Mechanics - Lecture 7, 3/7/2018

[Start 3/7 by finishing the "Lecture 6" notes.]

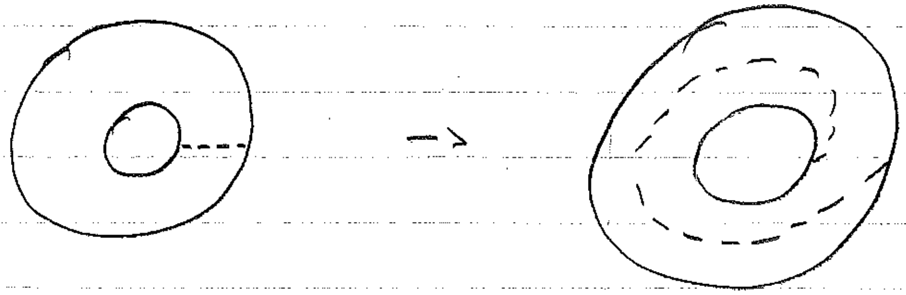
[Note: last pg of Lecture 6 notes uses term "Korn's inequality" for estimate

$$(*) \quad \int_\Omega |u|^2 \leq C_\Omega \int_\Omega |e(u)|^2$$

when either $u=0$ at $\partial\Omega$ or $\int_\Omega u\,dx = 0$ and $\int_\Omega \partial_i u_j$ is symmetric. These results are true, but really they are consequences of Korn's inequality. See later in these notes.]

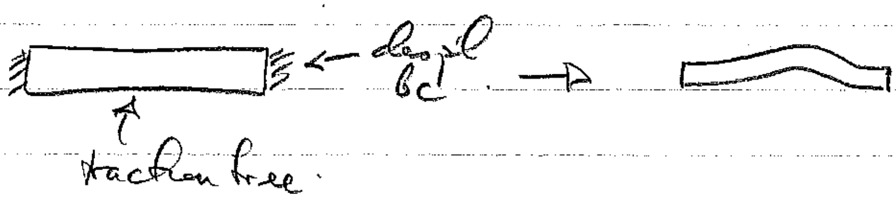
1st topic today: uniqueness.

- In nonlinear elasticity uniqueness is false, even when displacement is specified on entire bdry. Classical thought-expt due to F. John: 2D annulus. Imagine twisting inner circle by 2π. Should get soln to bvp with $x(X) = X$ on entire bdry, but $x(X) \neq X$ inside (see figure).



By contrast, we'll show that soln of linear elasticity with displacement bc is unique.

- In nonlinear elasticity, buckling demonstrates nonuniqueness for pbm assoc loading of a beam or column, where bc has displ type on part of bdry + traction-free on rest.

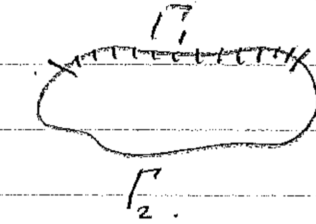


By contrast, we'll show that soln of lin elasticity is unique if displ is specified on any part of bdry.

- Recall simple pf of uniqueness for $\Delta u = f$ in Ω, $u = u_0$ at $\partial\Omega$: if $\exists$ two solns, then difference solves $\Delta u = 0$ in Ω, $u = 0$ at $\partial\Omega$. Multiply by u + integrate by pts $0 = \int_\Omega u \cdot \Delta u = -\int_\Omega |\nabla u|^2 \Rightarrow \nabla u = 0 \Rightarrow u = \text{const} \Rightarrow u = 0$ (by bc).

Similar arg't works for elasticity: consider bvp

$$\operatorname{div}(\alpha e(u)) = f \quad \text{in } \Omega$$
$$u = u_0 \quad \text{on } \Gamma_1$$
$$[\alpha e(u)] \cdot n = g \quad \text{on } \Gamma_2$$



where $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Soln is unique.

Pf: need only show $f = g = 0$, $u_0 = 0 \Rightarrow u \equiv 0$. Argue as for Laplace: with $\sigma = \alpha e(u)$,

$$0 = \int_\Omega \langle u, \operatorname{div}\sigma\rangle = -\int_\Omega \langle \nabla u, \sigma\rangle \quad \text{by Green's thm + bc}$$

$$= -\int \langle e(u), \sigma\rangle \quad \text{since } \sigma \text{ is symmetric}$$

$$= -\int \langle \alpha e(u), e(u)\rangle \quad \text{defn of } \sigma$$

$$\Rightarrow \quad e(u) \equiv 0 \qquad \text{since } \alpha \text{ is pos def.}$$

Now need Lemma: if $e(u) \equiv 0$ in connected region Ω then u is an "inf'l rigid motion" ie

$$u(x) = \sum' \omega_{ij} x_j + d_i$$

for some (constant) skew-symmetric ω_{ij} + some const d_i.

(Note: this is linear analogue of stmt that $F^TF \equiv I \Rightarrow x(X)$ is locally a rigid motion.)

Proof is easy in $\mathbb{R}^2$: $u_{1,1} \equiv 0,\ u_{2,2} \equiv 0 \Rightarrow u_1 = f(x_2)$
$u_2 = f(x_1)$

$u_{1,2} + u_{2,1} = 0 \Rightarrow f'(x_2) + g'(x_1) = 0$
$\Rightarrow f = \omega x_2 + \text{const}$
$g = -\omega x_1 + \text{const}$

Proof in $\mathbb{R}^3$ (or $\mathbb{R}^n$, $n \geq 3$) can be done similarly, by induction on dimn. Or, here's another (less intuitive) argt: observe that

$$\partial_{jk} u_i = \partial_j e_{ik} + \partial_k e_{ij} - \partial_i e_{jk}$$

Therefore $e(u) \equiv 0 \Rightarrow \nabla\nabla u \equiv 0 \Rightarrow u$ is linear in x. Since $e(u) = 0$, Du is skew-symmetric.

Wrap up pf of uniqueness: we were assuming $\operatorname{div} \sigma = 0$ in Ω, $u = 0$ at Γ_1, $\sigma \cdot n = 0$ at Γ_2.
We concluded u = inf'l rigid motion.
If $\Gamma_1 \neq \emptyset$ this forces $u = 0$.

What about pure traction pbm? Situation

is like Neumann pblm for Laplace eqn.
Recall: for $\Delta u = f$ in Ω, $\frac{\partial u}{\partial n} = g$ at $\partial\Omega$ we have a <u>consistency condn</u>: $\int_\Omega f = \int_{\partial\Omega} g$; when consistency holds, soln is unique up to a constant.

<u>Similar situation for lin elasticity</u>: traction pblm

$$-\mathrm{div}\,\sigma = f, \quad \sigma = \alpha e(u) \quad \text{in } \Omega$$
$$\sigma \cdot n = g \quad \text{at } \partial\Omega$$

can have soln only if

$$\int_{\partial\Omega} \langle g, \hat{u} \rangle \, ds + \int_\Omega \langle f, \hat{u} \rangle \, dx = 0 \quad \text{whenever } \hat{u} \text{ is an inf'l rigid motion.}$$

If it does have a soln, that soln is unique up to addition of an inf'l rigid motion.

<u>Pf of consistency</u>: if $e(\hat{u}) \equiv 0$ then

$$\int_\Omega \langle \hat{u}, f \rangle = -\int_\Omega \langle \hat{u}, \mathrm{div}\,\sigma \rangle = \cancel{+\int_\Omega \langle e(\hat{u}), \sigma \rangle}^{\,0} - \int_{\partial\Omega} \langle \hat{u}, \underbrace{\sigma \cdot n}_{g} \rangle.$$

Pf of uniqueness: same arg't as before (except now $\Gamma_1 = \phi$ so uniqueness is only true up to inf'l rigid motion)

What about existence? Again, sol'n is a lot like scalar Laplace eqn, or div-form scalar eqn $\nabla\cdot(\sigma(x)\nabla u) = f$. Main techniques

① var'l principles
② Lax-Milgram lemma
> very closely connected!

③ bdry integral techniques. (different)

Bdry integral methods are basically restricted to constant-coeff't setting (won't discuss them here).

Var'l prin + Lax-Milgram are simple + general; also form basis of most numerical schemes (eg finite elements), we'll focus on former.

Again, use scalar Laplace as guide. Sol'n to

$$\Delta u = f \text{ in } \Omega, \quad u = u_0 \text{ on part of } \partial\Omega\ (\Gamma_1)$$
$$\partial u/\partial n = g \text{ on rest of } \partial\Omega\ (\Gamma_2)$$

can be found using var'l prin

$$(*) \quad \min_{u=u_0 \text{ on } \Gamma_1} \int_\Omega \tfrac{1}{2}|\nabla u|^2 + fu \, dx - \int_{\Gamma_2} ug \, ds$$

Note: if $\Gamma_1 = \phi$ and data are inconsistent, then functional is unbdd below; we can drive it to $-\infty$ by taking u = suitable constant.

Existence via var'l prin uses convexity of fnl, plus lemma that it's bdd below. Key to latter is a pair of Poincaré-type ineq's

easy Poincaré ineq : $\int_\Omega |u|^2 \leq C \int_\Omega |\nabla u|^2$

provided $u = 0$ at $\partial\Omega$

hard Poincaré ineq : $\int_\Omega |u-\bar{u}|^2 \leq C \int_\Omega |\nabla u|^2$

where $\bar{u}$ = avg of u on Ω.

They assure us that although "energy" controls only $\int|\nabla u|^2$ directly, it also controls $\int|u|^2$ indirectly.

Situation for elasticity is just the same. Var'l prin is

$$\min_{u=u_0 \text{ on } \Gamma_1} \int_\Omega \tfrac{1}{2}\langle \alpha e(u), e(u)\rangle + \langle f, u\rangle \, dx - \int_{\Gamma_2} \langle u, g\rangle \, ds$$

Analogue of Poincaré ineq is consequence of Korn's $\neq$:

easy version: $\int_\Omega |\nabla u|^2 \le C \int_\Omega |e(u)|^2$

provided $u = 0$ at $\partial\Omega$

harder version: $\int_\Omega |\nabla u|^2 \le C \int_\Omega |e(u)|^2$

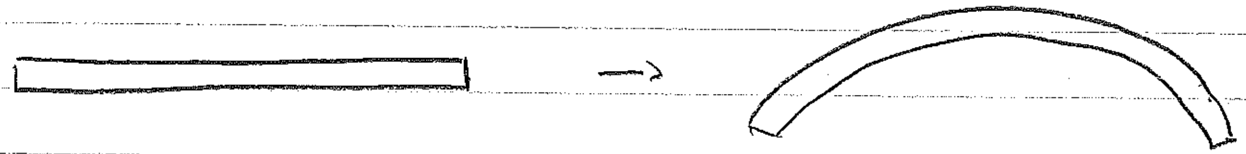
provided $\int_\Omega \nabla_i u^j$ is symmetric matrix.

They assure that although "energy" controls only $\int |e(u)|^2$ directly, it controls $\int |\nabla u|^2$ indirectly (and therefore also $\int |u|^2$ indirectly).

Same intuition on (hard) Korn ineq: it clearly implies

$$\int_\Omega |u - \hat{u}|^2 \le C \int_\Omega |e(u)|^2 \quad \text{for some inf'l rigid motion}$$

which is linear analogue of estimate (true, but much harder) that [small nonlin strain] => [close to rigid rotation]. Const depends on domain, of course, and long, thin domains $\Rightarrow$ very large constants



locally close to a rigid rotation,
But not globally!

"Easy Korn ineq" can be proved by an elementary integr by parts, or by an easy Fourier-transform-based argument. Here is the former:

for $u \in C_0^\infty(\Omega)$ (with Ω bdd),

$$\int_\Omega |e(u)|^2 = \int_\Omega \sum \left(\frac{\nabla_i u^j + \nabla_j u^i}{2}\right)^2$$

$$= \int_\Omega \tfrac{1}{2}|\nabla u|^2 + \int_\Omega \tfrac{1}{2} \sum \nabla_j u^i \nabla_i u^j$$

But since $u = 0$ near $\partial\Omega$,

$$\int_\Omega \nabla_j u^i \nabla_i u^j = \int_\Omega \nabla_i u^i \nabla_j u^j$$

So

$$\int_\Omega |e(u)|^2 = \tfrac{1}{2}\int |\nabla u|^2 + \tfrac{1}{2}\int |\mathrm{div}\, u|^2$$

Thus when $u=0$ at $\partial\Omega$ Korn's ineq with const C says

$$\int_\Omega |\nabla u|^2 \le C\left(\int_\Omega \tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}(\operatorname{div} u)^2 \right)$$

Evidently true with $C=2$.

"Hard Korn ineq" has interesting history:

- 1st "proof" by Korn, about 1910
- Friedrichs wrote a paper abt 1947 pointing out importance of this inequality, giving a new (still not simple) proof, + 1st "modern" proof of existence of solns (by Hilbert-space methods)
- Many proofs since Friedrichs. Some are very efficient but not so elementary (see eg Duvaut + Lions book). Also much study of other "coerciveness inequalities": when does control of selected combinations of $\partial_i u^j$ yield control of all derivatives individually (KT Smith, Aronszajn, others — 60's + 70's,

using pseudodifferential operators).

- A really simple, elementary proof was finally given by Oleinik + Kondratiev about 1989 (CRAS Paris Ser I, 1989, 483-487; also Rend. Mat. Appl. (7) 10, 1990, no 3, 641-666). Separate

I promised a discussion of existence via variational principle. Actually let's do a little less (I won't actually prove existence — though those who know the var'l pf of existence for Laplace's eqn will see what to do) and a little more (I'll discuss convergence of a typical finite element method calculation). Focus on:

linear elasticity in Ω (no body load)
$u = 0$ on part of $\partial\Omega$ (call it Γ_1)
$\sigma \cdot n = f$ on rest of $\partial\Omega$ (call it Γ_2).

for which var'l prin is

$$\min_{u=0 \text{ on } \Gamma_1} \int_\Omega \tfrac{1}{2}\langle \alpha e(u), e(u)\rangle - \int_{\Gamma_2} \langle u, f\rangle$$

Typical numerical method: minimize exactly the same functional in a finite dim'l space of functions V (chosen so that $u|_{\Gamma_1} = 0$ whenever $u \in V$). For example: V could consist of fns that are piecewise linear + cont's on a fixed triangulation of Ω.

Let $E[u] = \int_\Omega \frac{1}{2} \langle \alpha e(u), e(u) \rangle - \int_{\Gamma_2} \langle u, f \rangle$

+ let u_* = minimizer (i.e. actual soln of elasticity plm). Since E is quadratic + linear, Taylor exp. around u_* keeping only quadratic terms is exact. So for any v,

$$E(v) = E(u_*) + \boxed{\int_\Omega \langle \alpha e(u_*), e(v) - e(u_*) \rangle \, dx - \int_{\Gamma_2} \langle v - u_*, f \rangle \, ds}$$

1st var $\longrightarrow$

$$+ \frac{1}{2} \int_\Omega \langle \alpha (e(v) - e(u_*)), e(v) - e(u_*) \rangle$$

2nd var $\longrightarrow$

The eqns of elasticity (the EL eqns for E at u_*) assure that boxed terms (the 1st var) vanish, so

$$(*) \qquad \frac{1}{2}\int_\Omega \langle \alpha\, e(v-u_*),\, e(v-u_*)\rangle = E[v] - E[u_*].$$

If u_* can be approx well in the subspace V then RHS will be small at the best $v \in V$

Claim: if RHS is small, then v is close to u_* in H^1. This follows immediately from $(*)$ using positivity of α and the Korn ineq

$$(**) \qquad \int_\Omega |\nabla w|^2 \le C \int_\Omega |e(w)|^2 \quad \text{if } w=0 \text{ on } \Gamma_1.$$

Explanation of $(**)$: well, by the "basic" Korn ineq

$$\int_\Omega |\nabla w - \gamma|^2 \le C \int_\Omega |e(w)|^2$$

for some skew-symmetric, constant matrix γ. A Poincaré-type ineq then gives

$$\int_\Omega |w - (\gamma\cdot x + d)|^2 \le C \int_\Omega |e(w)|^2$$

and a standard "trace" theorem for $H^1(\Omega)$ gives

$$\int_{\Gamma_1} |w - (\gamma\cdot x + d)|^2 ds \le C \int_\Omega |e(w)|^2.$$

But $w = 0$ on Γ_1, and the ineq

$$(\gamma, d) \to \left(\int_{\Gamma_1} |\gamma \cdot x + d|^2 ds \right)^{1/2}$$

is a norm on the finite-dimensional space of all γ's + d's. So $\|\gamma\| + |d| \leq C' \int_\Omega |e(w)|^2$. Thus finally

$$\int_\Omega |\nabla w|^2 \leq C'' \int_\Omega |e(w)|^2$$

by triangle inequality.