Mechanics - 3/4/08 - Handout on Korn's 2^{nd} ineq.

On Korn's 2^{nd} $\neq$, namely:

(1) $\quad \int_\Omega \nabla_i u_j$ symmetric $\forall i,j \Rightarrow \int_\Omega |\nabla u|^2 \le C \int_\Omega |e(u)|^2$

Stmt is true for any "Lipschitz domain" ($\partial\Omega$ locally a Lipschitz graph, with Ω lying to one side). The following proof is essentially Oleinik + Kondratiev's CRAS note, but simplified slightly by hypothesis that $\partial\Omega$ is C^1 rather than Lipschitz.

Step 1: Observe it suffices to prove

(2) $\quad \int_\Omega |\nabla u|^2 \le C \left[\int_\Omega |e(u)|^2 + \int_\Omega |u|^2 \right]$

ie to show $H^1(\Omega)$ norm is equiv to $\left(\int_\Omega |e(u)|^2 + |u|^2 \right)^{1/2}$.
[Proof that (2) $\Rightarrow$ (1) can be used for many analogous pbms.]

Argue by contradiction: if (2) holds but (1) fails then $\exists$ seq. u^n st

$$\int_\Omega u_i^n = 0, \quad \int \nabla_i u_j^n \text{ symmetric}$$

$$\int_\Omega |\nabla u^n|^2 = 1, \quad \int |e(u^n)|^2 \to 0 \text{ as } n \to \infty.$$

Sequence is bdd in H^1 (here we use the ineq. $\int_\Omega |u-\bar{u}|^2 \le C \int_\Omega |\nabla u|^2$) so $\exists$ subseq. converging wkly in H^1 and <u>strongly</u> in L^2 to a limit u^*. May suppose wlog that whole seq. converges to u^*.

Evidently $e(u^*) = 0$. Applying (2) to $u^n - u^*$ gives

$$\int_\Omega |\nabla(u^n - u^*)|^2 \to 0$$

so actually $u^n \to u^*$ <u>strongly</u> in H^1. In particular

$$\int_\Omega |\nabla u^*|^2 = \lim_{n\to\infty} \int_\Omega |\nabla u^n|^2 = 1 .$$

But

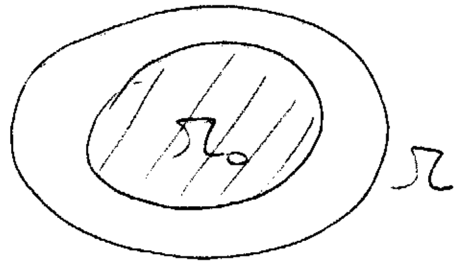
$$\int_\Omega \nabla_i u_j^* = \lim_{n\to\infty} \int_\Omega \nabla_i u_j^n \quad \text{is symmetric}$$

$$\int_\Omega u_i^* = \lim_{n\to\infty} \int_\Omega u_i^n = 0$$

and the last two relns, with $e(u^*) = 0$, imply $u^* \equiv 0$. This contradicts $\int_\Omega |\nabla u^*|^2 = 1$. So (1) holds after all.

<u>Step 2</u>: Observe it suffices to show, for some $\Omega_0 \subset\subset \Omega$

$$(3) \qquad \int_\Omega |\nabla u|^2 \le C \left[\int_\Omega |e(u)|^2 + \int_{\Omega_0} |\nabla u|^2 \right]$$



Indeed, suppose (3) is known. Then choose φ st

$$\varphi \equiv 1 \quad \text{on } \Omega_0$$
$$\varphi = 0 \quad \text{near } \partial\Omega$$

and apply Korn's 1st $\neq$ to $u\varphi$:

$$\int_{\Omega_0} |\nabla u|^2 \leq \int_{\Omega} |\nabla(u\varphi)|^2$$
$$\leq C \int_{\Omega} |e(u\varphi)|^2$$
$$\leq C \left[\int_{\Omega} |e(u)|^2 + |u|^2 \right]$$

<u>Step 3</u>: Preparing to prove (3), we seek a decomposition $u = w + (u - w)$ s.t.

- $\displaystyle\int_{\Omega} |\nabla w|^2 \leq C \int_{\Omega} |e(u)|^2$, so w is not problematic

- $\Delta(u_i - w_i) = 0$ on Ω, so we can use PDE to estimate $u - w$.

Main tool is the identity

$$(*) \qquad \frac{\partial^2 v_i}{\partial x_p \partial x_q} = \frac{1}{2}\left[\frac{\partial}{\partial x_p} e_{iq}(v) + \frac{\partial}{\partial x_q} e_{ip}(v) - \frac{\partial}{\partial x_i} e_{pq}(v)\right]$$

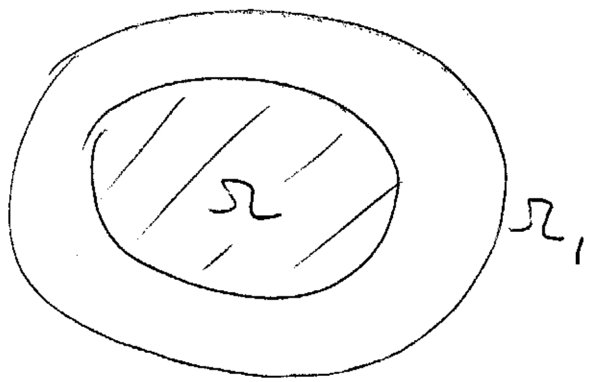
which gives

$$\Delta u_i = \sum_p{}' \frac{\partial}{\partial x_p} e_{ip}(u) - \frac{1}{2}\frac{\partial}{\partial x_i} e_{pp}(u) .$$

So a convenient construction is

$$\begin{aligned} \Delta w &= \operatorname{div} \tilde{\sigma} && \text{in } \Omega_1 \\ w &= 0 && \text{at } \partial\Omega_1 \end{aligned}$$

where $\Omega \subset\subset \Omega_1$



and

$$\tilde{\sigma}_{ij} = \begin{cases} e_{ij} - \frac{1}{2}(\operatorname{tr} e)\,\delta_{ij} & \text{in } \Omega \\ 0 & \text{in } \Omega_1 \setminus \Omega \end{cases}$$

This works since

$$\int_{\Omega_1} |\nabla w|^2 = -\int_{\Omega_1} w \operatorname{div} \tilde{\sigma} = \int_{\Omega_1} \langle \nabla w, \tilde{\sigma} \rangle \leq \left(\int_{\Omega_1} |\nabla w|^2\right)^{1/2} \left(\int_{\Omega} |\sigma|^2\right)^{1/2}$$

whence

$$\int_\Omega |\nabla w|^2 \le \int_{\Omega_1} |\nabla w|^2 \le \int_\Omega |\sigma|^2 \le C \int_\Omega |e(u)|^2$$

Step 4: Estimate $z = u - W$, taking advantage of fact that $\Delta z = 0$ in Ω (though we have no useful bc at $\partial\Omega$). We do this using

Lemma A: Let $\rho(x) = \text{dist}(x, \partial\Omega)$ and suppose $\Delta v = 0$ in Ω. Then

$$\int_\Omega \rho^2 |\nabla v|^2 \le C \int_\Omega v^2$$

Lemma B: Choosing $\Omega_0 \subset\subset \Omega$ conveniently, every f satisfies

$$\int_{\Omega \setminus \Omega_0} f^2 \le C \left[\int_\Omega \rho^2 |\nabla f|^2 + \int_{\Omega_0} f^2 \right]$$

(Proofs postponed to step 5).

Applying Lemma A to $v = e_{ij}(z)$ (note: $\Delta z_i = 0 \ \forall i \Rightarrow \Delta e_{ij}(z) = 0$) gives

$$\int_\Omega \rho^2 |\nabla e(z)|^2 \le C \int_\Omega |e(z)|^2$$

Recall from (4) that $\nabla\nabla z$ is representable in terms of $\nabla e(z)$, so we conclude

(5) $$\int_\Omega \rho^2 |\nabla\nabla z|^2 \le C \int_\Omega |e(z)|^2$$

Now apply Lemma B to $f = \nabla z$, and combine with (5), to see

$$\int_{\Omega\setminus\Omega_0} |\nabla z|^2 \le C\left[\int_\Omega \rho^2|\nabla\nabla z|^2 + \int_{\Omega_0}|\nabla z|^2\right]$$
$$\le C\left[\int_\Omega |e(z)|^2 + \int_{\Omega_0} |\nabla z|^2\right]$$

Thus evidently

(6) $$\int_\Omega |\nabla z|^2 \le C\left[\int_\Omega |e(z)|^2 + \int_{\Omega_0}|\nabla z|^2\right]$$

Now we're essentially done: in fact $|\nabla u|^2 \le 2(|\nabla w|^2 + |\nabla z|^2)$ by triangle ineq., and

$$\int_\Omega |\nabla w|^2 \le C\int_\Omega |e(u)|^2 \qquad \text{by step 3}$$

while

$$\int_\Omega |\nabla z|^2 \le C\left[\int_\Omega |e(u)|^2 + \int_{\Omega_0} |\nabla u|^2\right]$$

by (6) and several easy applns of triangle ineq. (remember: $u = z + w \Rightarrow z = u - w$)

whence $\int_\Omega |\nabla u|^2 \le C\left[\int_\Omega |e(u)|^2 + \int_{\Omega_0} |\nabla u|^2\right]$ as desired.

Step 5. Proofs of Lemmas A+B.

Pf of Lemma A: $\Delta v = 0 \Rightarrow \rho^2 v \Delta v = 0 \Rightarrow$

$$\operatorname{div}(\rho^2 v \nabla v) = 2\rho\nabla\rho \cdot v\nabla v + \rho^2 |\nabla v|^2$$

$$\Rightarrow \int \rho^2 |\nabla v|^2 \le 2\int \rho |\nabla\rho|\, |v| \, |\nabla v|$$

Now $|\nabla\rho| = 1$, and $2ab \le \delta a^2 + \frac{1}{\delta} b^2$, so for any $\delta > 0$ we have

$$\int_\Omega \rho^2 |\nabla v|^2 \le \delta \int_\Omega \rho^2 |\nabla v|^2 + \frac{1}{\delta}\int_\Omega v^2$$

Choosing $\delta < 1$ gives

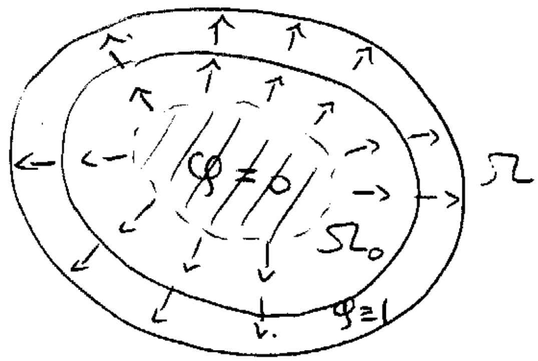
$$\int_\Omega \rho^2 |\nabla v|^2 \le C \int_\Omega v^2$$

as desired.

Pf of Lemma B: (Here is where we simplify Kondratiev-Oleinik by assuming $\partial\Omega$ is smooth enough). Assume $\vec{\xi} = \nabla\rho$ is C^1 in a nbhd of $\partial\Omega$. Choose cutoff fn φ, and subdomain $\Omega_0 \subset\subset \Omega$, s.t.

$\varphi \equiv 1$ on $\Omega \setminus \Omega_0$

$\varphi \equiv 0$ where $\vec{\xi} = \nabla\rho$ is not C^1



We have

$$\operatorname{div}\left(f^2 \rho \varphi \vec{\xi}\right) = 2f(\nabla f \cdot \xi)\varphi\rho + f^2(\nabla\rho\cdot\vec{\xi})\varphi + f^2\rho\varphi \operatorname{div}\vec{\xi} + f^2\rho\nabla\varphi\cdot\vec{\xi}$$

Making Ω_0 larger if necessary, we may suppose that

$$|\rho \operatorname{div}\vec{\xi}| \le 1/2 \quad \text{on } \Omega\setminus\Omega_0$$

and we know of course $\nabla\rho\cdot\vec{\xi} \equiv 1$ & $\varphi \equiv 1$ on $\Omega\setminus\Omega_0$.
So we get

$$\frac{1}{2}\int_{\Omega\setminus\Omega_0} f^2 \le \int_{\Omega} 2f(\nabla f\cdot\xi)\rho\varphi + \int_{\Omega} f^2\rho\nabla\varphi\cdot\vec{\xi} + \int_{\Omega_0} f^2\left(\rho\varphi \operatorname{div}\xi + (\nabla\rho\cdot\xi)\varphi\right)$$

$$\leq \delta \int_{\Omega} f^2 + \frac{1}{\delta} \int_{\Omega} \rho^2 |\nabla f|^2 + C \int_{\Omega_0} f^2$$

Choosing δ small again we get

$$\int_{\Omega \setminus \Omega_0} f^2 \leq C \left[\int_{\Omega} \rho^2 |\nabla f|^2 + \int_{\Omega_0} f^2 \right]$$

(with a new choice of C), as desired.