Wrinkling of thin elastic sheets – Lecture 3: The annulus problem

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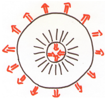
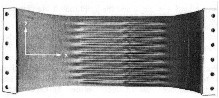
Today's focus

Annulus-shaped sheet, loaded by uniform tension at each boundary.

No wrinkling at larger radii; lots of wrinkling at smaller radii, to avoid compression. Free boundary where wrinkling starts.



Captures essential physics of the "drop on a sheet" (Huang et al, Science 2007) and the "stretched sheet" (Cerda & Mahadevan, PRL 2003)



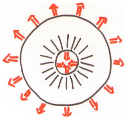
Focus of Davidovitch et al (PNAS 2011) and Bella & Kohn (CPAM 2014).

Key question: understand length scale and character of the wrinkling.

Today's presentation: "matching" upper & lower bounds on elastic energy. (Arguments provide strong hints but little pointwise information.)

Not discussed today: recent work by Bella (ARMA, in press), providing further insight on behavior near radius where wrinkling stops.

The big picture



Main result: excess energy is of order h. In other words, if

$$E_h = (\text{membrane energy}) + h^2(\text{bending energy}) + (\text{work done by loads})$$

then

$$\mathcal{E}_0 + C_1 h \leq \min E_h \leq \mathcal{E}_0 + C_2 h$$

where $\mathcal{E}_0$ is the min of the relaxed problem. Really two assertions:

- upper bound (requires a good ansatz, there's a surprise)
- lower bound (ansatz-free, provides interesting intuition)

Recent article with Peter Bella (Comm Pure Appl Math 67, 2014, 693-747): fully nonlinear treatment (large strains & rotations, general stress-strain law).

Today's discussion: von Karman version with Poisson's ratio 0 (similar ideas but easier & more transparent).

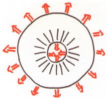
Outline

To be discussed:

- Mathematical formulation
- The relaxed problem
- The lower bound (ansatz-free, provides intuition)
- The upper bound (surprisingly, doesn't match pictures . . .)

sketching the essential ideas. Full details available on PCMI site (approx 7 pages).

Mathematical formulation



$$E_h = (\text{membrane energy}) + h^2(\text{bending energy}) + (\text{work done by loads})$$

$$\text{membrane energy} = \int_{\mathcal{A}} \left|e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3\right|^2 dx$$

$$\text{bending energy} = \int_{\mathcal{A}} |\nabla\nabla u_3|^2 \, dx$$

$$\text{work done by loads} = \int_{|x|=R_{\text{in}}} T_{\text{in}}\, w \cdot \frac{x}{|x|}\, ds - \int_{|x|=R_{\text{out}}} T_{\text{out}}\, w \cdot \frac{x}{|x|}\, ds$$

where $\mathcal{A}$ is the annulus. Parameters are $R_{\text{in}} < R_{\text{out}}$ (geometry), T_{in} and T_{out} (loads), and h (thickness).

Some restrictions are needed to make sure the annulus is wrinkled near R_{in} but not near R_{out}. They turn out to be

$$R_{\text{in}} T_{\text{in}} < R_{\text{out}} T_{\text{out}} \quad \text{and} \quad \frac{T_{\text{in}}}{T_{\text{out}}} > 2\frac{R_{\text{out}}^2}{R_{\text{in}}^2 + R_{\text{out}}^2}$$

Polar coordinates

The radial geometry suggests using polar coordinates, not only in space but also for the elastic deformation. Writing w_r and w_θ for the radial and tangential in-plane displacements, we have

$$\text{membrane} = \int_{\mathcal{A}} \left| \begin{matrix} \partial_r w_r + \frac{1}{2}(\partial_r u_3)^2 & * \\ * & r^{-1}(w_r + \partial_\theta w_\theta) + \frac{1}{2}r^{-2}(\partial_\theta u_3)^2 \end{matrix} \right|^2 r\,dr\,d\theta$$

in which the off-diagonal terms are

$$* = \frac{1}{2}\left(r^{-1}\partial_\theta w_r + \partial_r w_\theta - r^{-1}w_\theta + r^{-1}\partial_r u_3 \partial_\theta u_3 \right);$$

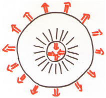
similarly

$$\text{bending} = \int_{\mathcal{A}} (|\partial_{rr} u_3|^2 + 2r^{-2}|\partial_{r\theta} u_3|^2 + r^{-4}|\partial_{\theta\theta} u_3|^2)\, r\,dr\,d\theta$$

and

$$\text{loads} = T_{\text{in}} \int_{r=R_{\text{in}}} w_r\, r\,d\theta - T_{\text{out}} \int_{r=R_{\text{out}}} w_r\, r\,d\theta.$$

The relaxed problem



For relaxed problem, we expect "infinitesimal wrinkling" for $r < L$, and "biaxial stretching" for $r > L$. Sheet has no reason to go out of plane or break radial symmetry. So for relaxed problem, $u_3 = 0$ and $w_\theta = 0$, and w_r depends only on r. Recalling from Lecture 2 that

$$\text{relaxed membrane energy} = (e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3)_+^2$$

we see that w_r minimizes

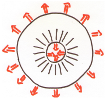
$$\int_{R_{\text{in}}}^{R_{\text{out}}} \left((\partial_r w_r)_+^2 + (r^{-1} w_r)_+^2 \right) r\,dr + T_{\text{in}} R_{\text{in}} w_r(R_{\text{in}}) - T_{\text{out}} R_{\text{out}} w_r(R_{\text{out}}).$$

We expect $\partial_r w_r > 0$ (rays should be in tension). Accepting this, the EL eqn (force balance) is

$$\partial_r(r \partial_r w_r) = r^{-1}(w_r)_+$$

with $2\partial_r w_r(R_{\text{in}}) = T_{\text{in}}$ and $2\partial_r w_r(R_{\text{out}}) = T_{\text{out}}$.

The relaxed problem – cont'd



The behavior in the wrinkled region is quite explicit: since prin strains are $\partial_r w_r > 0$ and $r^{-1} w_r$, edge of wrinkled region (call it $r = L$) is where $w_r = 0$. Within the wrinkled region, EL eqn becomes $\partial_r (r \partial_r w_r) = 0$, so

$$w_r = C \log(r/L) \quad \text{in the wrinkled region.}$$

Notice that

$$\text{compressive strain eliminated by wrinkling} = r^{-1} w_r.$$

It grows linearly as r decreases from L.

Outline

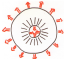
- Mathematical formulation
- The relaxed problem
- The lower bound (ansatz-free, provides intuition)
- The upper bound (surprisingly, doesn't match pictures . . .)

The lower bound – big picture

Lower bound says min $E_h \geq \mathcal{E}_0 + Ch$. Proof must be ansatz-free. Think of $E_h - \mathcal{E}_0$ as the excess energy due to positive h.

Step 1: Soln of $h = 0$ ("relaxed") problem is infinitesimally wrinkled but planar. So out-of-plane deformation costs membrane energy. Quantification: if excess energy is less than δh, then (using only membrane effects),

$$\int_{\mathcal{A}} |u_3|^2 \leq C\delta h$$



Step 2: The excess energy includes all the bending energy. So if u has excess energy less than δh, then

$$\int_{\mathcal{A}} |\nabla\nabla u_3|^2 \leq \delta h^{-1}.$$

Step 3: Use the interpolation inequality

$$\int_{\mathcal{A}} |\nabla u_3|^2 \leq C_1 \left(\int_{\mathcal{A}} |u_3|^2\right)^{1/2} \left(\int_{\mathcal{A}} |\nabla\nabla u_3|^2\right)^{1/2} + C_2 \int_{\mathcal{A}} |u_3|^2$$

to conclude that

$$\int_{\mathcal{A}} |\nabla u_3|^2 \leq C\delta.$$

Conclusion thus far: if δ is small then the deformation is almost planar.

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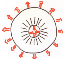

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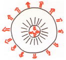

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to conclude that

$$\int_{\mathcal{A}} |\nabla u_3|^2 \leq C\delta.$$

Conclusion thus far: if δ is small then the deformation is almost planar.

The lower bound – big picture

Conclusion of Steps 1-3: if a deformation has excess energy $E_h - \mathcal{E}_0 \leq \delta h$ then it is almost planar, in the sense that $\int |\nabla u_3|^2 < C\delta$.

Step 4: For the unrelaxed energy, a deformation that's almost planar has energy that's much too large (greater than $\mathcal{E}_0$ by an order-one amount).

In fact: small excess energy $\Rightarrow$ close to relaxed solution. So circles assoc $r < L$ are shrunk ($w_r < 0$). For unrelaxed energy, compression costs membrane energy.

So δ can't be small, i.e. $E_h - \mathcal{E}_0 \geq ch$. Lower bound is proved.

The lower bound – more on step 1

In radial coordinates, if $\tilde{w}_r$ is soln of relaxed problem, then the excess energy of (w_r, w_θ, u_3) consists of its bending energy (which is positive) plus

$$\int_{\mathcal{A}} \left(\begin{array}{c} \text{unrelaxed membrane} \\ \text{energy of } (w_r, w_\theta, u_3) \end{array} \right) - \int_{\mathcal{A}} \left(\begin{array}{c} \text{relaxed membrane} \\ \text{energy of } (\tilde{w}_r, 0, 0) \end{array} \right) + \left(\begin{array}{c} \text{difference of} \\ \text{loading terms} \end{array} \right)$$

Claim: This is equal to

$$\int_{\mathcal{A}} (\partial_r \tilde{w}_r)(\partial_r u_3)^2 + (r^{-1}\tilde{w}_r)_+ (r^{-1}\partial_\theta u_3)^2 + \text{sum of perfect squares.}$$

The rest is easy: since $\partial_r \tilde{w}_r > 0$ (strictly) for all r, and $\tilde{w}_r > 0$ (strictly) for $r > (L + R_{\text{out}})/2$,

$$\int_{\mathcal{A}} (\partial_r u_3)^2 + \int_{r>(R_{\text{out}}+L)/2} (\partial_\theta u_3)^2 \leq C \text{ excess energy.}$$

Remembering that u_3 is arbitrary up to a constant (so it should be chosen with mean 0), we get (using a Poincare-type inequality along each ray) that

$$\int_{\mathcal{A}} u_3^2 \leq C \text{ excess energy}$$

as asserted by Step 1.

About the claim

$$\begin{array}{l}\text{unrelaxed membrane}\\ \text{energy of } (w_r, w_\theta, u_3)\end{array} = \left| \begin{matrix} \partial_r w_r + \frac{1}{2}(\partial_r u_3)^2 & * \\ * & r^{-1}(w_r + \partial_\theta w_\theta) + \frac{1}{2}(r^{-1}\partial_\theta u_3)^2 \end{matrix} \right|^2$$

$$\begin{array}{l}\text{relaxed membrane}\\ \text{energy of } (\tilde{w}_r, 0, 0)\end{array} = \left| \begin{matrix} \partial_r \tilde{w}_r & 0 \\ 0 & (r^{-1}\tilde{w}_r)_+ \end{matrix} \right|^2$$

Analogous to our claim, but more familiar: Consider minimizer $\tilde{\phi}$ of $\int_\Omega |\nabla \phi|^2 + \int_{\partial\Omega} \phi f$. For any α and ϕ, the analogue of our "excess energy" is

$$\text{excess} = \left(\int_\Omega |\nabla \phi + \alpha|^2 + \int_{\partial\Omega} \phi f \right) - \left(\int_\Omega |\nabla \tilde{\phi}|^2 + \int_{\partial\Omega} \tilde{\phi} f \right)$$

To estimate it, observe that

$$\begin{aligned} |\nabla \phi + \alpha|^2 &= |\nabla \tilde{\phi} + \nabla(\phi - \tilde{\phi}) + \alpha|^2 \\ &= |\nabla \tilde{\phi}|^2 + 2\langle \nabla \tilde{\phi}, \nabla(\phi - \tilde{\phi})\rangle + 2\langle \nabla \tilde{\phi}, \alpha \rangle + |\nabla(\phi - \tilde{\phi}) + \alpha|^2. \end{aligned}$$

Since $\int_\Omega 2\langle \nabla \tilde{\phi}, \nabla(\phi - \tilde{\phi})\rangle + \int_{\partial\Omega} (\phi - \tilde{\phi}) f = 0$, we get

$$\text{excess} = \int_\Omega 2\langle \nabla \tilde{\phi}, \alpha \rangle + |\nabla(\phi - \tilde{\phi}) + \alpha|^2$$

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- The upper bound (surprisingly, doesn't match pictures . . .)

The upper bound – first pass

Upper bound asserts existence of (w_r, w_θ, u_3) st $E_h \leq \mathcal{E}_0 + Ch$, in other words for which excess energy is of order h.

First idea (unsuccessful!) idea is to wrinkle on scale $h^{1/2}$. For such wrinkles the bending term will be of order $h^2 \cdot (h^{-1/2})^2 \sim h$. Ansatz:

$$\begin{aligned} w_r &= \tilde{w}_r \\ u_3 &= 2\sqrt{2\pi} h^{1/2} (-r\tilde{w}_r)^{1/2} \cos(\theta / h^{1/2}) \quad \text{for } r < L \end{aligned}$$

with $u_3 = 0$ for $r > L$. The tangential displacement w_θ should be chosen st

$$r^{-1} w_r + r^{-1} \partial_\theta w_\theta + \tfrac{1}{2}(r^{-1} \partial_\theta u_3)^2 = 0 \quad \text{pointwise}$$

which is possible since

$$\tfrac{1}{2} \int_0^{2\pi} (r^{-1} \partial_\theta u_3)^2 \, d\theta + 2\pi r^{-1} \tilde{w}_r = 0.$$

This doesn't work: excess energy is of order $h|\log h|$. In fact, ansatz has $|\partial_r u_3| \sim h^{1/2} |L - r|^{-1/2}$ and

$$\text{excess energy} \geq C \int_{\mathcal{A}} (\partial_r u_3)^2 \geq C \int_{r<L} h |L - r|^{-1} \, dr$$

which diverges. Truncation at $r \sim L - h$ still leaves excess $h|\log h|$.

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The upper bound – second pass

A better ansatz uses wrinkling on scale $\ell(r) \sim h^{1/2}(L-r)^{1/2}$ near $r = L$.

How is this possible? Wrinkles can refine dyadically. In Euclidean setting, consider wrinkles wrt y, with $\int |\partial_y u_3|^2\,dy = \frac{1}{2}a^2(x)$. Refinement from scale 2λ at $x = x_0$ to scale λ at $x = x_1$ is achieved by taking

$$u_3 = a(x)\left[f(x)\frac{\lambda}{\pi}\cos\left(\frac{\pi y}{\lambda}\right) + g(x)\frac{\lambda}{2\pi}\cos\left(\frac{2\pi y}{\lambda}\right)\right]$$

with $f^2 + g^2 = 1$, $f \equiv 1$ for $x < x_0$, and $g \equiv 1$ for $x > x_1$. Notice that $|\partial_x u_3| \sim |a_x|\lambda + a\lambda/(x_1 - x_0)$. Since change in local scale $\ell(x)$ satisfies $\Delta\ell/\Delta x \sim \lambda/(x_1 - x_0)$, this scheme achieves

$$|\partial_x u_3| \sim |a_x|\ell + a|\ell_x|.$$

Radial case is similar.

Does it work? Returning to linear setting, we need $\int_0^{2\pi} |\partial_\theta u_3|^2\,d\theta$ to vanish linearly near $r = L$. Use radial analogue of the above, with $a(r) \sim \sqrt{L-r}$. Choosing $\ell \sim h^{1/2}\sqrt{L-r}$ gives $a(r)|\ell'(r)| + |a'(r)|\ell(r) \sim h^{1/2}$, so that

$$\text{excess energy} \sim \int_{r \text{ near } L} |\partial_r u_3|^2 \sim h$$

as desired.

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Does it work? Returning to linear setting, we need $\int_0^{2\pi} |\partial_\theta u_3|^2 \, d\theta$ to vanish linearly near $r = L$. Use radial analogue of the above, with $a(r) \sim \sqrt{L-r}$. Choosing $\ell \sim h^{1/2}\sqrt{L-r}$ gives $a(r)|\ell'(r)| + |a'(r)|\ell(r) \sim h^{1/2}$, so that

$$\text{excess energy} \sim \int_{r \text{ near } L} |\partial_r u_3|^2 \sim h$$

as desired.

The upper bound – second pass

A better ansatz uses wrinkling on scale $\ell(r) \sim h^{1/2}(L-r)^{1/2}$ near $r = L$.

How is this possible? Wrinkles can refine dyadically. In Euclidean setting, consider wrinkles wrt y, with $\int |\partial_y u_3|^2 \, dy = \frac{1}{2}a^2(x)$. Refinement from scale 2λ at $x = x_0$ to scale λ at $x = x_1$ is achieved by taking

$$u_3 = a(x)\left[f(x)\frac{\lambda}{\pi}\cos\left(\frac{\pi y}{\lambda}\right) + g(x)\frac{\lambda}{2\pi}\cos\left(\frac{2\pi y}{\lambda}\right)\right]$$

with $f^2 + g^2 = 1$, $f \equiv 1$ for $x < x_0$, and $g \equiv 1$ for $x > x_1$. Notice that $|\partial_x u_3| \sim |a_x|\lambda + a\lambda/(x_1 - x_0)$. Since change in local scale $\ell(x)$ satisfies $\Delta\ell/\Delta x \sim \lambda/(x_1 - x_0)$, this scheme achieves

$$|\partial_x u_3| \sim |a_x|\ell + a|\ell_x|.$$

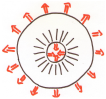
Radial case is similar.

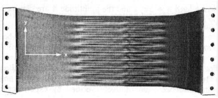
Does it work? Returning to linear setting, we need $\int_0^{2\pi} |\partial_\theta u_3|^2 \, d\theta$ to vanish linearly near $r = L$. Use radial analogue of the above, with $a(r) \sim \sqrt{L-r}$. Choosing $\ell \sim h^{1/2}\sqrt{L-r}$ gives $a(r)|\ell'(r)| + |a'(r)|\ell(r) \sim h^{1/2}$, so that

$$\text{excess energy} \sim \int_{r \text{ near } L} |\partial_r u_3|^2 \sim h$$

as desired.

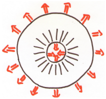
Stepping back

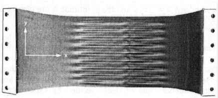




- The annulus example seems to capture the essential physics of the drop-on-sheet and stretched-sheet experiments.
- Its greater symmetry permits exact soln of relaxed problem.
- Ansatz-free lower bound combines
  - strict convexity of relaxed problem in tensile regime
  - an interpolation inequality (bending term enters here).
- Matching upper bound seems to require refinement of wrinkles.
  - Why do we not see this? Well, $|\log h|$ is almost a constant.

Stepping back – cont'd

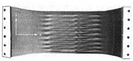




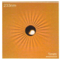
- Is the minimizer similar to our refinement-of-wrinkles ansatz?
  - Our arguments estimate the energy, but say little about ptwise character of minimizer.
  - Actual behavior near $r = L$ is probably rather different!
  - For latest progress, see P. Bella, *Transition between planar and wrinkled regions in a uniaxially stretched thin elastic film* (preprint).

Image credits

Images are from:



E. Cerda and L. Mahadevan, *Phys Rev Lett* 90 (2003) 074302



J. Huang et al, *Science* 317 (2007) 650–653