PDE for Finance Notes, Spring 2014 – Section 2.
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Solution formulas for the linear heat equation. Applications to barrier options.
Section 1 was relatively abstract – we listed many PDE's but solved just a few of them. This section has the opposite character: we discuss explicit solution formulas for the linear heat equation – both the initial value problem in all space and the initial-boundary-value problem in a halfspace. This is no mere academic exercise, because the (constant-volatility, constant-interest-rate) Black-Scholes PDE can be reduced to the linear heat equation. As a result, our analysis provides all the essential ingredients for valuing barrier options. The PDE material here is very standard – most of it can be found e.g. in Guenther & Lee (Section 5.4) or Strauss (Sections 2.4 and 3.1). Our discussion of the half-space problem with initial condition 0 is similar to that in Guenther & Lee (a well-organized presentation of this argument can also be found eg in the PDE book by F. John). For the financial topics (reduction of Black-Scholes to the linear heat equations; valuation of barrier options) see e.g. the "student guide" by Wilmott, Howison, and Dewynne.

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The heat equation and the Black-Scholes PDE. We've seen that linear parabolic equations arise as *backward* Kolmogorov equations, determining the expected values of various payoffs. They also arise as *forward* Kolmogorov equations, determinining the probability distribution of the diffusing state. The simplest special cases are the backward and forward linear heat equations $u_t + \frac{1}{2}\sigma^2 \Delta u = 0$ and $p_s - \frac{1}{2}\sigma^2 \Delta p = 0$, which are the backward and forward Kolmogorov equations for $dy = \sigma dw$, i.e. for Brownian motion scaled by a factor of σ. From a PDE viewpoint the two equations are equivalent, since $v(t,x) = u(T-t,x)$ solves $v_t - \frac{1}{2}\sigma^2 \Delta v = 0$, and final-time data for u at $t = T$ determines initial-time data for v at $t = 0$.

This basic example has direct financial relevance, because the Black-Scholes PDE can be reduced to it by a simple change of variables. Indeed, the Black-Scholes PDE is

$$V_t + rsV_s + \tfrac{1}{2}\sigma^2 s^2 V_{ss} - rV = 0.$$

It is to be solved for $t < T$, with specified final-time data $V(s,T) = \Phi(s)$. (Don't be confused: in the last paragraph s was time, but here it is the "spatial variable" of the Black-Scholes PDE, i.e. the stock price.) We claim this is simply the standard heat equation $u_t = u_{xx}$ written in special variables. To see this, consider the preliminary change of variables $(s,t) \to (x,\tau)$ defined by

$$s = e^x, \quad \tau = \tfrac{1}{2}\sigma^2(T-t),$$

and let $v(x,\tau) = V(s,t)$. An elementary calculation shows that the Black-Scholes equation becomes

$$v_\tau - v_{xx} + (1-k)v_x + kv = 0$$

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with $k = r/(\frac{1}{2}\sigma^2)$. We've done the main part of the job: reduction to a constant-coefficient equation. For the rest, consider $u(x,t)$ defined by

$$v = e^{\alpha x + \beta \tau} u(x, \tau)$$

where α and β are constants. The equation for v becomes an equation for u, namely

$$(\beta u + u_\tau) - (\alpha^2 u + 2\alpha u_x + u_{xx}) + (1 - k)(\alpha u + u_x) + ku = 0.$$

To get an equation without u or u_x we should set

$$\beta - \alpha^2 + (1 - k)\alpha + k = 0, \quad -2\alpha + (1 - k) = 0.$$

These equations are solved by

$$\alpha = \frac{1 - k}{2}, \quad \beta = -\frac{(k + 1)^2}{4}.$$

Thus,

$$u = e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k+1)^2 \tau} v(x, \tau)$$

solves the linear heat equation $u_\tau = u_{xx}$ with initial condition $u(x, 0) = e^{\frac{1}{2}(k-1)x} \Phi(e^x)$.

Remark. The preceding calculation might seem somewhat mysterious (i.e. unmotivated). There is a more intuitive, probabilistic explanation why the Black-Scholes PDE can be reduced by change of variables to the linear heat equation. You'll be asked to work it out in HW2.

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The initial-value problem. Consider the equation

$$f_t = \alpha \Delta f \quad \text{for } x \in R^n, t > 0 \tag{1}$$

with specified data $f(x, 0) = f_0(x)$. Its solution is:

$$f(x, t) = (4\pi\alpha t)^{-n/2} \int e^{-|x-y|^2/4\alpha t} f_0(y)\, dy. \tag{2}$$

Why? Because (1) can be viewed as the forward Kolmogorov equation for $dy = \sigma dw$ when $\alpha = \frac{1}{2}\sigma^2$. Let's take $\alpha = \frac{1}{2}$ for simplicity, so $\sigma = 1$. The probability of a Brownian particle being at x at time t, given that it started at y at time 0, is $(2\pi t)^{-n/2} e^{-|x-y|^2/2t}$. If the initial probability distribution is $f_0(y)$ then the probability of being at x at time t is $(2\pi t)^{-n/2} \int e^{-|x-y|^2/2t} f_0(y)\, dy$, exactly our formula specialized to $\alpha = 1/2$.

We have, in effect, used our knowledge about Brownian motion to write down a specific solution of (1). To know it is the *only* solution, we must prove a uniqueness theorem. We'll address this in the next Section.

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For what class of initial data f_0 is the solution formula (2) applicable? From our probabilistic argument it might appear that f_0 has to be a probability density (positive, with integral 1). In fact however there is no such limitation. It isn't even necessary that f_0 be integrable. To prove that the solution formula works much more generally, one must verify (a) that it solves the equation, and (b) that it has the desired initial value. The proof of (a) is easy, by differentiating under the integral. The proof of (b) is more subtle. Most textbooks present it assuming f_0 is continuous, but the standard argument also works somewhat more generally, e.g. if f_0 is piecewise continuous.

There is however one requirement: the solution formula must make sense. This requires a modest restriction on the growth of f_0 at ∞, to make the integral on the right hand side of (2) converge. For example, if $f_0(x) = \exp(c|x|^2)$ with $c > 0$ then the integral diverges for $t > (4\alpha c)^{-1}$. The natural growth condition is thus

$$|f_0(x)| \le M e^{c|x|^2} \tag{3}$$

as $|x| \to \infty$.

Solutions growing at spatial infinity are uncommon in physics but common in finance, where the heat equation arises by a logarithmic change of variables from the Black-Scholes PDE, as shown above. The payoff of a call is linear in the stock price s as $s \to \infty$. This leads under the change of variable $x = \log s$ to a choice of f_0 which behaves like e^{cx} as $x \to \infty$. Of course this lies well within what is permitted by (3).

What option payoffs are permitted by (3)? Since $x = \log s$, the payoff must grow no faster as $s \to \infty$ than $M \exp\left(c(\log s)^2\right)$. This condition is pretty generous: it permits payoffs growing like any power of s as $s \to \infty$, though it excludes growth like e^s.

Discontinuous initial conditions are relatively uncommon in physics, but common in finance. A digital option, for example, pays a specified value if the stock price at maturity is greater than a specified value, and nothing otherwise. This corresponds to a discontinuous choice of f_0. Notice that even if f_0 is discountinuous, the solution $f(x, t)$ is smooth for $t > 0$. This can be seen by differentiating under the integral in the solution formula.

We have reduced the Black-Scholes equation to the heat equation, and we have given an explicit solution formula for the heat equation. Unraveling all this gives an explicit solution for the Black-Scholes equation. Of course in the end this is just the familiar formula, giving the value of an option as the discounted risk-neutral expected payoff.

It may seem we haven't gained much. And indeed, for vanilla options we haven't. The PDE viewpoint is much more useful, however, for considering barrier options.

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The initial-boundary value problem for a halfspace. Now consider

$$u_t = u_{xx} \quad \text{for } t > 0 \text{ and } x > x_0, \text{ with } u = g \text{ at } t = 0 \text{ and } u = \phi \text{ at } x = x_0. \tag{4}$$

Since this is a boundary-value problem, we must specify data both at the initial time $t = 0$ and at the spatial boundary $x = 0$. We arrived at this type of problem (with t replaced by $T - t$) in our discussion of the backward Kolmogorov equation when we considered a payoff defined at an exit time. The associated option-pricing problems involve barriers. If the option becomes worthless at when the stock price crosses the barrier then $\phi = 0$ (this is a knock-out option). If the option turns into a different instrument when the stock price crosses the barrier then ϕ is the value of that instrument. (When $\phi = 0$, (4) can also be viewed as a forward Kolmogorov equation, for $\sqrt{2}$ times Brownian motion with an absorbing boundary condition at $x = 0$.)

Please notice that if u is to be continuous (up to the boundary, and up to the initial time) then the boundary and initial data must be *compatible*, in other words they must satisfy $g(0) = \phi(0)$. When the data are incompatible, the solution is bound to be singular near $x = t = 0$ even if g and ϕ are individually smooth. The incompatible case is directly relevant to finance. For example, the pricing problem for a down-and-out call has incompatible data if the strike price is below the barrier. Such options become difficult to hedge if, near the time of maturity, the stock price wanders near the barrier.

The pricing problem can be decomposed, by linearity, into two separate problems: $u = v + w$ where v solves

$$v_t = v_{xx} \quad \text{for } t > 0 \text{ and } x > x_0, \text{ with } v = g \text{ at } t = 0 \text{ and } v = 0 \text{ at } x = x_0 \qquad (5)$$

(in other words: v solves the same PDE with the same initial data but boundary data 0) and w solves

$$w_t = w_{xx} \quad \text{for } t > 0 \text{ and } x > x_0, \text{ with } w = 0 \text{ at } t = 0 \text{ and } w = \phi \text{ at } x = x_0 \qquad (6)$$

(in other words: w solves the same PDE with the same boundary data but initial condition 0). Careful though: in making this decomposition we run the risk of replacing a compatible problem with two incompatible ones. As we'll see below, the solution formula for (5) is very robust even in the incompatible case. The formula for (6) is however much less robust. So in practice our decomposition is most useful when $\phi(0) = 0$, so that the w-problem has compatible data. This restriction represents no real loss of generality: if $\phi(0) = c \neq 0$ then our decomposition can be used to represent $u - c$, which solves the same PDE with data $\phi - c$ and $g - c$.

The half-space problem with boundary condition 0. We concentrate for the moment on v. To obtain its solution formula, we consider the whole-space problem with the *odd reflection of g* as initial data. Remembering that $x_0 = 0$, this odd reflection is defined by

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x > 0 \\ -g(-x) & \text{if } x < 0 \end{cases}$$

(see Figure 1). Notice that the odd reflection is continuous at 0 if $g(0) = 0$; otherwise it is discontinuous, taking values $\pm g(0)$ just to the right and left of 0.

Let $\tilde{v}(x, t)$ solve the whole-space initial-value problem with initial condition $\tilde{g}$. We claim
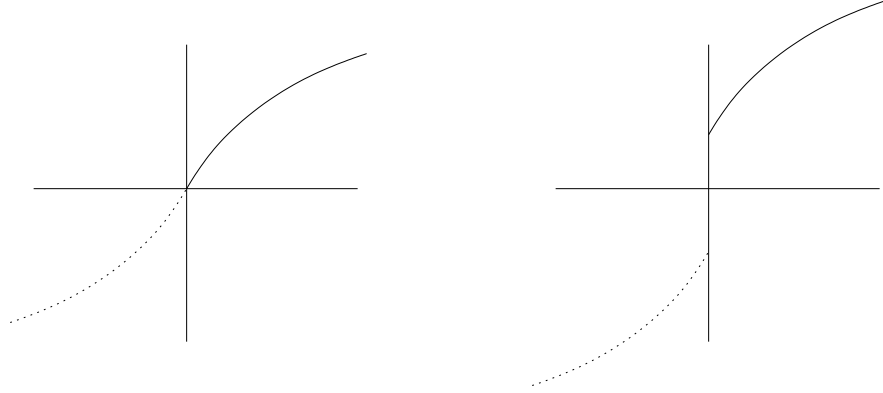
Figure 1: *Odd reflection. Note that the odd reflection is discontinuous at 0 if the original function doesn't vanish there.*

- $\tilde{v}$ is a smooth function of x and t for $t > 0$ (even if $g(0) \neq 0$);

- $\tilde{v}(x,t)$ is an odd function of x for all t, i.e. $\tilde{v}(x,t) = -\tilde{v}(-x,t)$.

The first bullet follows from the smoothing property of the heat equation. The second bullet follows from the uniqueness of solutions to the heat equation, since $\tilde{v}(x,t)$ and $-\tilde{v}(-x,t)$ both solve the heat equation with the *same* initial data $\tilde{g}$. (Please accept this uniqueness result for now; we'll prove it in the next Section.)

We're essentially done. The oddness of $\tilde{v}$ gives $\tilde{v}(0,t) = -\tilde{v}(0,t)$, so $\tilde{v}(0,t) = 0$ for all $t > 0$. Thus

$$v(x,t) = \tilde{v}(x,t), \quad \text{restricted to } x > 0$$

is the desired solution to (5). Of course it can be expressed using (2): a formula encapsulating our solution procedure is

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\begin{aligned}
v(x,t) &= \int_0^\infty k(x-y,t)g(y)\,dy + \int_{-\infty}^0 k(x-y,t)(-g(-y))\,dy \\
&= \int_0^\infty [k(x-y,t) - k(x+y,t)]g(y)\,dy
\end{aligned}
$$

where $k(x,t)$ is the fundamental solution of the heat equation, given by

$$k(z,t) = \frac{1}{\sqrt{4\pi t}}e^{-z^2/4t}. \tag{7}$$

In other words

$$v(x,t) = \int_0^\infty G(x,y,t)g(y)\,dy$$

with

$$G(x,y,s) = k(x-y,t) - k(x+y,t). \tag{8}$$

Fortunately $G(x,y,s) = G(y,x,s)$ so we don't have to try to remember which variable (x or y) we put first. The function G is called the "Green's function" of the half-space problem.

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Based on our discussion of the forward Kolmogorov equation, we recognize $G(x, y, t)$ as giving the probability that a Brownian particle starting from y at time 0 reaches position x at time t without first reaching the origin. (I'm being sloppy: the relevant random walk is not Brownian motion but $\sqrt{2}$ times Brownian motion.)

The half-space problem with initial condition 0. It remains to consider w, defined by (6). It solves the heat equation on the half-space, with initial value 0 and boundary value $\phi(t)$.

The solution w is given by

$$w(x, t) = \int_0^t \frac{\partial G}{\partial y}(x, 0, t - s)\phi(s)\, ds \tag{9}$$

where $G(x, y, t)$ is the Green's function of the half-space problem given by (8). Using the formula derived earlier for G, this amounts to

$$w(x, t) = \int_0^t \frac{x}{(t - s)\sqrt{4\pi(t - s)}} e^{-x^2/4(t-s)} \phi(s)\, ds$$

Notice that the integral is quite singular near $x = t = 0$. That's why the w-problem is best applied to compatible data ($\phi(0) = 0$).

The justification of (9) is not difficult, but it's rather different from what we've done before. To represent the value of w at location x_0 and time t_0, consider the function ψ which solves the heat equation *backward in time* from time t_0, with final-time data concentrated at x_0 at time t_0. We mean ψ to be defined only for $x > 0$, with $\psi = 0$ at the spatial boundary $x = 0$. In formulas, our definition is

$$\psi_\tau + \psi_{yy} = 0 \quad \text{for } \tau < t_0 \text{ and } y > 0, \text{ with } \psi = \delta_{x_0} \text{ at } \tau = t_0 \text{ and } \psi = 0 \text{ at } y = 0.$$

A formula for ψ is readily available, since the change of variable $s = t_0 - \tau$ transforms the problem solved by ψ one considered earlier for v:

$$\psi(y, \tau) = G(x_0, y, t_0 - \tau). \tag{10}$$

What's behind our strange-looking choice or ψ? Two things. First, the choice of final-time data gives

$$w(x_0, t_0) = \int \psi(y, t_0)w(y, t_0)\, dy.$$

(The meaning of the statement "$\psi = \delta_{x_0}$ at time t_0" is precisely this.) Second, if w solves the heat equation forward in time and ψ solves it backward in time then

$$\begin{aligned}
\frac{d}{ds}\int_0^\infty \psi(y, s)w(y, s)\, dy &= \int_0^\infty \psi_s w + \psi w_s\, dy \\
&= \int_0^\infty -\psi_{yy}w + \psi w_{yy}\, dy \\
&= \int_0^\infty -(\psi_y w)_y + (\psi w_y)_y\, dy \\
&= (-\psi_y w + \psi w_y)\big|_0^\infty . \tag{11}
\end{aligned}$$

(I've used here that the heat equation backward-in-time is the formal adjoint of the heat equation forward-in-time; you saw this before in the discussion of the forward Kolmogorov equation, which is always the formal adjoint of the backward Kolmogorov equation.) Because of our special choice of ψ the last formula simplifies: ψ and ψ_y decay rapidly enough at ∞ to kill the "boundary term at infinity," and the fact that $\psi = 0$ at $y = 0$ kills one of the two boundary terms at 0. Since $w(0, s) = \phi(s)$ what remains is

$$\frac{d}{ds} \int_0^\infty \psi(y, s) w(y, s) \, dy = \psi_y(0, s) \phi(s).$$

We're essentially done. Substitution of (10) in the above gives, after integration in s,

$$\int_0^\infty \psi(y, t_0) w(y, t_0) \, dy - \int_0^\infty \psi(y, 0) w(y, 0) = \int_0^{t_0} G_y(x_0, 0, t_0 - s) \phi(s) \, ds.$$

The first term on the left is just $w(x_0, t_0)$, by our choice of ψ, and the second term on the left vanishes since $w = 0$ at time 0, yielding precisely the desired solution formula (9).

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Barrier options. The preceding analysis provides the main ideas needed for pricing any (European) barrier option. By definition, a barrier option is like a vanilla option except that it acquires or loses its value if the stock price goes above or below a specified barrier X:

An **up-and-out** option loses its value if the stock price crosses X from below prior to maturity.

A **down-and-out** option loses its value if the stock price crosses X from above prior to maturity.

An **up-and-in** option pays only if the stock price crosses X from below prior to maturity.

A **down-and-in** option pays off only if the stock price crosses X from above prior to maturity.

For example, a down-and-out call with strike price K, maturity T, and barrier X, pays $(s - K)_+$ if the stock price stays above X, and nothing if the stock price dips below X prior to maturity. The corresponding down-and-in call has no value until the stock price crosses the barrier X from above; if that ever happens then it behaves like a standard call thereafter.

To connect our PDE's with their financial applications, let's discuss in detail the case of a down-and-out barrier call with $X < K$. This case has compatible data (i.e. the payoff vanishes at $s = X$), making it easier than the case $X > K$. The value of this barrier call is

$$V(s, t) = C(s, t) - \left(\frac{s}{X}\right)^{(1-k)} C(X^2/s, t)$$

where $k = r/(\frac{1}{2}\sigma^2)$ and $C(s, t)$ is the value of the ordinary European call with strike K and maturity T. One can, of course, check by mere arithmetic that this formula solves the PDE

and the boundary conditions. But we prefer to show how this formula emerges from our understanding of the initial-boundary-value problem for the heat equation.

Recall that under the change of variables

$$s = e^x, \quad \tau = \tfrac{1}{2}\sigma^2(T-t), \quad V(s,t) = e^{\alpha x + \beta \tau} u(x, \tau)$$

with $\alpha = (1-k)/2$, $\beta = -(k+1)^2/4$, the Black-Scholes PDE becomes

$$u_\tau = u_{xx}.$$

Restricting $s > X$ is the same as restricting $x > \log X$, so the linear heat equation is to be solved for $x > \log X$, with $u = 0$ at $x = \log X$. Its initial value $u_0(x) = u(0, x)$, is obtained from the payoff of the call by change of variables: $u_0(x) = e^{-\alpha x}(e^x - K)_+$.

Since the boundary condition is now at $x = \log X$, we can impose the condition $u = 0$ at $x = \log X$ via odd reflection about $\log X$. That is, we look for a solution of $u_t = u_{xx}$ for all x, with the property that

$$u(x', t) = -u(x, t) \quad \text{when } x' \text{ is the reflection of } x \text{ about } \log X.$$

Such a solution must satisfy $u(\log X, t) = 0$, since the condition of odd symmetry gives $u(\log X, t) = -u(\log X, t)$.

Let's be more explicit about the condition of odd symmetry. Two points $x' < \log X < x$ are related by reflection about $\log X$ if $x - \log X = \log X - x'$, i.e. if $x' = 2\log X - x$. So a function $u(x, t)$ has odd symmetry about $\log X$ if it satisfies

$$u(2\log X - x, t) = -u(x, t) \quad \text{for all } x.$$

OK, the plan is clear: First (a) extend the initial data by odd symmetry about $x = \log X$; then (b) solve the linear heat equation for $t > 0$ and all x. Carrying out step (a): the desired initial data is $u_0(x) = e^{-\alpha x}(e^x - K)_+$ for $x > \log X$; moreover our assumption that $X < K$ assures that $u_0(x) = 0$ for $x \le \log X$. So the extended initial data is

$$f_0(x) = u_0(x) - u_0(2\log X - x).$$

Carrying out step (b) is of course trivial: the solution $f(x, t)$ is given by the convolution of f_0 with the fundamental solution, i.e. by (2).

To make the value of the option explicit without an orgy of substitution, we use the fact that our PDE's are linear. So the value $V(s, t)$ of our barrier option is the difference of two terms. The first corresponds under the change of variables to

$$f_1(x, t) = \text{solution of the whole-space heat equation with initial data } u_0(x),$$

i.e. the first term is the value $C(s, t)$ of the ordinary European call. (We note for later use the relation $f_1(x, \tau) = C(e^x, t)e^{-\alpha x - \beta \tau}$.) The second term corresponds under the change of variables to

$$\begin{aligned} f_2(x, t) &= \text{solution of the whole-space heat equation with initial data } u_0(2\log X - x) \\ &= f_1(2\log X - x), \end{aligned}$$

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so it is

$$
\begin{aligned}
e^{\alpha x+\beta\tau} f_2(x,\tau) &= e^{\alpha x+\beta\tau} f_1(2\log X - x, t) \\
&= e^{\alpha x+\beta\tau} e^{-(\alpha[2\log X - x]+\beta\tau)} C(e^{2\log X - x}, t) \\
&= X^{-2\alpha} s^{2\alpha} C(X^2/s, t) = (s/X)^{1-k} C(X^2/s, t).
\end{aligned}
$$

The solution formula asserted above is precisely the difference of these two terms.

We close with a comment about the associated down-and-in call. (Remember: it has no value until the stock price crosses the barrier X from above; if this ever happens then it behaves like a standard call thereafter.) At first we seem to have to solve the Black-Scholes PDE with value of a vanilla call as its boundary condition. But actually there's no need for such hard work. Indeed, it's obvious that

$$
\text{down-and-out call} + \text{down-and-in call} = \text{standard call}
$$

since the two portfolios are equivalent. So the value of a down-and-in call is just the difference between the value of the standard call and the down-and-out call.