MECHANICS – Problem Set 2, distributed 2/7/18, due 2/21/2018

(1) **The bending stiffness of xerox paper.** Recall our discussion of "the xerox paper problem" from Lecture 2: consider a standard 8.5×11 sheet of paper, held at one edge so the tangent there is vertical. We showed that if $r(s) = (\cos\theta(s), \sin\theta(s), 0)$ describes its profile then

$$A\theta'' + f_0 s \cos\theta(s) = 0$$

on $0 < s < L$, with boundary conditions

$$\theta'(0) = 0, \quad \theta(L) = -\pi/2,$$

where $s = 0$ corresponds to the free edge and $s = L$ corresponds to the edge being held. Here L = 11 inches is the length of the paper, f_0 is the gravitational constant (i.e. $(0, -f_0, 0)$ is the force per unit length due to gravity), and A is the bending stiffness of the paper (i.e. the relation between bending moment and curvature is $m_3 = A\theta'$).

Clearly the profile depends only on the ratio A/f_0. *Estimate the value of this ratio for a standard sheet of paper.* There is more than one way to approach this. You could (but you don't have to) proceed as follows:

(a) Using Matlab, you can solve the ODE $\theta'' + s\cos\theta(s) = 0$ for $s > 0$, with "initial condition" $\theta(0) = \theta_0, \theta'(0) = 0$, for various choices of $\theta_0 > 0$. It is clear from the equation that $\theta'' < 0$, so $\theta(s)$ decreases. Eventually – say, at $s = S(\theta_0)$ – it reaches $\theta(s) = -\pi/2$.

(b) Our paper has a known length L. So consider

$$\tilde{\theta}(\tilde{s}) = \theta\left(\frac{S}{L}\tilde{s}\right)$$

where $S = S(\theta_0)$. It has the desired boundary conditions

$$\tilde{\theta}'(0) = 0, \quad \tilde{\theta}(L) = -\pi/2,$$

and it solves the equation

$$\left(\frac{L}{S}\right)^3 \tilde{\theta}'' + \tilde{s}\cos\tilde{\theta}(\tilde{s}) = 0.$$

Thus it solves our PDE with A/f_0 replaced by $(L/S)^3$. The profile of the sheet of paper with this choice of A/f_0 is obtained by integrating (using Matlab again) the ODE

$$r_s = (\cos\tilde{\theta}(s), \sin\tilde{\theta}(s)), \quad 0 \le s \le L.$$

(c) Plot the profiles you get from part (b), for various values of θ_0. About what should θ_0 be to get something that resembles the profile of the xerox paper? What do you conclude about A/f_0? (I don't expect an exact answer, just a ballpark estimate.)

(2) **A variational perspective on bifurcation of the elastica.** Recall from the Lecture 2 notes that equilibrium configurations of the elastica (with length 1 and the physical constant A set to 1) are critical points of the functional

$$E[\theta] = \int_0^1 \frac{1}{2}\theta_s^2 + \lambda \cos\theta \, ds,$$

and that (to leading order) the bifurcation diagram is described by $\theta(s) = g\phi(s)$ with

$$\lambda - \lambda_1 = \frac{\pi^2}{32} g^2 \tag{1}$$

where $\phi(s) = \sin\left(\frac{\pi}{2}s\right)$ and $\lambda_1 = \pi^2/4$. Give another "derivation" of (1) by (i) assuming that $\theta(s) = g\phi(s)$ for some g, (ii) estimating $E[\theta]$ as a function of g, using the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$, then (iii) considering the condition that g be a critical point of the resulting expression. (I put "derivation" in quotes, because a proper explanation why it's sufficient to consider $\theta = g\phi$ requires the arguments of the Lecture 2 notes.)

(3) **Bifurcation of an imperfect elastica.** Consider an imperfect elastica, with (constant) intrinsic curvature δ. This means the constitutive law is $m_3 = A(\theta' - \delta)$. We take the length to be 1, and the boundary conditions to be the same as considered in Lecture 2: the left side ($s = 0$) is clamped in a horizontal position, while the right side ($s = 1$) is loaded horizontally. For simplicity we set $A = 1$.

(a) Show that the associated boundary value problem is

$$\theta'' + \lambda \sin\theta = 0, \quad \theta(0) = 0, \; \theta'(1) = \delta.$$

(b) Show that solutions of this boundary-value problem are critical points of

$$E = \int_0^1 \tfrac{1}{2}(\theta' - \delta)^2 + \lambda \cos\theta \, ds$$

subject to boundary condition $\theta(0) = 0$. (Note that I have not imposed $\theta'(1) = \delta$; you must explain why a critical point satisfies this "natural boundary condition.")

(c) Consider the associated linear problem

$$u'' + \lambda_0 u = f, \quad u(0) = 0, \; u'(1) = g$$

with $\lambda_0 = \pi^2/4$. Show that for a solution to exist, the data must satisfy $\int_0^1 f(s)\phi(s)\, ds = g$ with $\phi(s) = \sin(\frac{\pi}{2}s)$. [More is true: when this condition holds a solution exists, and is unique up to an additive multiple of $\phi(s)$. You'll need this in part (d); I'm not asking you to prove it, but if you've taken PDE then you should know how to give a proof.]

(d) Seek a formal solution for the configuration of the buckled structure by means of a perturbation expansion

$$\begin{aligned}
\theta &= 0 + \epsilon\theta^{(1)} + \epsilon^2\theta^{(2)} + \dots \\
\delta &= 0 + \epsilon\delta^{(1)} + \epsilon^2\delta^{(2)} + \dots \\
\lambda &= \pi^2/4 + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \dots
\end{aligned}$$

Reconcile your answer with your physical intuition about which way the elastica should buckle (depending on the sign of δ).

(e) Liapunov-Schmidt reduction says that the equilibrium equation can be expressed in the form

$$f(x, \mu; \delta) = 0$$

with the notation

$$\begin{aligned} \theta &= x\phi + \tilde{\theta}, \quad \tilde{\theta} \perp \phi \\ \mu &= \lambda - \pi^2/4. \end{aligned}$$

Show that your answer to (d) is consistent with f having a Taylor expansion near 0 of the form

$$f(x, \mu; \delta) \approx x^3 + c_1 \mu x + c_2 \delta$$

for suitable choices of the constants c_1 and c_2.

(f) Give a variational perspective on this problem, analogous to the one requested in Problem 2 for the case $\delta = 0$.