

Mechanics - Lecture 6, 2/28/2018

[2/28 class will begin with end of "Lecture 5" notes - pp 5.12 - 5.15]

We've been doing "fully nonlinear" elasticity. Now let's pass to linear elasticity. This entails two separate linearizations:

1st geometrical linearization ("small strain theory") - we suppose

$$\chi_\delta(X) = X + \delta u(X)$$

with δ small, Here u is the "infinitesimal elastic displacement". Evidently

$$F = I + \delta \cdot Du$$

$$\Rightarrow (F^T F)^{1/2} = I + \frac{\delta}{2} (Du + Du^T) + \mathcal{O}(\delta^2)$$

so to leading order, $(F^T F)^{1/2} - I$ is δ times the linear elastic strain

$$e(u) = \frac{1}{2} (Du + Du^T)$$

2nd physical linearization ("linear stress-strain law")

Using hyperelasticity as a starting pt (spatially homogeneous, for simplicity) with $W(F) = 0$ & $W \geq 0$ (so rest state is a local min of elastic energy), we have

$$W(F) = \delta^2 \langle \alpha e(u), e(u) \rangle + \mathcal{O}(\delta^3)$$

where

$$\langle \alpha e, e \rangle = \sum' \alpha_{ijkl} e_{ij} e_{kl}$$

is a pos def (symmetric) quadratic form on symmetric tensors.

Dropping terms of higher order in δ (and rescaling by δ^2) we expect a vari' l prin of the form $\delta E = 0$ where

$$E = \int_{\Omega} \langle \alpha e(u), e(u) \rangle dx + \left[\text{terms assoc} \right. \\ \left. \text{to loads} \right]$$

so equil eqn is

$$\operatorname{div}(\alpha e(u)) + f = 0$$

and we recognize

$$\sigma = \alpha e(u)$$

as the stress tensor in the linearly elastic setting.

Notes: a) σ is symmetric

b) we lose the ability to distinguish between rot + deformed corals when we linearize.

(Put differently: The Cauchy + Piola-Kirchhoff stresses agree to 1st order in δ ; that's why σ is symmetric.)

Summary: in linear elasticity, basic unknown is "displacement" $u(x)$. Assoc lin strain is $e(u) = \frac{1}{2}(Du + Du^T)$. Stress is $\sigma = \alpha e(u)$. Eqs of equil are

$$\operatorname{div} \sigma + f = 0, \quad \sigma = \alpha e(u)$$

augmented by suitable bc, for example

$$\textcircled{1} u = u_0 \text{ at } \partial\Omega \text{ ("displacement bc")}$$

$$\textcircled{2} \sigma \cdot n = g \text{ at } \partial\Omega \text{ ("traction bc")}$$

or ③ $u \cdot n = 0, (\sigma \cdot n)_{\text{tan}} = 0$ at $\partial\Omega$
("lubricated bc")

We discussed isotropy in nonlinear setting. Analogue for linear elasticity is

Hooke's law α is isotropic $\Leftrightarrow \alpha(R^T e R) = R^T (\alpha \cdot e) R$
for any e (symmetric)
and any R (rotation)

(More generally: R is a symmetry of the material if $\alpha(R^T e R) = R^T (\alpha \cdot e) R$ for any e . Intuition: rotating material \Leftrightarrow keep material fixed but rotate coordinates.)

Lemma: The general isotropic Hooke's law can be expressed as

$$\sigma_{ij} = 2\mu e_{ij} + \lambda (\text{tr} e) \delta_{ij}$$

where $\lambda + \mu$ are constants ("Lamé moduli").

Explan (not quite a proof): Hooke's law isotropic \Leftrightarrow energy $\langle \alpha e, e \rangle$ is a quadratic, isotropic

function of symm tensors e . Fact of linear algebra: all such fns are obtained by starting from

$$e_{ij} e_{kl}$$

and "contracting indices" in pairs. There are two distinct ways to do this:

$$a) \sum_{i,k} e_{ii} e_{kk} = (\text{tr} e)^2$$

$$b) \sum_{i,j} e_{ij} e_{ij} = \text{tr}(e^2) = |e|^2$$

So

$$\langle \lambda e, e \rangle = 2\mu |e|^2 + \lambda (\text{tr} e)^2,$$

for some $\lambda + \mu$.

There are other common reprs of an isotropic Hooke's law, and good reasons to use them:

1st in terms of Young's modulus E + Poisson's ratio ν

$$\sigma_{ij} = \frac{E}{1+\nu} \left(e_{ij} + \frac{\nu}{1-2\nu} (\text{tr} e) \delta_{ij} \right)$$

2nd in terms of bulk modulus K and shear modulus μ ,

$$\sigma_{ij} = 3K \left(\frac{1}{3} (\text{tr } e) \delta_{ij} \right) + 2\mu \left(e_{ij} - \frac{1}{3} (\text{tr } e) \delta_{ij} \right)$$

Positivity requires

$\mu > 0$	$E > 0$	$\mu > 0$
$3\lambda + 2\mu > 0$	$-1 < \nu < 1/2$	$K > 0$
for Lamé moduli	for Young's modulus + Poisson's ratio	for bulk + shear moduli

Of course one set of parameters determines the others, eg

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad \mu = \frac{E}{2(1+\nu)}$$

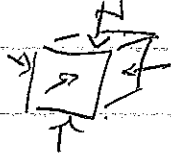
$$\lambda = K - \frac{2}{3}\mu$$

Meaning of these parameters:

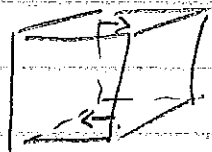
- Bulk + shear moduli describe action of α on multiples of identity + trace-free matrices ("pure shears") separately

$$\sigma_{ij} = 3K \underbrace{\left(\frac{1}{3} \text{tr } e \delta_{ij} \right)}_{\text{proj'n of } e \text{ onto mult of } I} + 2\mu \underbrace{\left(e_{ij} - \frac{1}{3} (\text{tr } e) \delta_{ij} \right)}_{\text{proj'n of } e \text{ orthog to } I}$$

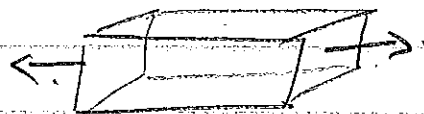
Note: bulk modulus measures vol change due to hydrostatic pressure



shear modulus gives response to any pure shear, e.g. $\sigma_{12} = T$, $\sigma_{ij} = 0$ otherwise
 $\Rightarrow e_{12} = \frac{1}{2\mu} \sigma_{12}$



• Poisson's ratio + Young's modulus describe behavior under uniaxial tension ($\sigma_{11} = T$, other $\sigma_{ij} = 0$).



$$\Rightarrow e_{11} = \frac{1}{E} T$$

$$e_{22} = e_{33} = -\frac{\nu}{E} T$$

Most materials have $\nu > 0$, so uniaxial tension produces contraction in the other two axes. Cork has $\nu \approx 0$, which is why it is used for closing wine bottles.

There are lots of scalar or 2D reductions of

elastostatics that help generate intuition; see
eg Howell, Kozlov, Ockendon et al

- antiplane strain (Sec 2.3)
 - torsion (Sec 2.4, 2.5)
- } reductions to
scalar 2nd order
pde
- plane strain (Sec 2.6) - a 2nd order system
but equiv to a 4th order
scalar pde pbm.

var' principles: soln of elastostatics with
Dir bc (displacement fixed at $\partial\Omega$) achieves

$$\min_{u=0 \text{ at } \partial\Omega} \left\{ \int_{\Omega} \frac{1}{2} \langle \alpha e(u), e(u) \rangle - \langle u, f \rangle dx \right.$$

Traction bc (bc $\sigma \cdot n = F$ at $\partial\Omega$) - we
have similar var' principle

$$\min_u \left\{ \int_{\Omega} \frac{1}{2} \langle \alpha e(u), e(u) \rangle - \langle u, f \rangle dx \right. \\ \left. - \int_{\partial\Omega} \langle u, F \rangle \right.$$

(we'll discuss these + their implications next week). These are the elastic analogues of the well-known vari' l prin's for Laplace's eqn

$$-\Delta u = f \text{ in } \Omega, \quad u = g \text{ at } \partial\Omega \iff u \text{ achieves min of } \int_{\Omega} \frac{1}{2} |\nabla w|^2 - wf \, dx$$

among all fns w s.t. $w = g$ at $\partial\Omega$

$$-\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = F \text{ at } \partial\Omega \iff u \text{ achieves min of } \int_{\Omega} \frac{1}{2} |\nabla w|^2 - wf \, dx - \int_{\partial\Omega} wF \, dA$$

among all fns w (no bc) on Ω

Some things we'll need to sort out:

- for traction bc there are consistency conditions on $f + F$, namely

$$\int_{\Omega} \langle u, f \rangle \, dx + \int_{\partial\Omega} \langle u, F \rangle \, dA = 0$$

whenever $e(u) \equiv 0$

- $e(u) \equiv 0$ iff $u(x) = \sum_i c_{ij} x_j + d_i$ with $c_{ij} + d_i$ constant + c_{ij} skew-symmetric (such u is an "infinitesimal rigid motion")

• Korn's inequality:

$$\int_{\Omega} |u|^2 \leq C_{\Omega} \int_{\Omega} |\operatorname{er}u|^2$$

provided either

a) $u = 0$ on $\partial\Omega$ ("Korn's 1st case")

or

b) $\int_{\Omega} u \, dx = 0$ and $\int_{\Omega} \partial_i u_j$ is a symmetric matrix

("Korn's 2nd inequality").

Note: all these statements are analogues for elasticity of results that are standard & well-known in setting of Laplace's eqn.