**PDE for Finance, Spring 2011 – Homework 5**
**Distributed 4/4/11, due 4/18/11.**

Problem 1 is a classic example (due to Merton) of optimal asset allocation. Problems 2-4 reinforce our discussion of optimal stopping and American options.

1) Consider the following asset-allocation problem. Two investment opportunities are available. One is risk-free, earning (constant) interest $r$. The other is lognormal, with (constant) drift $\mu$ and volatility $\sigma$, i.e. it satisfies $dp = \mu p ds + \sigma p dw$. You start at time $t$ by investing wealth $x$. Your control is the weighting of your portofolio between these two assets, i.e.

$$\alpha(s) = \text{fraction of wealth invested in the risky asset at time } s$$

subject to $0 \leq \alpha \leq 1$. You never withdraw from or add to the portfolio, and you have a fixed horizon $T$. Your goal is to maximize the utility of your portfolio value at time $T$; in other words, your value function is

$$u(x,t) = \max_{\alpha(s)} E_{y(t)=x} \left[ h(y(T)) \right]$$

where $y(s)$ is the value of the portfolio at time $s$.

(a) Find the HJB equation satisfied by $u$.

(b) Find the solution – and the optimal investment strategy – if your utility is $h(y) = y^\gamma$ with $0 < \gamma < 1$.

(c) Find the solution – and the optimal investment strategy – if your utility is $h(y) = \log y$.

2) Example 2 of the Section 6 notes discusses when to sell a stock. The goal proposed in the notes was to maximize the discounted wealth realized by the sale, i.e.

$$\max_{\tau} E_{y(0)=x} \left[ e^{-r\tau}(x-a) \right]$$

A different goal would be to maximize the discounted *utility* of wealth realized by the sale, i.e.

$$\max_{\tau} E_{y(0)=x} \left[ e^{-r\tau} h(x-a) \right]$$

where $h$ is your utility.

(a) Consider the utility $h(y) = y^\gamma$ with $0 < \gamma < 1$. (This is concave only for $y > 0$, but that's OK – it would clearly be foolish to sell at a price that realizes a loss.) Find the value function and the optimal strategy.

(b) The example in the notes corresponds to $\gamma = 1$. Using $\gamma < 1$ corresponds to introducing risk-averseness, and decreasing $\gamma$ corresponds to increasing the risk-averseness. How is this reflected in the $\gamma$-dependence of the optimal strategy?

3) In Example 2 of the Section 6 notes we assumed $\mu < r$. Let's explore what happens when $\mu \geq r$. All other conventions of Example 2 remain in effect: the asset price satisfies $dy = \mu y dt + \sigma y dw$ and the value function is $u(x) = \max_\tau E_{y(0)=x}[e^{-r\tau}(y(\tau) - a)]$.

(a) Show that if $\mu > r$ then $u = \infty$.

(b) Show that if $\mu = r$ then $u(x) = x$.

4) For a lognormal underlying with continuous dividend yield $d$, the risk-neutral process is $dy = (r - d)ydt + \sigma ydw$. The value of a perpetual American call with strike $K$ is thus

$$u(x) = \max_{\tau} E_{y(0)=x} \left[ e^{-r\tau} (y(\tau) - K)_+ \right]$$

where $r$ is the risk-free rate.

(a) How is this problem related to Example 2 of the Section 6 notes?

(b) Find the value of this option, and the optimal exercise rule, for $d > 0$.

(c) Show that as $d \to 0$ the value approaches $u(x) = x$.