metric of $\mathbb{R}^3$, expressed in local coords X rather than Eucl coords x, since $dx = F\,dX \Rightarrow |dx|^2 = |F\,dX|^2 = \langle F^T F\,dX, dX\rangle$.

<u>In language of differential geometry</u>: a pos symmetric matrix valued fn $g_{\alpha\beta}(X)$ can be expressed as $F^T F$ iff it is a "flat metric" (ie the standard metric expressed in a weird coord system).
<u>Major result of differential geometry</u>: This is true (locally) iff its "Riemann curvature tensor" vanishes. (This is a nonlinear pde system involving $g_{\alpha\beta}$ and its 1^{st} + 2^{nd} derivatives.)

3) Which deformations do no stretching or shrinking? <u>Ans</u>: strain $= 0 \Leftrightarrow F^T F = \underline{I}$

$\iff \partial x/\partial X$ is a rotn at each x.
Assuming a little smoothness, this $\Rightarrow$
$x(X)$ is a <u>rigid motion</u> (another classical result from differential geometry).

4) What can we say about defs that do <u>little stretching</u>? <u>Much</u> <u>harder</u> <u>question</u>! Studied by F. John in 1960's + revisited by Friesecke, James, + Mueller about 2002. They showed:

$$\inf_{R \in SO(3)} \|Du - R\|^2_{L^2(\Omega)} \leq C \int_\Omega \mathrm{dist}^2(Du, SO(3))$$

(a "nonlinear Korn inequality"), + they used it to provide rigorous justifn of plate + shell theory

(5) What about elastic membranes? These are described by maps $\mathbb{R}^2 \to \mathbb{R}^3$. Polar decomp (really, SVD) gives $F_{i\alpha} = \frac{\partial x^i}{\partial X_\alpha} = R \cdot U$ ← $(F^T F)^{1/2}$ now 2x2

where now R is an isometric immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$.

Which defs do no stretching or shrinking?

Image is then a 2D surface in $\mathbb{R}^3$ isometric to $\mathbb{R}^2$. Nec + suff't condn (assuming C^2) is that Gaussian curvature vanish. Fact of diff'l geometry: such surfaces are "developable" (directions with prin curvature zero line up, forming straight lines that stay in the surface).

OK, now let's turn to stress (analysis of forces; "statics"; Cauchy's thm is closely linked to the fact that the natural objects to integrate over 2D surfaces are 2-forms.)

We consider 2 types of forces:

- body force $\vec{f}(x)$ per unit deformed volume, acting at (deformed) position x
- surface force $\vec{T}(x;\vec{n})$ per unit deformed area, acting on plane $\perp$ unit vector $\vec{n}$ at (deformed) position x. (Sign convention: $\vec{T} = -p\vec{n}$ with $p > 0$ for hydrostatic pressure)

Intuition: a spatial volume is acted upon by body force (eg gravity) and also a surface force (applied by rest of body).

Note convention: $\vec{f}$ = force/unit deformed vol. Assoc force/unit ref vol is $f \cdot \det\left(\frac{\partial x}{\partial X}\right)$. (In lecture 1 we used f for force/unit ref length; that was a <u>different</u> convention.)

Clearly $\vec{T}(\vec{n}) = -\vec{T}(-\vec{n})$ ("law of equal reaction")

<u>Cauchy's Thm:</u>

a) $\vec{T}(x, \vec{n})$ is linear in $\vec{n}$, ie

$$T_i(x, \vec{n}) = \sum_{j=1}^{3} \tau_{ij}(x)\, n_j$$

b) The matrix $\tau_{ij}(x)$ is symmetric, ie

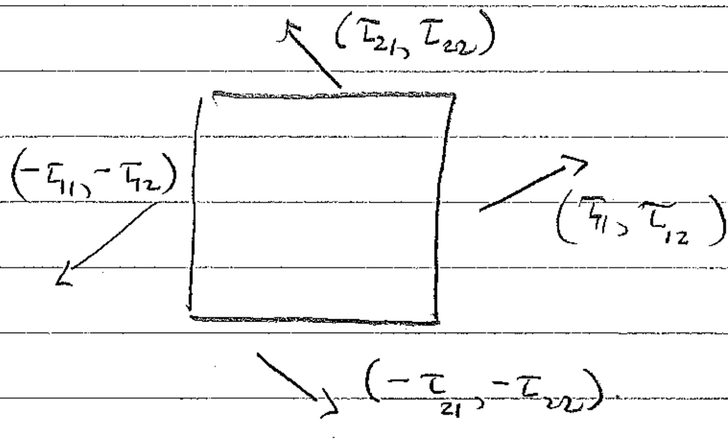
$$\tau_{ij} = \tau_{ji} \quad \text{for all } i \neq j$$

We call τ the <u>Cauchy stress tensor</u>. (In dynamics, $\vec{T} = \vec{T}(x, t; \vec{n})$; same result holds, with $\tau = \tau_{ij}(x, t)$.)

Intuition behind (a): it says essentially that stress is a <u>2-form</u>. (Proof resembles discn in H. Whitney's book "Geometric Integration Theory" – 1st few pages only! – why 2-forms are the natural objects to integrate over surfaces.)

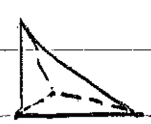
Intuition behind (b): if it weren't so then a small cube of material would experience a net torque. Visualization in 2D:

if $\tau_{12} \neq \tau_{21}$
forces on faces
would be
as shown
[figure has
$\tau_{21} < 0$, $\tau_{12} > 0$]



Pf of Cauchy's thm, assuming everything is in static equilibrium (the dynamic case is only slightly different)

Pf of (a): Fix x_0 + $\vec{n}$. Consider tetrahedron (undeformed coords) V with corner x_0 + 3 faces $\perp$ axes (call them S_1, S_2, S_3) + 4^{th} face $\perp \vec{n}$ (call it S_0). Figure, for $\vec{n} = \frac{1}{\sqrt{3}}(1,1,1)$:



Geometry $\Rightarrow$ $|S_i| = n_i |S_0|$
Outward normal to S_i is $-e_i$

Balance of forces $\Rightarrow$

$$0 = \int_{S_0} T(x,\vec{n})\, dA + \sum_i \int_{S_i} T(x,-e_i)\, dA + \int_V f\, d\mathrm{vol}$$

where dA = surface area. Assuming a little regularity, divide by $|S_0|$ + take small-vol limit to get

$$T(x,\vec{n}) + \sum n_i T(x,-e_i) = 0 .$$

(Vol integral scales like vol not area, so it $\to 0$ in limit if f is bdd). Thus:

$$T(x,\vec{n}) = \sum_j \tau_{ij}(x) n_j \quad , \text{ with } \tau_{ij} = T_i(x, e_j).$$

[Proof in dynamic setting is almost the same; sole difference: there's an additional body force assoc acceleration x_{tt}.]

We also see equil pde at this stage:

$$(*) \quad \sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_j} + f_i = 0$$

since total force on any volume must vanish.

Pf of (b): equil eqn came from balance of forces; symmetry comes from balance of torques: in static setting

$$\int_V x \wedge f \, dvol + \int_{\partial V} x \wedge (T \cdot n) \, dA = 0$$

where $a \wedge b$ is vector cross-product of $a, b \in \mathbb{R}^3$:

$$(a \wedge b)_i = \sum_{j,k}' \varepsilon_{ijk} a_j b_k = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

PDE version of balance of torques is

$$(**) \quad \sum_{j,k}' \varepsilon_{ijk} x_j f_k + \sum_{j,k,l}' \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} x_j T_{kl} \right) = 0$$

Simplify using bal of forces (*) to get

$$\sum_{jkl}' \varepsilon_{ijk} \delta_{jl} T_{kl} = 0 \quad \text{for each } i$$

ie

$$\sum_{jk}' \varepsilon_{ijk} T_{kj} = 0 \quad \text{for each } i$$

whence T is symmetric.

Less coordinate-bound version of this calcn:

$$\int_{\partial V} x_i T_j(n) - x_j T_i(n) = \int_{\partial V} \sum_k (x_i \tau_{jk} - x_j \tau_{ik}) n_k$$

$$= \int_V \sum_k \frac{\partial}{\partial x_k} (x_i \tau_{jk} - x_j \tau_{ik})$$

$$= \int_V \tau_{ji} + x_i (\operatorname{div} \tau)_j$$

$$- \tau_{ij} - x_j (\operatorname{div} \tau)_i$$

$$= \int_V (\tau_{ji} - \tau_{ij}) + (x_j f_i - x_i f_j)$$

Now, balance of torques says

$$0 = \int_{\partial V} (x_i T_j - x_j T_i) + \int_V x_i f_j - x_j f_i$$

for all $i \neq j$. Combined with calcn above we have

$$\int_V (\tau_{ij} - \tau_{ji})\, dx = 0$$

True for all $V \Rightarrow \tau_{ij} = \tau_{ji}$.

Done with stress + strain. Also done with balance laws (at least for statics). Next wk: constitutive modeling.