Mechanics - Lecture 5, 2/21/2018

(There are no "Lecture 4 notes" - on 2/14 we finished the "Lecture 3 notes".)

Continuing 3D nonlinear elasticity: already discussed stress + strain; need to do

a) constitutive laws (relns btwn stress + strain)
b) "reference" vs "deformed" (Lagrangian vs Eulerian) coords
c) examples

These notes do (a) + (b); HW 3 does (c).

Topics (a) + (b) are connected, because

A) var'l principles are easier to state + solve in reference coords, but

B) measurements + most typical loads (eg pressure) come to us as force per unit deformed area

Explaining (A): We'll see that a convenient approach to constitutive modeling is to insist that

$$\text{elastostatics} \iff \delta E = 0, \text{ where}$$

$$E = \int_\Omega W(\partial x/\partial X)\,dX + \int_\Omega V(x(X))\,dX$$

with Ω = reference config'n + $W(F)$ a fn of matrices ("elastic energy density") satisfying certain structural conditions. Assoc pde

$$\sum_{\alpha=1}^{3} \frac{\partial}{\partial X_\alpha}\left[\frac{\partial W}{\partial F_{i\alpha}} \left(\partial x/\partial X\right) \right] - \frac{\partial V}{\partial x_i}(x(X)) = 0$$

expresses balance of forces in ref. coords

$\frac{\partial W}{\partial F_{i\alpha}}(\partial x/\partial X)$ = force in dirn i per unit ref area, acting on surface $\perp$ α^{th} coord vector in ref coords

$=_{\text{defn}}$ "1st Piola-Kirchhoff stress tensor" $P_{i\alpha}$

$-\frac{\partial V}{\partial x_j}(x(X))$ = force per unit ref vol acting at location $x(X)$.

Viewpt is convenient because we're solving a pde on a fixed domain Ω.

Explaining (B): Recall that

T_{ij} = force in dirn i per unit deformed area, acting on surface $\perp$ j^{th} coord vector in deformed coords and that τ is symmetric ($P_{i\alpha}$ is not symmetric!)

A common bc cond is "constant pressure p_0" which means

$$\tau \cdot \vec{n} = -p_0 \vec{n} \quad \text{at } \partial x(\Omega),$$

where $\vec{n}$ = unit normal to $\partial x(\Omega)$. Simple to say in Eulerian coords, messy to write in Lagr coords.

"Traction free" bc is easy in both contexts

$$\tau \cdot \vec{n} = 0 \text{ at } \partial x(\Omega), \quad \vec{n} = \text{unit normal to } \partial x(\Omega)$$
$$\Updownarrow$$
$$P \cdot N = 0 \text{ at } \partial \Omega, \quad N = \text{unit normal to } \partial\Omega$$

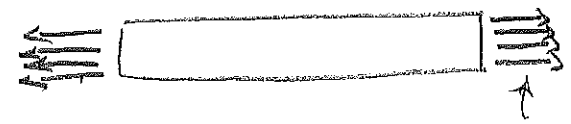
but specified (nonzero) traction is more subtle; the statements

$$P \cdot N = f \quad (\text{"dead load" } f)$$

vs

$$\tau \cdot n = f \quad (\text{"live load" } f)$$

are different. (Dead loads are hard to apply in practise, since load must maintain its direction + magnitude / unit ref area regardless of deformation. One scheme: use long springs, eg for uniaxial tension



springs attached to
a distant frame.)

Key to translation btwn Eulerian + Lagrangian:

$$\sum_{j=0}^{3} \frac{\partial}{\partial x_j}(T_{ij}) + f_i = 0 \qquad \text{(with } f = \text{force per unit deformed vol)}$$

$$\Updownarrow$$

$$\sum_{\alpha=1}^{3} \frac{\partial}{\partial X_\alpha}(P_{i\alpha}) + f_i^R = 0$$

with $f^R = f \det\left(\frac{\partial x}{\partial X}\right)$
= force / unit vol in
rest coords

<u>Claim</u> $P_{i\alpha} = \det\left(\frac{\partial x}{\partial X}\right) \sum_k T_{ik} \frac{\partial X_\alpha}{\partial x_k}$

(In matrix notation: $P = J\,T(F^{-1})^T$ where $F_{i\alpha} = \partial x_i / \partial X_\alpha$

and $J = \det F$.)

<u>Pf of Claim</u>: $\sum_j \frac{\partial}{\partial x_j}(T_{ij}) + f_i = 0 \iff \int -\sum_j T_{ij} \frac{\partial \varphi}{\partial x_j} + \sum f_i \varphi \; dx = 0$

for all cptly sptd φ. (here i is fixed)

Change to X vars:

$$\int \left[-\sum_{j,\alpha}' \tau_{ij} \frac{\partial \varphi}{\partial X_\alpha} \frac{\partial X_\alpha}{\partial x_j} + f_i \varphi \right] \left(\det \frac{\partial x}{\partial X} \right) dX = 0.$$

$$\sum_\alpha' \frac{\partial}{\partial X_\alpha} P_{i\alpha} + f_i^R = 0 \quad \text{with} \quad P_{i\alpha} = \sum_j' \tau_{ij} \frac{\partial X_\alpha}{\partial x_j} \det \frac{\partial x}{\partial X}.$$

[Warning: as noted earlier, $P_{i\alpha}$ is not symmetric.]

Another place where Eulerian vs Lagr. is relevant is the comparison btwn elastodynamics + fluid dynamics.

Eqns of elastodynamics, in Lagr vars:

$$\sum_\alpha \frac{\partial}{\partial X_\alpha} (P_{i\alpha}) + m(X) g_i = m(X) \ddot{x}_i$$

where $m(X)$ = mass per unit ref vol (density)
$m(X) g$ = body force per unit ref vol.
$\ddot{x}$ = 2nd deriv in t, holding ref. posn fixed.

To recognize cons of mass + momentum as consequences, we should write this in Eulerian coords. It's impt to distinguish

$$\frac{Df}{Dt} = \text{deriv of } f \text{ wrt } t, \text{ holding } X \text{ fixed}$$

$$\frac{\partial f}{\partial t} = \text{deriv of } f \text{ wrt } t, \text{ holding } x \text{ fixed.}$$

By chain rule:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \sum_i v_i \frac{\partial f}{\partial x_i} \qquad \text{where } v_i = \frac{Dx_i}{Dt}$$

(we wrote $\dot{x}_i$ for v_i on page 4.5)

<u>Claim</u>: Eqns of elastodynamics in Eulerian coords are

$$\frac{\partial \rho}{\partial t} + \sum_j \frac{\partial}{\partial x_j}(\rho v_j) = 0 \qquad \text{[cons of mass]}$$

$$\rho\left(\frac{\partial v_i}{\partial t} + v \cdot \nabla_x v^i\right) = \sum_j \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i \qquad \text{[cons of momentum]}$$

where

$$\rho(x,t) = m(X(x,t)) \det\left(\frac{\partial X}{\partial x}\right) = \text{mass per unit deformed vol}$$

$$v(x,t) = \text{velocity (at deformed position } x, \text{ time } t)$$

<u>Pf of 1st eqn</u>: $\quad 0 = \dfrac{d}{dt}\displaystyle\int_B m \, dX \qquad$ for any region $B \subset \Omega$

$$= \frac{d}{dt}\int_{x(B,t)} \rho \, dx$$

$$= \int_{x(B,t)} \rho_t \, dx + \int_{\partial x(B,t)} \rho \, v \cdot n$$

$$= \int_{x(B,t)} \rho_t + \mathrm{div}_x(\rho v) \, dx$$

True for all $B \Rightarrow \rho_t + \mathrm{div}_x(\rho v) = 0$.

Note that this eqn is "purely kinematic" (ie it makes no use of eqns of motion, forces, etc)

Pf of 2nd eqn: recall that $\mathrm{div}_X P = J \, \mathrm{div}_x \Sigma$, with $J = \det(\partial x / \partial X)$. So

$$m \ddot{x} = \mathrm{div}_X P + mg \iff J^{-1} m \, \ddot{x} = \mathrm{div}_x \Sigma + J^{-1} mg$$

Now, $J^{-1} m = \rho$ (by defn) + $\ddot{x} = \dot{v} = \frac{Dv}{Dt} = v_t + v \cdot \nabla v$

Substn gives the asserted 2nd eqn (cons of momentum).

OK, let's turn to constitutive modeling

Two viewpts are possible:

(a) "Cauchy elasticity": specify Cauchy stress as fn of position + def gradient

$$\tau = \hat{\tau}(X, F)$$

(b) "Hyperelasticity": specify $P_{i\alpha} = \dfrac{\partial W(X;F)}{\partial F_{i\alpha}}$ where

W = "energy density"

Viewpt (b) is more restrictive — but therefore more useful (since it seems to be adequate).

Often the body is homogeneous in its ref confign; then constit. reln is indep of X

Always require structural condn of "frame indifference"

a) for Cauchy elasticity: $\hat{\tau}(RF) = R\,\hat{\tau}(F)R^T$ for all orn-preserving rotations R

b) for hyperelasticity: $W(F) = W(RF)$ for all orientation-preserving rotns R.

Interptn:
a) observer in rotated coord system sees same basic stress-strain law

b) rotation does no work

(HW 3 will ask you to show equiv of these relns, for hyperelasticity.)

Many materials are <u>isotropic</u>. In hyperelasticity

isotropy $\Leftrightarrow$ $W(F) = W(FR)$ for any orientation preserving rotn.

$\Leftrightarrow$ $W(F) = \varphi(\lambda_1, \lambda_2, \lambda_3)$ where $\{\lambda_j\}$ are the "principal stretches" (eigs of $(F^TF)^{1/2}$), and φ is a <u>symmetric</u> function of its arguments

Corresp assertion for Cauchy elasticity:

isotropy $\Leftrightarrow$ $\hat{\tau}(FR) = \hat{\tau}(F)$ for all R.

For strings it was natural to ask that $\nu \to N(\nu)$ be monotone increasing. Similarly, for 3D elasticity it is natural to impose some structural conditions, eg

a) eqns of elastostatics are elliptic (eqns of elastodynamics are hyperbolic) or, more globally,

b) var'l prin of elastostatics achieves its minimum.

Discn of (b) would take us too far afield (keyword: "W should be quasiconvex."). Essence of (a) is that $F \to W(F)$ should be (strictly) "rank one convex"

$$\sum \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}} \xi_i \xi_j \eta_\alpha \eta_\beta \geq C |\xi|^2 |\eta|^2.$$

for all $\xi, \eta \in \mathbb{R}^3$. Intuition about why it matters: for a const coefft linear pde system

$$\sum_{\alpha, j, \beta} \frac{\partial}{\partial X_\alpha} \left(A_{i\alpha j\beta} \frac{\partial x_j}{\partial X_\beta} \right) = f_i \quad \text{in } \mathbb{R}^n$$

we can try to solve by Fourier transform

$$- \sum_{\alpha, \beta, j} k_\alpha k_\beta A_{i\alpha j\beta} \hat{x}_j(k) = \hat{f}_i(k)$$

Cond (*) assures that the matrix $\sum_{\alpha, \beta} k_\alpha k_\beta A_{i\alpha j\beta}$ (which has to be inverted) is pos definite.

Rank-one convexity is weaker than convexity. Why not just assume elastic energy is a convex fn of F? Ans: it's not compatible with frame indifference + condn that W(F) be min when F is a rotn

$$W \text{ is min at } F \in SO(3) \quad \text{(only)}$$
$$+ \; SO(3) \text{ is not convex}$$
$$\Rightarrow W \text{ cannot be convex}$$

But convexity is still a convenient tool for identifying suitable energy fns. For example

(*) if $W(F) =$ convex fn of F
+ convex fn of $\det F$

then W is rank-one convex (indeed, "quasi-convex"), and such structure is compatible with the desired behavior. (How to find a frame-indifferent convex fn of F? Use this result, proved eg in Ciarlet's book as Thm 4.9-1: if $W(F) = \varphi(\lambda_1, \lambda_2, \lambda_3)$ where λ_j are eigs of $(F^T F)^{1/2}$ + φ is symmetric, convex, + nondecreasing in each var, then W is a convex fn of F.)

A simple (extreme) case of (*) is the <u>incompressible</u> <u>neoHookean</u> material

$$W(F) = (\lambda_1^2+\lambda_2^2+\lambda_3^2-3) \quad \text{restr by } \det F = \lambda_1\lambda_2\lambda_3 = 1$$

(note: $\lambda_1^2+\lambda_2^2+\lambda_3^2 = \mathrm{tr}(F^TF) = \sum F_{i\alpha}^2$ is obviously convex!).

There's much more to say here – the main tool for getting rank-one convex W is to use "polyconvexity" – but this would take us too far afield. See Ch 4 of Ciarlet for that.

We briefly touched on incompressibility just now. In fact rubber is nearly incompressible and the incompr neoHookean law is widely used as a model constitutive law. What does eqn of elastostatics look like in this setting? Short ans: pde includes an unknown fn – the <u>pressure</u> – as a Lagrange multiplier for constraint of incompressibility; assoc constit law is therefore

$$T = -p\,I + T^*(F)$$

where T^* is given by an elastic energy by rules discussed earlier (but restricted to $\det F = 1$) + p is determined by balance of forces.

<u>Explain this</u>: starting from constrained var'l prin

$$0 = \delta_{\det(\partial x/\partial X)=1} \int_\Omega W(\partial x/\partial X)\, dX$$

$\Rightarrow$ by method of Lagr mult, EL eqn is formally

$$\delta \int_\Omega W(\partial x/\partial X) + \gamma(X)\left[\det\left(\frac{\partial x}{\partial X}\right) - 1\right] dX = 0$$

for some (unknown) $\gamma(X)$. Assoc pde is

$$\sum_\alpha{}' \frac{\partial}{\partial X_\alpha}\left(\frac{\partial W}{\partial F_{i\alpha}} + \gamma(X)\frac{\partial(\det F)}{\partial F_{i\alpha}}\right) = 0.$$

Our task is to show that Cauchy stress assoc to Piola-Kirchhoff stress $\gamma(X)\frac{\partial \det F}{\partial F_{i\alpha}}$ has the form $-p(x) I$. (In fact will show $p = -\gamma$). Key is Cramer's rule, which says $\frac{\partial(\det F)}{\partial F_{i\alpha}}$ = matrix of minors = $J(F^{-1})^T$

Recalling that $P = J\tau(F^{-1})^T$ i.e. $\tau = J^{-1}PF^T$ we get

$$P = \gamma \frac{\partial \det F}{\partial F_{i\alpha}} \Rightarrow \tau = J^{-1}(\gamma J (F^{-1})^T) F^T = \gamma I$$

as asserted.

Returning to general (compressible) case, here's another viewpt that is sometimes useful. Recall that for isotropic elasticity,

$$W(F) = \varphi(\lambda_1, \lambda_2, \lambda_3)$$

where φ is a symmetric fn of 3 vars + $\{\lambda_j\}$ are eigs of $(F^TF)^{1/2}$. Any such fn has the alternative repn

$$(*) \quad W(F) = \Psi(I, II, III)$$

where I, II, III are the elementary symmetric fns of the eigenvalues of $F^TF = C$, i.e.

$$I = \mathrm{tr}\, C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$II = \tfrac{1}{2}\{(\mathrm{tr}\, C)^2 - \mathrm{tr}(C^2)\} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2$$

$$III = \det C = \lambda_1^2\lambda_2^2\lambda_3^2$$

Advantage of this repn: Piola-Kirchhoff stress involves a polynomial expression in derivs of Ψ and components of F.

There is an analogous framework for constitutive modeling in Cauchy elasticity:

$$(**) \quad \Sigma(F) = \varphi_0 I + \varphi_1 B + \varphi_2 B^2$$

where $B = FF^T$ and φ_i are suitable fns of I, II, III. (Note: $B = FF^T$ & $C = F^TF$ have the same eigenvalues, though their eigenvectors are generally different.) HW3 will ask you to show $(*) \Rightarrow (**)$.

HW3 will also have pbms exploring

- how expts can be used to identify the constitutive law
- how nonlinearities of elasticity explain observable effects, eg rln btwn pressure + radius when blowing up a balloon.

(Lecture 6 will turn to linear elasticity.)