MECHANICS – Problem Set 3, distributed 2/21/18, due 3/21/18. *I'm allowing longer than usual, since this problem set is relatively long and 3/14 is spring break.*

These problems provide practice with basic concepts of 3D nonlinear elasticity, and explore various reductions including (i) incompressible fluid dynamics, (ii) elastic membranes, and (iii) balloons.

(1) A homogeneous *elastic fluid* is a hyperelastic material with an energy function $W(F) = h(\det F)$. Show that the Cauchy stress is then $\tau = -p(\rho)I$, where $p(\rho) = -h'(\rho_R/\rho)$. [Here ρ_R is the density in Lagrangian, assumed constant, and ρ is the density in Eulerian variables.] Show that in this case the equations of elastodynamics are precisely the compressible Euler equations

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla p(\rho) + \rho f$$

$$\frac{\partial \rho}{\partial t} + \sum \frac{\partial}{\partial x_i}(\rho v_i) = 0 \ .$$

[Note: to calculate $\partial W/\partial F_{i\alpha}$ when $W(F) = h(\det F)$ you'll to use Cramer's Rule, which says that $\frac{\partial(\det F)}{\partial F} = (\det F)(F^T)^{-1}$.]

(2) Consider a hyperelastic material, whose Piola-Kirchhoff stress tensor is given by $P_{i\alpha} = \partial W/\partial F^i_\alpha$. Show that if W is frame-indifferent (i.e. if $W(F) = W(RF)$ for all orientation-preserving rotations R) then the associated Cauchy stress τ satisfies $\tau(RF) = R\tau(F)R^T$.

(3) Consider a homogeneous, isotropic, hyperelastic material with energy function $W(F) = \psi(I_1, I_2, I_3)$, where I_1, I_2, I_3 are the elementary symmetric functions of $B = FF^T$ ($I_1 = \operatorname{tr} B$, $I_2 = \frac{1}{2}[(\operatorname{tr} B)^2 - tr(B^2)]$, $I_3 = \det B$). Show that the associated Cauchy stress has the form $\tau = \phi_0 I + \phi_1 B + \phi_2 B^2$ with

$$\begin{aligned}
\phi_0 &= 2\frac{\partial \psi}{\partial I_3}\det F \\
\phi_1 &= 2\frac{\partial \psi}{\partial I_1}(\det F)^{-1} + 2\frac{\partial \psi}{\partial I_2}(\operatorname{tr} B)(\det F)^{-1} \\
\phi_3 &= -2\frac{\partial \psi}{\partial I_2}(\det F)^{-1} \ .
\end{aligned}$$

(4) Rubber can be modelled as a homogeneous, isotropic, *incompressible* hyperelastic material. The energy function for such a material has the form $W(F) = \psi(I_1, I_2)$, since all deformations must satisfy the constraint $\det F = 1$. Its Cauchy stress has the form $\tau = -pI + \phi_1 B + \phi_2 B^2$, where ϕ_1, ϕ_2 have the form derived in Problem 3. Let's explore how W can be determined experimentally, using relatively simple experiments on thin membranes.

Consider a sheet (in reference coordinates) of length $2A$, width $2B$, and thickness $2h$, with $A, B \gg h$. Consider deformations of the form

$$x_i = \lambda_i X_i \ , \quad i = 1, 2, 3,$$

which can be maintained by edge tractions alone (i.e. for which the the faces $X_3 = \pm h$ are traction-free). Show that

$$\begin{aligned}
I_1 &= \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} \\
I_2 &= \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \lambda_1^2\lambda_2^2
\end{aligned}$$

and that the Cauchy stress is

$$\begin{aligned}
\tau_{11} &= 2(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2})(\frac{\partial\psi}{\partial I_1} + \lambda_2^2\frac{\partial\psi}{\partial I_2}) \\
\tau_{22} &= 2(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2})(\frac{\psi}{\partial I_1} + \lambda_1^2\frac{\psi}{\partial I_2}) \\
\tau_{33} &= 0 \\
\tau_{ij} &= 0 \quad i \neq j \ .
\end{aligned}$$

Conclude that $\frac{\partial\psi}{\partial I_1}$ and $\frac{\partial\psi}{\partial I_2}$ satisfy

$$\begin{aligned}
\frac{\partial\psi}{\partial I_1} &= \frac{1}{2(\lambda_1^2 - \lambda_2^2)}\left(\frac{\lambda_1^2\tau_{11}}{\lambda_1^2 - 1/\lambda_1^2\lambda_2^2} - \frac{\lambda_2^2\tau_{22}}{\lambda_2^2 - 1/\lambda_1^2\lambda_2^2}\right) \\
\frac{\partial\psi}{\partial I_2} &= \frac{-1}{2(\lambda_1^2 - \lambda_2^2)}\left(\frac{\tau_{11}}{\lambda_1^2 - 1/\lambda_1^2\lambda_2^2} - \frac{\tau_{22}}{\lambda_2^2 - 1/\lambda_1^2\lambda_2^2}\right) \ .
\end{aligned}$$

Thus by measuring the dependence of τ_{11} and τ_{22} on λ_1 and λ_2 one can determine the function ψ.

(5) Consider a spherical rubber balloon (such as you might buy in a toy store). To a reasonable approximation we may:

- consider the reference domain to be a thin spherical annulus $\Omega = \{x \ : \ r_0 - \epsilon < |X| < r_0 + \epsilon\}$;
- consider the air pressure in the balloon to be a constant p;
- ignore the atmospheric pressure outside the balloon;
- consider experiments that are volume-controlled (fixing the volume of the interior of the balloon) or pressure-controlled (fixing the air pressure in the balloon).

From common experience, it is difficult to start blowing up a balloon, but then it gets easier, though eventually as the balloon gets large the blowing gets hard again (unless it bursts). This suggests a pressure-volume relation of the type shown in figure 1 below.

(a) Assume the rubber is hyperelastic and incompressible. Show that variational principle associated with a pressure-controlled experiment involves the energy $E = \int_\Omega W(F)\ dX - p$(volume inside balloon). (In other words, check that this gives the correct equilibrium and boundary conditions.) What variational principle is associated with a volume-controlled experiment?

(b) Consider the limit $\epsilon \to 0$ and assume the deformation is uniform expansion (i.e. the sphere $X = r_0$ is mapped by $x(X) = \lambda X$ to a sphere of radius λr_0). Suppose W has the form $\Phi(\lambda_1, \lambda_2, \lambda_3)$ where λ_1, λ_2, and λ_3 are the principal stretches (eigenvalues of $(F^T F)^{1/2}$). Show that when restricted to the case of "uniform expansion" the pressure-controlled variational principle takes the form $E(\lambda) = c_1 F(\lambda) - c_2 p\lambda^3$ with

$$F(\lambda) = \Phi(\lambda, \lambda, \lambda^{-2}).$$

What are the constants c_1 and c_2?

figure 1

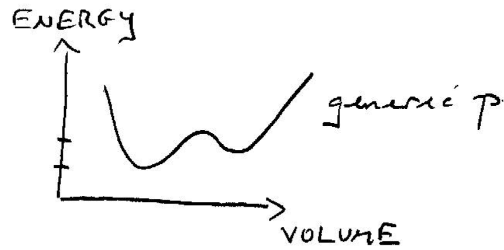

figure 2

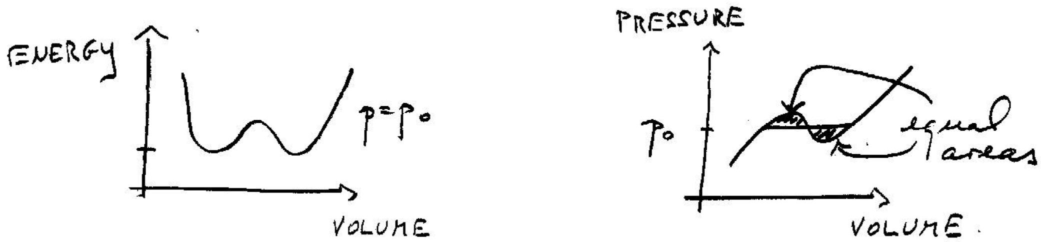

figure 3

(c) Two commonly-used constitutive laws for rubber are the *neo-Hookean* energy

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = a(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$$

with $a > 0$, and the *Mooney-Rivlin* energy

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = a(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + (a/K)(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3)$$

with $a > 0$ and $K > 0$ (typically $4 < K < 8$). Are these laws consistent with the nonmonotone pressure-volume relation shown in figure 1?

(d) Let's think about the 1D energy $E(\lambda)$, using the non-monotonicity of the pressure-volume relation (as shown in Figure 1) but not using any special formula for F (such as those in part b). Evidently, certain values of the pressure p are consistent with 3 different volumes rather than just one. For such p, E must have "double-well" structure, as shown in Figure 2. Show that the two wells have exactly the same depth precisely when $p = p_0$ satisfies the "equal area rule" sketched in Figure 3.

(e) In real pressure-controlled experiments, as p crosses the value p_0, the balloon size changes (relatively suddenly) so that the volume occupies the deeper well (the energetically preferred state). How can this be reconciled with our 1D model?