Ergodicity of truncated stochastic Navier Stokes with deterministic forcing and dispersion

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Abstract

Turbulence in idealized geophysical flows is a very rich and important topic. The anisotropic effects of explicit deterministic forcing, dispersive effects from rotation due to the β-plane and F-plane, and topography together with random forcing all combine to produce a remarkable number of realistic phenomena. These effects have been studied through careful numerical experiments in the truncated geophysical models. These important results include transitions between coherent jets and vortices, and direct and inverse turbulence cascades as parameters are varied, and it is a contemporary challenge to explain these diverse statistical predictions. Here we contribute to these issues by proving with full mathematical rigor that for any values of the deterministic forcing, the β and F-plane effects and topography, with minimal stochastic forcing, there is geometric ergodicity for any finite Galerkin truncation. This means that there is a unique smooth invariant measure which attracts all statistical initial data at an exponential rate. In particular, this rigorous statistical theory guarantees that there are no bifurcations to multiple stable and unstable statistical steady states as geophysical parameters are varied in contrast to claims in the applied literature. The proof utilizes a new statistical Lyapunov function to account for enstrophy exchanges between the statistical mean and the variance fluctuations due to the deterministic forcing. It also requires careful proofs of hypoellipticity with geophysical effects and uses geometric control theory to establish reachability. To illustrate the necessity of these conditions, a two dimensional example is developed which has the square of the Euclidean norm as the Lyapunov function, and is hypoelliptic with nonzero noise forcing, yet fails to be reachable or ergodic.
Keywords: Unique stochastic invariant measure, exponential attraction, beta plane, topography, general dispersion.

1 Introduction

Turbulence in idealized geophysical flows is a very rich and important topic with numerous phenomenological predictions and idealized numerical experiments. The anisotropic effects of explicit deterministic forcing, the β-effect due to the earth's curvature, and topography together with random forcing all combine to produce a remarkable number of realistic phenomena [1, 2, 3]. These include the formation of coherent jets and vortices, and direct and inverse turbulent cascades as parameters are varied [1, 2, 3]. It is well known that careful numerical experiments indicate interesting statistical bifurcations between jets and vortices as parameters vary [4, 5, 6, 7, 8, 9], and it is a contemporary challenge to explain these approximate statistical theories [10, 11, 12, 4]. However, careful numerical experiments and statistical approximations are only possible or valid for large finite times so the ultimate statistical steady state of these turbulent geophysical flows remain elusive. Here we contribute to these issues by proving with full mathematical rigor that for any values of the deterministic forcing, the β-plane effect, and topography and with precise minimal stochastic forcing for any finite Galerkin truncation of the geophysical equations, there is a unique smooth invariant measure which attracts all statistical initial data at an exponential rate, that is geometric ergodicity. The rate constant depends on the geophysical parameters and could involve a large preconstant.

Next we introduce the equations for geophysical flows which we consider in this paper. Here we investigate the ergodicity of a finite Galerkin truncation of geophysical flow on a periodic domain $\mathbb{T}^2 = [-\pi, \pi]^2$, with general dissipation, β-plane effect, stratification effect, topography, deterministic forcing and a minimal stochastic forcing. Without truncation, the model is given by [3]:

$$\begin{aligned} dq + \nabla^{\perp}\psi \cdot \nabla q dt &= D(\Delta)\psi dt + f(\mathbf{x})dt + dW_t, \\ q &= \Delta\psi - F^2\psi + h(\mathbf{x}) + \beta y. \end{aligned} \tag{1.1}$$

In the equation [3] above:

- q is the potential vorticity. ψ is the stream function. It determines the vorticity by $\omega = \Delta\psi$, and the flow by $u = \nabla^{\perp}\psi = (-\partial_y\psi, \partial_x\psi)$. Here $\mathbf{x} = (x, y)$ denotes the spatial coordinate.

- The operator $D(\Delta)\psi = \sum_{j=0}^{l}(-1)^j\gamma_l\Delta^l\psi$ stands for a general dissipation operator. We assume here $\gamma_l \geq 0$ and at least one $\gamma_l > 0$. This term can include: 1) Newtonian (eddy) viscosity, $\nu\Delta^2\psi$, 2) Ekman drag dissipation, $-d\Delta\psi$, 3) radiative damping, $d\psi$, 4) hyper-viscosity dissipation, which could be a higher order power of Δ and any positive combination of these.

- Here $f(\mathbf{x})$ is the external deterministic forcing. The random forcing W_t is a Gaussian random field. Its spectral formulation will be given explicitly soon afterwards by (1.4).

- βy is the β-plane approximation of the Coriolis effect and $h(x)$ is the periodic topography.

- The constant $F = L_R^{-1}$, where $L_R = \sqrt{gH_0}/f_0$ is the Rossby radius which measures the relative strength of rotation to stratification.

Note if one considers for example the atmospheric wind stress on the ocean, the equation in (1.1) naturally has both deterministic and stochastic components to the forcing.

The main goal of the present study is the following Theorem informally stated here

Theorem 1.1. *Consider any finite Galerkin truncation of* (1.1) *and the associated statistical solutions. Then for any deterministic forcing, f, topography, h, and β, F, with any minimal stochastic forcing involving at least two modes in a precise fashion, there is a unique smooth invariant measure which attracts all other statistical solution at large time at an exponential rate.*

Note that the requirement of minimal stochastic forcing is essentially the same as in [13, 14] without considering any geophysical effects.

Here is the outline of the remainder of the paper. After a few preliminary subsections on the Galerkin truncation and spectral formulation of (1.1), we build on earlier works on geometric ergodicity for finite dimensional systems [15, 16, 17, 18] to yield a general strategy to prove the precise version of the above theorem, which involves the following three key steps:

1) The existence of a stochastic Lyapunov function for (1.1).

2) Hypoellipticity of the generator.

3) Reachability through a suitable cadlag control process [18].

In Section 2, following [19], we construct a new stochastic Lyapunov function for the Galerkin truncation of (1.1) with both deterministic and stochastic forcing involving the square of the mean enstrophy plus the trace of the covariance matrix for the fluctuating enstrophy. In Section 3 we verify hypoellipticity of the generator of the Galerkin truncation of (1.1). Our proof reveals the subordinate role of the linear geophysical effects compared to the nonlinear triad interaction. In Section 4, we use geometric control theory [18, 20] to prove reachability by a suitable cadlag control. Our inductive proof follws similar reasoning as our verification of hypoellipticity in Section 4 and reveals a parallel structure. In Section 5, we construct a specific example of a two dimensional stochastic system which has the quadratic Euclidean energy as a Lyapunov function, is hypoelliptic with non-zero forcing everywhere, yet violates the reachability criterion and fails to be ergodic. A concluding discussion and further research directions are discussed in Section 6. The appendix fills an important gap for rigorous hypoellipticity in [16, 18] by deriving higher order bounds for the system.

1.1 Galerkin truncation

In order to implement (1.1) in numerics, we need to do a Galerkin truncation. One way to achieve this is letting $q = q_\Lambda + \beta y$, where q_Λ has Fourier modes only within a symmetry finite indices set $\mathcal{I} \subset \mathbb{Z}^2/\{(0,0)\}$:

$$q_\Lambda = \sum_{k\in\mathcal{I}} q_k e_k(\mathbf{x}), \quad e_k(\mathbf{x}) = \frac{e^{ik\cdot\mathbf{x}}}{2\pi}.$$

We say $\mathcal{I}$ is symmetric if $k \in \mathcal{I}$, then $-k \in \mathcal{I}$. One practical choice of $\mathcal{I}$ can be of form

$$\mathcal{I} = \{k \in \mathbb{Z}^2/\{(0,0)\}\big|\ |k| \leq N\} \quad \text{or} \quad \mathcal{I} = \{k \in \mathbb{Z}^2/\{(0,0)\}\big|\ |k_1| \leq N, |k|_2 \leq N\}, \qquad (1.2)$$

with N being a large number. Let $\mathbf{P}_\Lambda$ be the projection of $L^2(\mathbb{T}^2)$ onto the finite subspace spanned by $\{e_k(\mathbf{x}), k \in \mathcal{I}\}$. We can then project (1.1) to the modes in $\mathcal{I}$ using a truncation operator . The truncated model is then

$$\begin{aligned} dq_\Lambda &= -\mathbf{P}_\Lambda(\nabla^\perp \psi_\Lambda \cdot \nabla q_\Lambda)dt - \beta(\psi_\Lambda)_x dt + D(\Delta)q_\Lambda dt + f_\Lambda(\mathbf{x})dt + dW_\Lambda(t), \\ q_\Lambda &= \Delta\psi_\Lambda - F^2\psi_\Lambda + h_\Lambda(\mathbf{x}). \end{aligned} \tag{1.3}$$

$\psi_\Lambda = \sum_{k\in\mathcal{I}} \psi_k e_k(\mathbf{x})$ is the truncation of ψ, and likewise we can have spectral formulations of the truncated relatively vorticity ω_Λ, external forcing f_Λ, topography h_Λ. In particular, we model the Gaussian random field as

$$W_\Lambda(t) = \sum_{k\in\mathcal{I}} \sigma_k W_k(t). \tag{1.4}$$

The $W_k(t)$ above are independent complex Wiener processes except for conjugating pairs, where $\sigma_k = \sigma^*_{-k}, W_k = W^*_{-k}$. One simple way to achieve this is letting $B^r_k(t), B^i_k(t)$ to be independent real Wiener processes, and

$$W_k(t) = \frac{1}{\sqrt{2}}(B^r_k(t) + iB^i_k(t)), \quad W_{-k}(t) = \frac{1}{\sqrt{2}}(B^r_k(t) - iB^i_k(t))$$

for $k \in \mathcal{I}_+ = \{k \in \mathcal{I} : k_2 > 0\} \cup \{k \in \mathcal{I} : k_2 = 0, k_1 > 0\}$. The corresponding incompressible flow field is $u_\Lambda = \nabla^\perp\psi_\Lambda$, while its underlying basis will be $\tilde{e}_k = \frac{ik^\perp}{|k|} e_k$.

1.2 Spectral formulation

Another way to obtain and study (1.3) is projecting (1.1) onto each Fourier mode. In fact, it suffices to derive equations for any one of q_k, ψ_k,u_k or ω_k, since the others can then be determined quite easily by the following linear relation:

$$\omega_k = \frac{|k|^2(q_k - h_k)}{F^2 + |k|^2}, \quad \psi_k = \frac{-q_k + h_k}{F^2 + |k|^2}, \quad u_k = -\frac{|k|(q_k - h_k)}{F^2 + |k|^2}\omega_k.$$

We chose to project (1.3) onto the Fourier modes of q_Λ. The resulting formula for q_k is

$$dq_k(t) = \frac{-d_k + i\beta k_1}{F^2 + |k|^2}(q_k(t) - h_k)dt + \sum_{m+n=k, m,n\in\mathcal{I}} (a_{m,n} q_m q_n - b_{m,n} h_n q_m)dt + f_k dt + \sigma_k dW_k(t), \tag{1.5}$$

with the three wave interaction coefficients $a_{m,n}, b_{m,n}$ and the general damping d_k given by

$$b_{m,n} := \frac{\langle n^\perp, m\rangle}{2\pi|n|^2 + 2\pi F^2}, \quad a_{m,n} = \frac{\langle n^\perp, m\rangle}{4\pi}\left(\frac{1}{|m|^2 + F^2} - \frac{1}{|n|^2 + F^2}\right), \quad d_k = \sum_j \gamma_j |k|^{2j}.$$

It is easy to see that

$$\langle n^\perp, m\rangle = \langle (n+m)^\perp, m\rangle = -\langle m^\perp, n\rangle$$

so $a_{m,m} = 0$, $a_{m,n} = a_{n,m} = a_{n,m+n} = -a_{-m,n}$, moreover the triad conservation property $a_{m,n} + a_{n,-m-n} + a_{-m-n,m} = 0$ since the sum is

$$\frac{\langle n^\perp, m\rangle}{4\pi}\left(\frac{1}{|m|^2 + F^2} - \frac{1}{|n|^2 + F^2} + \frac{1}{|n|^2 + F^2} - \frac{1}{|n+m|^2 + F^2} + \frac{1}{|m+n|^2 + F^2} - \frac{1}{|m|^2 + F^2}\right).$$

Also note that the damping $d_k \geq d_0 := \sum_j \gamma_j > 0$.

1.3 Ergodicity with minimal stochastic forcing

When $\beta = 0$ and $F \equiv 0$, (1.3) is essentially the stochastic Navier Stokes equation with finite projection. [13, 14] have considered the minimum number of stochastically forced modes in order for the system to be ergodic, and the least amount of forced modes is given by the following assumption

Assumption 1.2. *Let $\mathcal{I}_0 = \{k \in \mathcal{I} | \sigma_k \neq 0\}$ be the symmetric subset of stochastically forced modes, suppose that the following increasing sequence of index sets*

$$\mathcal{I}_j = \mathcal{I}_0 \cup \{m + n | m, n \in \mathcal{I}_{j-1} \text{ with } \langle m^\perp, n\rangle \neq 0, |m| \neq |n|\},$$

has $\mathcal{I}_N = \mathcal{I}$ for certain $N < \infty$.

The intuition behind Assumption 1.2 is straightforward: the stochastic forcing over modes in $\mathcal{I}_0$ can influences other modes by the the three waves interaction in (1.5), and the way that $\mathcal{I}_j$ expands follows exactly the same rules. In particular, if $\mathcal{I}$ is connected by the neighboring relation on $\mathbb{Z}^2$ and includes $\{k : \max\{|k_1|, |k_2|\} = 1\}$ as a subset, like the ones in (1.2), then Assumption 1.2 holds as long as $\mathcal{I}_0$ contains modes $(0, 1)$ and $(1, 1)$. The verification is given by Lemma A.1. According to [13, 18, 14], Assumption 1.2 is the minimal requirement for (1.5) to be ergodic without geophysical effects. Our paper shows this minimal stochastic forcing assumption is still sufficient when general geophysical effects are considered.

Following our derivations from the previous two subsections, we can view $q_\Lambda(t)$ as a process $\mathbf{q} \in \mathbb{C}^{\mathcal{I}}$. Geometric ergodicity of diffusion processes in finite-dimensional spaces is relatively well understood. The following theorem is a version of [15, Theorem 2.3].

Theorem 1.3. *Let X_n be a Markov chain in a space E such that*

1. *There is a Lyapunov function $\mathcal{E} : E \mapsto \mathbb{R}^+$ for the Markov process X_n with compact sub-level sets, while $\mathbb{E}\mathcal{E}(X_t) \leq e^{-\gamma t}\mathbb{E}\mathcal{E}(X_0) + K$ for certain $\gamma, K > 0$.*

2. *Minorization: for any compact set B, there is a compact set $C \supset B$ such that the minorization condition holds for C. That is, there is a probability measure ν with $\nu(C) = 1$, and a $\eta > 0$ such that for any given set A*

$$\mathbb{P}(X_n \in A | X_{n-1} = x) \geq \eta\nu(A)$$

   for all $x \in C$.

Then there is a unique invariant measure π and a constant $r \in (0, 1), \kappa > 0$ such that

$$\|\mathbb{P}^\mu(X_n \in \cdot\,) - \pi\|_{tv} \leq \kappa r^n \left(1 + \int \mathcal{E}(x)\mu(dx)\right).$$

Here $\mathbb{P}^\mu(X_n \in \cdot\,)$ is the law of X_n given $X_0 \sim \mu$; and $\|\cdot\|_{tv}$ denotes the total variation distance, which is $\|\mu - \nu\|_{tv} = \int |p(x) - q(x)|dx$, assuming μ and ν has density p and q.

As for diffusion processes in $\mathbb{R}^d$, the minorization condition can be achieved by the following proposition, which is a combination of [15, Lemma 2.7], [21], Theorem 4.20 [22] and Lemma 3.4 of [16].

Proposition 1.4. *Let X_t be a diffusion process in $\mathbb{R}^d$ that follows*

$$dX_t = Y(X_t)dt + \sum_{k=1}^{n} \Sigma_k(X_t) \circ dB_k. \tag{1.6}$$

In above, B_k are independent 1D Wiener processes, and $\circ$ stands for Stratonovich integral. Y and Σ_k are smooth vector fields with at most polynomial growth for all derivatives. Assume moreover that for any $T > 0, k > 0, p > 0$ and initial condition, the following growth condition holds

$$\mathbb{E}\sup_{t\leq T}|X_t|^p < \infty, \quad \mathbb{E}\sup_{t\leq T}\|J_{0,t}^{(k)}\|^p < \infty, \quad \mathbb{E}\sup_{t\leq T}\|J_{0,t}^{-1}\|^p < \infty. \tag{1.7}$$

Here $J_{0,t}$ is the Frechet derivative flow: $J_{0,t}v = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(X_t^{x_0+\epsilon v} - X_t^{x_0})$, and $J_{0,t}^{(k)}$ are the higher order derivatives. Then X_t satisfies the minorization assumption if the following two hold:

- *Hypoellipticity: let $\mathcal{L}$ be the Lie algebra generated by $\{Y, \Sigma_1, \ldots, \Sigma_n\}$. Let $\mathcal{L}_0$ be the ideal of $\{\Sigma_1, \ldots, \Sigma_n\}$ inside $\mathcal{L}$, which is essentially the linear space spanned by*

  $$\{\Sigma_i, [\Sigma_i, \Sigma_j], [\Sigma_i, Y], [[\Sigma_i, \Sigma_j], \Sigma_k], [[\Sigma_i, Y], \Sigma_k], \ldots\}.$$

  The diffusion process is hypoelliptic if $\mathcal{L}_0 = \mathbb{R}^d$ at each point.

- *Reachability: there is a point $x^* \in \mathbb{R}^d$, such that with any $\epsilon > 0$ there is a $T > 0$, such that from any point $x_0 \in \mathbb{R}^d$ we can find an cadlag control process b_k such that the solution to the following ODE initialized at x_0*

  $$dx_t = Y(x_t)dt + \sum_{k=1}^{n} \Sigma_k(x_t)b_k dt \tag{1.8}$$

  satisfies $|x_T - x^*| < \epsilon$. Here by cadlag control, we mean $b_k(t)$ is continuous from right, has left limit and locally bounded.

Moreover, with arbitrary initial condition, X_t has a smooth density with respect to the Lebesgue measure. So if π is an invariant measure, it has smooth density.

In Section 2-4 we will verify these conditions one by one, while the growth condition is verified in the Appendix. So in conclusion we have

Theorem 1.5. *The truncated general model in* (1.3) *with all geophysical effects is geometrically ergodic, and the invariant measure has smooth density with respect to the Lebesgue measure, as long as the minimal stochastic forcing condition Assumption 1.2 holds.*

Remark 1.6. *First, in Proposition 1.4 the Stratonovich integral makes no difference with Itô integral in our following discussion, because Σ_k are constant vector fields. Second, in the original proofs of [13, 16, 18], the growth bound* (1.7) *is not verified. This is not very rigorous because the application of Hörmander theorem, like [23, Theorem 38.16], [24] and [25], all require the underlying vector fields to have bounded derivatives. The moment controls* (1.7) *are needed for unbounded vector fields, like stochastic Navier Stokes, see [21]. On the other hand, this technical gap can be easily closed by some arguments in [26], which is demonstrated in our Appendix.*

Remark 1.7. *If we don't require smoothness of the density, but just ergodicity, the authors doubt that it suffices to verify the Hörmander condition near point x^* instead of all points. This can be told from Theorem 2.3.3 of [24], and the fact that the proof of [21] does not use information away from the initial point.*

2 Lyapunov functions

2.1 Statistical energy conservation form

Following the procedures documented in [19], we can rewrite (1.3) into following form:

$$d\mathbf{q} = [L + D]\mathbf{q} + B(\mathbf{q}, \mathbf{q}) + \mathbf{F} + \Sigma d\mathbf{W}(t). \tag{2.1}$$

Here $\mathbf{q}, \mathbf{F}$ and $\mathbf{W}$ are $|\mathcal{I}|$-dim complex valued vectors with components being q_k, F_k, W_k. The operators above are given by

- L is a skew symmetric matrix. Its diagonal entries are $L_{kk} = \frac{i\beta k_1}{|k|^2+F^2}$, and off diagonal entries are $L_{km} = -b_{k-m,m}h_{k-m}$. Note that $L_{mk} = -b_{m-k,m}h_{m-k} = -L^*_{km}$.
- D is a diagonal negative definite matrix. Its diagonal entries are $\frac{-d_k}{|k|^2+F^2}$.
- B is a quadratic form. Its k-th component is $[B(\mathbf{p}, \mathbf{q})]_k = \sum a_{m,n}p_m q_n$. It satisfies the relation

$$\begin{aligned}
\langle B(\mathbf{q}, \mathbf{q}), \mathbf{q}\rangle &= \sum_{k\in\mathcal{I}} q^*_k \sum_{m+n=k} a_{m,n}q_m q_n \\
&= \sum_{m,n} a_{m,n}q_m q_n q_{-m-n} \\
&= \frac{1}{3}\sum_{m,n} q_m q_n q_{-m-n}(a_{m,n} + a_{n,-m-n} + a_{-m-n,m}) = 0
\end{aligned}$$

  due to the triad conservation property listed below (1.5).
- $\mathbf{F}$ is a constant vector, it has components of form $F_k = -\frac{-d_k+i\beta k_1}{F^2+|k|^2}h_k + f_k$.
- Σ is a diagonal matrix with entries $\Sigma_{kk} = \sigma_k$.

In many applications, statistical quantities are considered for their ensemble mean and fluctuations [27, 28, 29]. We denote the ensemble mean field as $\bar{\mathbf{q}} = \mathbb{E}\mathbf{q}$, then the potential vorticity field has the Reynold's decomposition $\mathbf{q} = \bar{\mathbf{q}} + \sum_{k\in\mathcal{I}} Z_k(t)\mathbf{e}_k$. The $\mathbf{e}_k$ is the canonical unit vector with 1 at its k-th component, which corresponds to e_k in the Fourier decomposition. The exact mean field equation is the following:

$$\frac{d\bar{\mathbf{q}}}{dt} = (L + D)\bar{\mathbf{q}} + B(\bar{\mathbf{q}}, \bar{\mathbf{q}}) + \sum_{m,n} R_{m,n}B(\mathbf{e}_m, \mathbf{e}_n) + \mathbf{F}.$$

$R_{m,n}$ in the above equation is the covariance covariance matrix $R_{m,n} = \mathbb{E} Z_m Z_n^*$. This matrix follows an ODE, as derived in [19]:

$$\frac{dR}{dt} = L_v R + R L_v^* + Q_F + Q_\sigma.$$

The matrix L_v is given by

$$[L_v]_{m,n} = \langle [L+D]\mathbf{e}_m + B(\bar{\mathbf{q}}, \mathbf{e}_m) + B(\mathbf{e}_m, \bar{\mathbf{q}}), \mathbf{e}_n \rangle.$$

Matrix Q_σ expresses energy transfers due to external stochastic forcing, so it is a diagonal matrix with entries $[Q_\sigma]_{k,k} = |\sigma_k|^2$. The energy flux is represented by Q_F as

$$[Q_F]_{m,n} = \overline{Z_i Z_j Z_n} \langle B(\mathbf{e}_i, \mathbf{e}_j), \mathbf{e}_n \rangle + \overline{Z_i Z_j Z_m} \langle B(\mathbf{e}_i, \mathbf{e}_j), \mathbf{e}_n \rangle.$$

One key property of Q_F is that $\mathrm{tr}(Q_F) = 0$ [19, 27, 28, 29].

2.2 Stochastic Lyapunov functions

The statistical energy conservation form (2.1) combined with [19] gives us a straight forward Lyapunov function, which is the total potential enstrophy. Define

$$\mathcal{E} = \int \frac{1}{2} |q_\Lambda|^2 d\mathbf{x} = \frac{1}{2} |\mathbf{q}|^2 = \frac{1}{2} \sum_{k \in \mathcal{I}} |q_k|^2,$$

then $\mathbb{E}\mathcal{E} = \frac{1}{2} \int |\bar{\mathbf{q}}|^2 d\mathbf{x} + \frac{1}{2}\mathrm{tr}(R)$. Applying the Theorem in [19], the time derivative of $\mathbb{E}\mathcal{E}$ is given by

$$\frac{d}{dt} \mathbb{E}\mathcal{E} = \langle D\bar{\mathbf{q}}, \bar{\mathbf{q}} \rangle + \langle \bar{\mathbf{q}}, \mathbf{F} \rangle + \mathrm{tr}(DR) + \mathrm{tr} Q_\sigma.$$

Using the spectral decomposition of $\bar{\omega}$

$$\langle D\bar{\mathbf{q}}, \bar{\mathbf{q}} \rangle + \mathrm{tr}(DR) = -\sum_k d_k (|\bar{q}_k|^2 + R_{k,k}) \leq -2d_0 \mathbb{E}\mathcal{E},$$

here we recall $d_0 = \sum \gamma_l \leq d_k$ for all k. As a consequence, $\mathcal{E}$ is a Lyapunov function because

$$\frac{d}{dt} \mathbb{E}\mathcal{E} \leq -2d_0 \mathbb{E}\mathcal{E} + \mathbf{Re} \langle \bar{\mathbf{q}}, \mathbf{F} \rangle + \frac{1}{2}\mathrm{tr}(Q_\sigma) \leq -d_0 \mathbb{E}\mathcal{E} + \frac{1}{2d_0} |\mathbf{F}|^2 + \frac{1}{2}\mathrm{tr}(Q_\sigma). \tag{2.2}$$

In the derivation above, we used that $2\mathbb{E}\mathcal{E} \geq |\bar{\mathbf{q}}|^2$, and then applied Young's inequality. Then from (2.2), it suffices to apply Grönwall's inequality to see $\mathcal{E}$ is a Lyapunov function.

As a matter of fact, we can use (2.2) and show that $\mathcal{E}^{2^n}$ and $\exp(\lambda \mathcal{E})$ for λ below a threshold are all Lyapunov functions. This is verified in the Appendix.

When there is no topography, $h \equiv 0$, the total energy will also be a Lyapunov function. The total energy is given by $E = \frac{1}{2} \int (|\nabla \psi_\Lambda|^2 + F^2 |\psi_\Lambda|^2) d\mathbf{x} = \frac{1}{2} \sum_{k \in \mathcal{I}} |v_k|^2$, where $v_k = C_k^{-1} q_k$ with $C_k = \sqrt{|k|^2 + F^2}$. The dynamics of v_k can be derived from (1.5) by a linear transform.

$$dv_k(t) = \frac{-d_k + i\beta k_1}{F^2 + |k|^2} v_k(t) dt + \sum_{m+n=k, m,n \in \mathcal{I}} a_{m,n} C_m C_n C_k^{-1} v_m v_n dt + C_k^{-1} f_k dt + C_k^{-1} \sigma_k dW_k(t).$$

We can as well rewrite this dynamic into a statistical energy conservation form like (2.1), because for all $m+n=k$,

$$a_{m,n}C_mC_nC_k^{-1}+a_{n,k}C_nC_kC_m^{-1}+a_{k,m}C_kC_mC_n^{-1}=0.$$

The remaining derivation for the dissipation of E is identical to the one of $\mathcal{E}$. If we need to include topography we need another equation for a large scale flow in the original dynamics as discussed extensively in [3].

3 Hypoellipticity

In order to demonstrate hypoellipticity, we verify the Hörmander brackets condition. The complication here is that the variables are complex valued, and complex conjugacy requires that $q_k=q_{-k}^*$. Therefore, we decompose the Fourier modes into their real and complex part,

$$q_k=R_k+iI_k,\quad h_k=h_k^r+ih_k^i,\quad F_k=F_k^r+iF_k^i.$$

Notice that $R_k=R_{-k}$ and $I_k=-I_{-k}$. We also partition $\mathcal{I}$ into $\mathcal{I}_+$ and $\mathcal{I}_-$, where

$$\mathcal{I}_+=\{k\in\mathcal{I}|k_2>0\}\cup\{k\in\mathcal{I}|k_2=0,k_1>0\},\quad \mathcal{I}_-=\{k\in\mathcal{I}|-k\in\mathcal{I}_+\}.$$

It is clear that the vector ω is determined once its components with $k\in\mathcal{I}_+$ are, so we will focus on the indices set $\mathcal{I}_+$ instead of $\mathcal{I}$. In this formulation, we regard (1.5) as a real valued process $(R_k,I_k)_{k\in\mathcal{I}_+}$, where the dynamics is given by

$$\begin{aligned}
dR_k&=\left(-\frac{d_k|k|^2}{F^2+|k|^2}R_k-\frac{k_1\beta}{F^2+|k|^2}I_k\right)dt+F_k^rdt+\frac{\sigma_k}{\sqrt{2}}dB_k^r(t)\\
&+\sum_{m+n=\pm k}a_{m,n}(R_mR_n-I_mI_n)dt+\sum_{m-n=\pm k}a_{m,n}(R_mR_n+I_mI_n)dt\\
&-\sum_{m+n=\pm k}b_{m,n}(h_n^rR_m-I_mh_n^i)dt-\sum_{m-n=\pm k}b_{m,n}(h_n^rR_m+h_n^iI_m)dt,\\
&=Y_k^r(\mathbf{q})dt+\frac{\sigma_k}{\sqrt{2}}dB_k^r(t); \qquad (3.1)\\
dI_k&=\left(-\frac{d_k|k|^2}{F^2+|k|^2}I_k+\frac{k_1\beta}{F^2+|k|^2}R_k\right)dt+F_k^idt+\frac{\sigma_k}{\sqrt{2}}dB_k^i(t)\\
&+\sum_{m+n=\pm k}a_{m,n}(R_mI_n+I_mR_n)dt+\sum_{m-n=\pm k}a_{m,n}(R_mI_n-I_mR_n)dt\\
&-\sum_{m+n=\pm k}b_{m,n}(R_mh_n^r-I_mh_n^i)dt-\sum_{m-n=\pm k}b_{m,n}(R_mh_n^i-I_mh_n^r)dt\\
&=Y_k^i(\mathbf{q})dt+\frac{\sigma_k}{\sqrt{2}}dB_k^i(t). \qquad (3.2)
\end{aligned}$$

Proposition 3.1. *With the minimal forcing condition, Assumption 1.2, the Fourier coefficients $(\omega_k)_{k\in\mathcal{I}_+}$, which follow (3.1)-(3.2), are jointly hypoelliptic.*

Proof. Define the linear span of vector fields

$$L_0 = \operatorname{span}\left\{ \frac{\partial}{\partial I_k}, \frac{\partial}{\partial R_k} \middle| k \in \mathcal{I}_0 \right\},$$

and define an expansion inductively by

$$L_n = \operatorname{span}\left\{ X_1, [X_2, [X_1, Y]] | X_i \in L_{n-1} \right\}.$$

Here the vector field Y is given

$$Y = \sum_k Y_k^r(\mathbf{q}) \frac{\partial}{\partial R_k} + \sum_k Y_k^i(\mathbf{q}) \frac{\partial}{\partial I_k}.$$

Following Proposition 1.4, in order to show hypoellipticity, we essentially need to show $L_n = \mathbb{R}^{2\mathcal{I}_+}$ for some n, since $L_n \subset \mathcal{L}_0$.

For that purpose, we define the set $\mathcal{I}_n^+$ in the following sense, where $\mathcal{I}_0^+ = \{k \in \mathcal{I}_+ | k \in \mathcal{I}_0 \text{ or } -k \in \mathcal{I}_0\}$.

$$\mathcal{I}_n^+ = \mathcal{I}_{n-1}^+ \cup \{k \in \mathcal{I}_+ : k = (m+n)^+ \text{ or } (m-n)^+, m, n \in \mathcal{I}_{n-1}, a_{m,n} \neq 0\}.$$

Here $(m+n)^+$ is $\{m+n, -m-n\} \cap \mathcal{I}_+$. It is easy to see that $\mathcal{I}_n^+ = \mathcal{I}_n \cap \mathcal{I}_+$.

Our proof is closed once we can show that $\frac{\partial}{\partial R_k}, \frac{\partial}{\partial I_k} \in L_j$ for all $k \in \mathcal{I}_j^+$ inductively. This claim holds explicitly for $j = 0$. If $m, n \in \mathcal{I}_{j-1}^+$, while $\frac{\partial}{\partial R_i}, \frac{\partial}{\partial I_i} \in L_{j-1}$ for $i = m, n$, then

$$\begin{aligned}
\left[\frac{\partial}{\partial R_m}, Y\right] =& \frac{-d_m |m|^2}{|m|^2 + F^2} \frac{\partial}{\partial R_m} - \frac{m_1 \beta}{|m|^2 + F^2} \frac{\partial}{\partial I_m} \\
&+ \sum_{n \in \mathcal{I}_+} a_{m,n} R_n \frac{\partial}{\partial R_{(m+n)^+}} + a_{m,n} R_n \frac{\partial}{\partial R_{(m-n)^+}} + a_{m,n} I_n \frac{\partial}{\partial I_{(m+n)^+}} + a_{m,n} I_n \frac{\partial}{\partial I_{(m-n)^+}} \\
&- \sum_{n \in \mathcal{I}_+} b_{m,n} h_n^r \frac{\partial}{\partial R_{(m+n)^+}} - b_{m,n} h_n^r \frac{\partial}{\partial R_{(m-n)^+}} - b_{m,n} h_n^i \frac{\partial}{\partial I_{(m+n)^+}} - b_{m,n} h_n^i \frac{\partial}{\partial I_{(m-n)^+}},
\end{aligned}$$

and symmetrically

$$\begin{aligned}
\left[\frac{\partial}{\partial I_m}, Y\right] =& \frac{-d_m |m|^2}{|m|^2 + F^2} \frac{\partial}{\partial I_m} + \frac{m_1 \beta}{|m|^2 + F^2} \frac{\partial}{\partial R_m} \\
&+ \sum_{n \in \mathcal{I}_+} -a_{m,n} I_n \frac{\partial}{\partial R_{(m+n)^+}} + a_{m,n} I_n \frac{\partial}{\partial R_{(m-n)^+}} + a_{m,n} R_n \frac{\partial}{\partial I_{(m+n)^+}} - a_{m,n} R_n \frac{\partial}{\partial I_{(m-n)^+}} \\
&+ \sum_{n \in \mathcal{I}_+} b_{m,n} h_n^i \frac{\partial}{\partial R_{(m+n)^+}} - b_{m,n} h_n^i \frac{\partial}{\partial R_{(m-n)^+}} - b_{m,n} h_n^r \frac{\partial}{\partial I_{(m+n)^+}} + b_{m,n} h_n^r \frac{\partial}{\partial I_{(m-n)^+}}.
\end{aligned}$$

As a consequence,

$$
\begin{aligned}
\left[\frac{\partial}{\partial R_n}, \left[\frac{\partial}{\partial R_m}, Y\right]\right] &= a_{m,n}\frac{\partial}{\partial R_{(m+n)^+}} + a_{m,n}\frac{\partial}{\partial R_{(m-n)^+}},\\
\left[\frac{\partial}{\partial I_n}, \left[\frac{\partial}{\partial R_m}, Y\right]\right] &= a_{m,n}\frac{\partial}{\partial I_{(m+n)^+}} + a_{m,n}\frac{\partial}{\partial I_{(m-n)^+}},\\
\left[\frac{\partial}{\partial R_n}, \left[\frac{\partial}{\partial I_m}, Y\right]\right] &= a_{m,n}\frac{\partial}{\partial I_{(m+n)^+}} - a_{m,n}\frac{\partial}{\partial I_{(m-n)^+}},\\
\left[\frac{\partial}{\partial I_n}, \left[\frac{\partial}{\partial I_m}, Y\right]\right] &= -a_{m,n}\frac{\partial}{\partial R_{(m+n)^+}} + a_{m,n}\frac{\partial}{\partial R_{(m-n)^+}}.
\end{aligned}
\tag{3.3}
$$

It is clear that these four vector fields span $\frac{\partial}{\partial R_{(m+n)^+}}, \frac{\partial}{\partial R_{(m-n)^+}}, \frac{\partial}{\partial I_{(m+n)^+}}, \frac{\partial}{\partial I_{(m-n)^+}}$, which completes our induction. □

4 Reachability

In [14], the reachability issue is easy to demonstrate. If fact, this method still works if there are no deterministic forcing or topography. Because when $f_k \equiv 0, h_k \equiv 0$, we can simply let the stochastic forcing in (1.8), and consider the total enstrophy function $\mathcal{E}$, its time derivative is given by (2.2)

$$
\frac{d\mathcal{E}}{dt} \leq -\nu\mathcal{E}
$$

As a consequence, $\sum |q_k(t)|^2 \leq e^{-\nu t}\sum |q_k(0)|^2$, so for large enough t, $\mathbf{q}$ is close to the origin. Yet with general geophysical effects, this argument no longer holds, and we need to introduce a more complicate framework.

4.1 Geometric control

In geometry control theory [20], the question of reachability has been discussed in a more general setting. Let the underlying state space M be a simply connected n-dimensional smooth manifold, and TM denotes its tangent space. Let $F: M \times U \to TM$ be a mapping, such that for each u in the control space U, $F(\,\cdot\,, u): M \mapsto TM$ is an analytic vector field. Let $\mathcal{F} = \{F(\,\cdot\,, u)\}$ be the family of vector fields generated by F. In the context of our system (3.1) and (3.2), $M = \mathbb{R}^{2|I_+|}$ and $U = \mathbb{R}^{2|I_0^+|}$, while

$$
F(\mathbf{q}, \mathbf{u}) = Y(\mathbf{q}) + \sum_{k\in I_0^+} u_k^r \frac{\partial}{\partial R_k} + u_k^i \frac{\partial}{\partial I_k}.
$$

A continuous curve $x(t) \in M$, $t \in [0, T]$, is an *integral curve* of $\mathcal{F}$, if there exists a partition $0 = t_0 < t_1 < \cdots < t_m = T$, and $X_1, \ldots, X_m \in \mathcal{F}$, such that

$$
\dot{x}(t) = X_i(x(t)), \quad t \in (t_{i-1}, t_i).
$$

The *reachable set* from a point $x_0 \in M$, denoted by $\mathcal{A}_{\mathcal{F}}(x_0, T)$, is the set of terminal points $x(T)$ of integral curves of $\mathcal{F}$ that originates at x_0. In particular, we will write the integral curve generated by a vector field $\{X\}$ at x_0 as $\exp(tX)x_0$. We will also denote set of points reachable from x_0 before T as $\mathcal{A}_{\mathcal{F}}(x_0, \leq T) = \cup_{t\leq T}\mathcal{A}_{\mathcal{F}}(x_0, t)$.

In order to verify the reachability condition of Proposition 1.4, the interested question here is when will $\overline{\mathcal{A}_{\mathcal{F}}(x, T)}$ be M for all x, in which case we say $\mathcal{F}$ is *time-T strongly controllable*. We will say $\mathcal{F}$ is *strongly controllable*, if $\overline{\mathcal{A}_{\mathcal{F}}(x, \leq T)} = M$ for any $T > 0$. Roughly speaking, one sufficient condition to guarantee controllability is letting $\{F(x, \cdot)\} = T_xM$, because then the integral curves can go to any desired direction. In [20], a set of techniques is developed to gradually expand $\mathcal{F}$ eligibly, so that $\{\widetilde{F}(x, \cdot)\} = \mathcal{R}^n$ where $\widetilde{F}$ is an expansion of $\mathcal{F}$. We call an expansion $\mathcal{F}'$ of $\mathcal{F}$ *eligible*, if

$$\mathcal{A}_{\mathcal{F}'}(x, \leq T) \subset \overline{\mathcal{A}_{\mathcal{F}}(x, \leq T)}, \quad x \in M.$$

It is relatively intuitive to see that $\mathcal{F}'$ will be confined in the time zero ideal I of Lie algebra generated by $\mathcal{F}$. Here $I = Lie(X - Y, [X, Y] : X, Y \in \mathcal{F})$, which is essentially $\mathcal{L}_0$ if we consider the control problem (1.8). The largest eligible extension of $\mathcal{F}$ within I will be called the *strong Lie saturate*. Its equivalence to controllability is given by the following Theorem

Theorem 4.1 (Theorem 12 of Section 3.3 [20])**.** *$\mathcal{F}$ is strongly controllable if and only if the strong Lie saturate is equal to $Lie(\mathcal{F})$ and $I_x(\mathcal{F}) = T_xM$.*

Moreover, we have the equivalence of strong controllability and time-T strong controllability as follow:

Theorem 4.2 (Theorem 13 of Section 3.4 [20])**.** *If $\mathcal{F}$ is strongly controllable and $I_x(\mathcal{F}) = T_xM$, then $\mathcal{A}_{\mathcal{F}}(x, T) = M$ for all x and $T > 0$.*

Note in [20], there is an additional concept called "Lie-determined systems". We do not introduce this notion since all analytic vector fields are Lie-determined systems.

Due to these two theorems, in order to show strong controllability, it suffices for us find an eligible expansion $\mathcal{F}'$ of $\mathcal{F}$ such that it spans the tangent space. [20] have introduced three ways to do eligible expansions. The first eligible way of expansion is by taking closure of the control space

Theorem 4.3 (Theorem 5 of Section 3.2.1 [20])**.** *Let $\mathcal{F}_1$ be the topological closure of a smooth family of vector files $\mathcal{F}$, then it is an eligible expansion.*

The second eligible way of expansion is doing a convex cone.

Theorem 4.4 (Theorem 8 of Section 3.2.2 [20])**.** *Let $\mathcal{F}_1$ be the convex cone $\sum_{i=1}^{n} \lambda_i X_i$ with $\lambda_i \geq 0, \sum \lambda_i \leq 1$, and $X_i \in \mathcal{F}$, then it is an eligible expansion.*

The third way of expansion is by considering a strong normalizer. A smooth diffeomorphism Φ on M is a *strong normalizer* for $\mathcal{F}$, or simply called a *normalizer*, if

$$\Phi(\mathcal{A}_{\mathcal{F}}(\Phi^{-1}(x), \leq T)) \subset \overline{\mathcal{A}_{\mathcal{F}}(x, \leq T)}, \quad x \in M, T \geq 0.$$

We denote the set of all strong normalizer as $\mathcal{N}_S(\mathcal{F})$. If X is vector field and Φ is a diffeomorphism, we will use $\Phi_\#(X)$ to denote the vector field $(\Phi_* \circ X) \circ \Phi^{-1}$. One easy way to tell a diffeomorphism is a normalizer is the following

Lemma 4.5 (The Lemma after Definition 5 in Section 3.2.2 [20]). *A diffeomorphism* Φ *is a strong normalizer for* $\mathcal{F}$ *if both* $\Phi(x)$ *and* $\Phi^{-1}(x)$ *belong to* $\overline{\mathcal{A}_{\mathcal{F}}(x, \leq T)}$ *for all* x *and* $T > 0$.

One particular choice of $\Phi(x)$ for the application of this lemma, is letting $\Phi(x) = \exp(tX)x$, if $\lambda X \in \mathcal{F}$ for all $\lambda \in \mathbb{R}$.

The following theorem shows that normalizers can expand the eligible controls.

Theorem 4.6 (Theorem 9 of Section 3.2.3 [20]). *Let* $\mathcal{F}_1 = \{\Phi_\#(X) : \Phi \in \mathcal{N}_S(\mathcal{F}), X \in \mathcal{F}\}$, *then it is an eligible expansion.*

In a simplified setting, we can consider polynomial vector fields on $\mathbb{R}^d$. A vector field $A(x)$ is called a *polynomial vector field* if the each coordinate is a polynomial in the variable $x_1, \ldots, x_n$. In general $A(x)$ can be written as

$$A(x) = \sum_{k=0}^{p} A^{(k)}(x) \tag{4.1}$$

where $A^{(k)}(x)$ is a homogenous polynomial vector field of order k, so $A^{(k)}(\lambda x) = \lambda^k A^{(k)}(x)$. One useful fact of polynomial vector fields is the following:

Lemma 4.7. *Let* $A(x)$ *be a polynomial vector field of highest order* p *with decomposition* (4.1). *Suppose* $\{A, \lambda X, \lambda \in \mathbb{R}\} \subset \mathcal{F}$, *and* X *is a constant vector field. We consider the Lie bracket as a linear mapping from vector fields on* $\mathbb{R}^d$ *to themselves, and write* $adV : Y \mapsto [V, Y]$. *Let* $\mathcal{F}' = \mathcal{F} \cup \{\lambda(adV)^p A^p, \lambda > 0\}$, *then it is an eligible expansion of* $\mathcal{F}$.

Proof. Since both X and $-X$ are in $\mathcal{F}$, for any fixed $\lambda > 0$, $\exp(\pm\lambda X)x \in \mathcal{A}_{\mathcal{F}}(x)$, so by Lemma 4.5, $\exp(\lambda X)$ is a normalizer. By Theorem 4.6, we can expand $\mathcal{F}$ by including $\exp(\lambda X)_\# A$. By taking the k-th order derivative of λ over the expression $\exp(\lambda X)_* A \exp(-\lambda X)$, one will find the following Taylor expansion, which holds for general vector fields X and A, as long as the sum converges,

$$\exp(\lambda X)_\# A = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (adX)^k A.$$

In our particular case, the convergence is trivial, since the series are nonzero only for the first p terms. This is because X is a constant vector field, so $[X, V]$ is of order $k-1$ if V is of order k, and $[X, V] = 0$ if V is a constant vector field as well. Due to Theorem 4.4, we can also include $\lambda^{-p} \exp(\lambda X)_\# A$ for the expansion. Then by Theorem 4.3, we can eligibly include the limit of $\lambda' \lambda^{-p} \exp(\lambda X)_\# A$ with any fixed $\lambda' > 0$ and $\lambda \to \infty$, which is clearly $\frac{\lambda'}{k!}(adX)^p A^{(p)}$ in this case. □

Using the argument of Lemma 4.7, one important result of [20] is that if A is of an odd order, then hypoellipticity implies controllability, when the controls are over constant vector fields. The fact that A is odd, makes $(adX)^p A$ and $(ad(-X))^p A$ constant vector fields of opposite sign, so one can gradually expand $\mathcal{F}$ like in the proof of hypoellipticity. Unfortunately for us, the system (3.1-3.2) is of order two. But still we can show controllability if we carefully do the expansions.

Proposition 4.8. *Under Assumption 1.2, the joint system (3.1-3.2) is strongly controllable, and also time-T strongly controllable for all $T > 0$.*

Proof. Recall that $\mathcal{I}_n^+$ is defined as follow, $\mathcal{I}_0^+ = \{k \in \mathcal{I}_+ | k \in \mathcal{I}_0 \text{ or } -k \in \mathcal{I}_0\}$ and

$$\mathcal{I}_n^+ = \mathcal{I}_{n-1}^+ \cup \{k \in \mathcal{I}_+ : k = (m+n)^+ \text{ or } (m-n)^+, m, n \in \mathcal{I}_{n-1}, a_{m,n} \neq 0\}.$$

Based on Theorems 4.1 and 4.2, if suffices to show that starting from

$$\mathcal{F}_0 = \text{span}\left\{Y, \frac{\partial}{\partial I_k}, \frac{\partial}{\partial R_k}\middle| k \in \mathcal{I}_0^+\right\},$$

we can find a sequence of eligible expansions such that

$$\mathcal{F}_j \supset \text{span}\left\{Y, \frac{\partial}{\partial I_k}, \frac{\partial}{\partial R_k}\middle| k \in \mathcal{I}_j^+\right\}.$$

This can be easily obtained by induction. Suppose the relation above holds for $\mathcal{F}_j$, then for any $k \in \mathcal{I}_{j+1}^+$, there are $m, n \in \mathcal{I}_j^+$ so that k is $(m+n)^+$ or $(m-n)^-$, while $a_{m,n} \neq 0$. Let

$$X = c\frac{\partial}{\partial R_m} + \frac{\partial}{\partial R_n},$$

which is a constant vector field in $\mathcal{F}_j$. Apply Lemma 4.7, we can expand $\mathcal{F}_j$ eligibly by adding $\lambda(adX)^2Y$, $\lambda > 0$. According to 3.3,

$$\begin{aligned}(adX)^2Y &= c\left[\frac{\partial}{\partial R_n}, \left[\frac{\partial}{\partial R_m}, Y\right]\right] + c\left[\frac{\partial}{\partial R_m}, \left[\frac{\partial}{\partial R_n}, Y\right]\right] \\ &= 2ca_{m,n}\frac{\partial}{\partial R_{(m+n)^+}} + 2ca_{m,n}\frac{\partial}{\partial R_{(m-n)^+}},\end{aligned}$$

because $a_{m,n} = a_{n,m}$ and $a_{m,m} = a_{n,n} = 0$. Following the same line, if we let

$$X' = c\frac{\partial}{\partial I_m} + \frac{\partial}{\partial I_n},$$

we can eligibly add

$$(adX')^2Y = -2ca_{m,n}\frac{\partial}{\partial R_{(m+n)^+}} + 2ca_{m,n}\frac{\partial}{\partial R_{(m-n)^+}}.$$

Since c here can be any real number, by Theorem 4.4, we can expand $\mathcal{F}_j$ eligibly into $\mathcal{F}_{j+1}$ so that it includes span$\{\frac{\partial}{\partial R_{(m+n)^+}}, \frac{\partial}{\partial R_{(m-n)^+}}\}$, and likewise also span$\{\frac{\partial}{\partial I_{(m+n)^+}}, \frac{\partial}{\partial I_{(m-n)^+}}\}$, using the another half of (3.3). This concludes the induction and also the proof. □

5 Reachability is necessary for ergodicity

Although in our turbulent system (1.3), the proof for hypoellipticity was easily extended to one for reachability, thanks to [20], in general reachability is not replaceable by hypoellipticity. In particular, reachability guarantees that the the state space is irreducible while hypoellipticity only shows the regularity of the transition density.

In this section, we will show a simple concrete diffusion process in $\mathbb{R}^2$ that 1) has the quadratic Euclidean norm squared as a Lyapunov function; 2) is hypoelliptic and the stochastic forcing is never zero; 3) is reducible so it is not ergodic. The basic intuition for the reducibility part is showing there are two regions that are separated by the drift vector fields, so the diffusion process cannot enter one from another.

Consider the following diffusion process $z_t = (x_t, y_t)$

$$d\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \Phi(y_t)(9x_t - x_t^3) \\ -y_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dB_t^1 + \begin{bmatrix} \sin\psi(x_t) \\ \cos\psi(x_t) \end{bmatrix} \circ dB_t^2 = Y(z_t)dt + \Sigma_1 dB_t^1 + \Sigma_2(z_t) \circ dB_t^2. \tag{5.1}$$

Here Φ is one plus the cumulative distribution function of the standard normal distribution:

$$\Phi(y) = 1 + \int_{-\infty}^{y} \phi(u)du = 1 + \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

We only need to use that Φ is smooth, takes value in $(1, 2)$ and is strictly increasing. The function $\psi(x)$ in (5.1) is a combination of three mollifiers:

$$\psi(x) = \begin{cases} \frac{\pi}{2}\exp(-1/(1-x^2)), & |x| \le 1, \\ 0, & 1 \le |x| \le 2, \\ \frac{\pi}{2}\exp(-1/(|x|-2)), & |x| \ge 2. \end{cases}$$

We only need to use that ψ is smooth, ψ is bounded by $\pi/2$ and positive, and $\dot{\psi}$ is bounded by π.

5.1 Lyapunov function and attractors

The Itô integral form of (5.1) is

$$d\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \Phi(y_t)(9x_t - x_t^3) + \frac{1}{4}\dot{\psi}(x_t)\sin 2\psi(x_t) \\ -y_t - \frac{1}{2}\dot{\psi}(x_t)\sin(\psi(x_t))^2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dB_t^1 + \begin{bmatrix} \sin\psi(x_t) \\ \cos\psi(x_t) \end{bmatrix} dB_t^2.$$

Using the Itô formula, we find the time derivative of $V_t := \frac{1}{2}(x_t^2 + y_t^2)$ is bounded by:

$$\begin{aligned} \mathcal{L}V_t &= \Phi(y_t)(9x_t^2 - x_t^4) + \pi(|x_t| + |y_t|) + \sin^2\psi(x_t) - y_t^2 + \cos^2\psi^2(x_t) \\ &\le 18x_t^2 - x_t^4 - y_t^2 + \pi(|x_t| + |y_t|) + 1 \\ &= -\frac{1}{2}x_t^2 - \frac{1}{2}y_t^2 - \left(\frac{19}{2} - x_t^2\right)^2 - \frac{1}{2}(|y_t| - \pi)^2 - \frac{1}{2}(|x_t| - \pi)^2 + 1 + (\frac{19}{2})^2 + \pi^2 \\ &\le -V_t + k_v. \end{aligned}$$

So by Dynkin's formula and Gronwall's inequality, we have that V_t is a Lyapunov function.

$$\mathbb{E}V_t \le e^{-t}V_0 + k_v.$$

5.2 Hypoellipticity Verification

By Proposition 1.4, it suffices to check the following three vector fields span $\mathbb{R}^2$ at each (x, y):

$$\Sigma_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} \sin\psi(x) \\ \cos\psi(x) \end{bmatrix}, \quad [Y, \Sigma_1] = \left[\begin{bmatrix} \Phi(y)(9x - x^3) \\ -y \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]. \tag{5.2}$$

When $|x| \notin [1, 2]$, $\sin\psi(x) > 0$, the first two vector fields in (5.2) already span $\mathbb{R}^2$. For $|x| \in [1, 2]$, notice that

$$\left[\begin{bmatrix} \Phi(y)(9x - x^3) \\ -y \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} \phi(y)(9x - x^3) \\ -1 \end{bmatrix}$$

as $\phi(y) > 0$ and $x(x - 3)(x + 3)$ has no root in $[1, 2]$, the vector field above spans $\mathbb{R}^2$ with Σ_1.

5.3 Reducibility from "walls"

The intuition behind system (5.1) is setting up two attracting fixed points, $(\pm 3, 0)$ so one cannot be reached from another following the flow. This is achieved by building two walls: $A_+ = \{1 \le x \le 2\}, A_- = \{-2 \le x \le -1\}$ and show that no points can go from the right of A_+ to the left of it. This can be explicitly illustrated as the following lemma, which can be used as a proof for reducibility.

Lemma 5.1. *Let $B_+ = \{1 < x\}$ then starting from any $z_0 = (x_0, y_0) \in B_+$, $x_t \ge 1$ a.s.*

Proof. Since x_t is continuous, if x_t ever reach B_+^c, it has to cross A_+. So by the strong Markov property it suffices to show that for any $z_0 \in A_+$, $x_t \ge 1$ a.s. This is equivalent of showing that if we denote the exiting time of A_+ as τ, i.e.:

$$\tau := \inf\{t : x_t \notin A_+\},$$

then $x_\tau \ne 1$ a.s. Note for any $T \ge 0$, before time τ, $x_t \in A_+$, so:

$$\begin{aligned} x_{\tau\wedge T} &= x_0 + \int_0^{\tau\wedge T} \Psi(y_t)(9x_t - x_t^3)dt + \frac{1}{4}\dot{\psi}(x_t)\sin 2\psi(x_t)dt + \sin\psi(x_t)dW_t^2 \\ &= x_0 + \int_0^{\tau\wedge T} \Psi(y_t)(9x_t - x_t^3)dt \ge x_0 > 1. \end{aligned}$$

□

Remark 5.2. *In [21], Remark 2.2 has another simple example in 1D with hypoellipticity but reducible:*

$$dx_t = \sin x_t + \cos x_t \circ dB_t.$$

The bracket condition holds, because $[\sin x, \cos x] = -1$, while intervals like $[2n\pi - \frac{3}{2}\pi, 2n\pi + \frac{3}{2}\pi]$ are not reachable from one and another. But stochastic forcing here is zero at $(n + \frac{1}{2})\pi$, while our 2D model has forcing nonzero everywhere, and the Euclidean norm squared is a Lyapunov function.

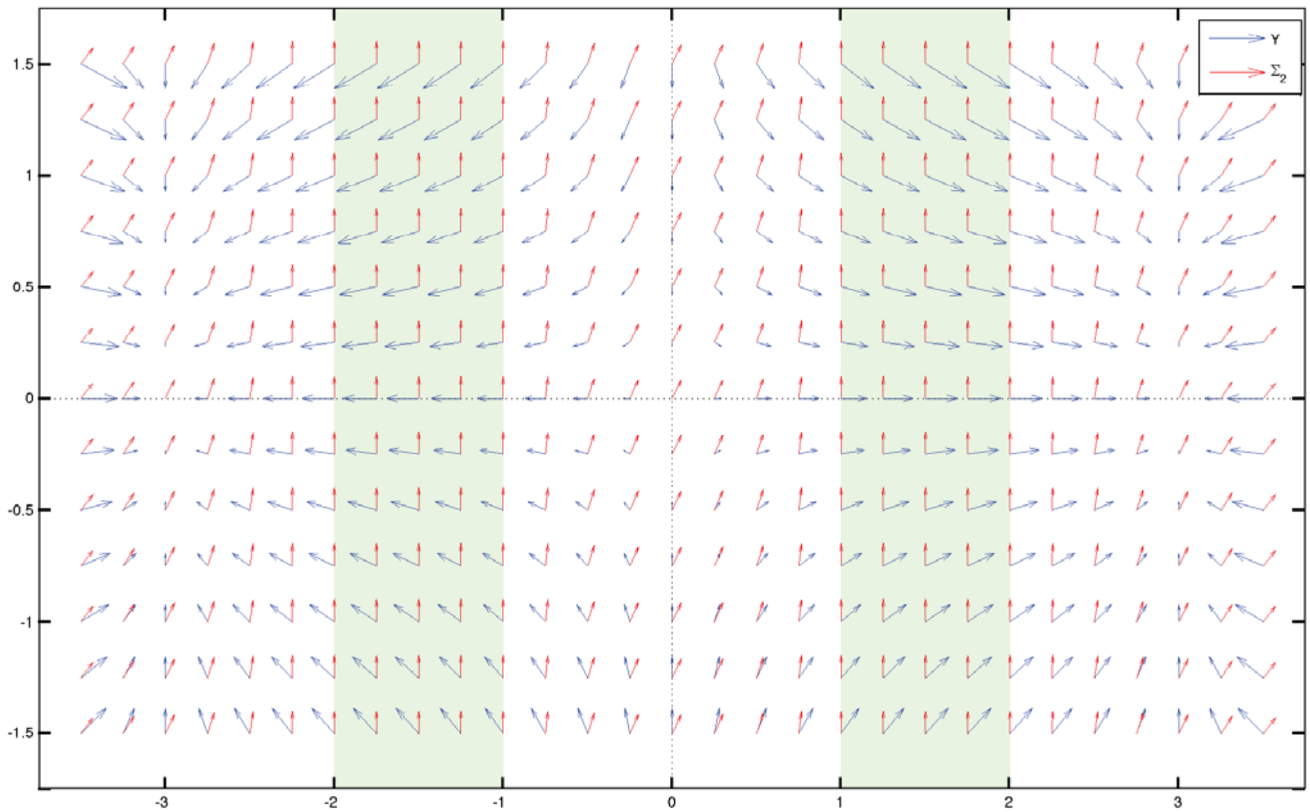

Figure 5.1: The plot of the vector fields Y and Σ_2. Recall that Σ_1 is the constant vector field of direction $[0, 1]$, so it is omitted here. The flow can go in the blue direction, the red direction and the inverse of red direction. In the right shaded area A_+, the flow can only go rightward. The vectors are rescaled to fit into the page.

6 Concluding discussion

The main result of the paper for truncated geophysical turbulence models is geometric ergodicity with a unique invariant measure and minimal stochastic forcing for all geophysical parameters involving deterministic forcing, topography, and the β-plane and F-plane effects. This theorem provides a mathematically rigorous framework to discuss and explain the ultimate statistical steady state in the competition between jets and coherent vortices in the wide variety of numerical experiments with random stochastic and deterministic forcings and dissipation operators. In particular, this rigorous theory guarantees that there are no bifurcations to multiple statistical steady states as geophysical parameters are varied. Future problems which should be addressed by the same approach include the extension to geophysical models on the sphere where forcing two stochastic modes is not enough [3], two-layer models with baroclinic instability [19, 13] and various equations for rotating and stratified turbulence in three space dimensions [19, 13, 14]. The extension of the results here to the infinite dimensional setting [14] is a major challenge for future work.

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A Miscellaneous claims

Lemma A.1. *If $\mathcal{I}$ is connected through neighbors on $\mathbb{Z}^2$ and contains $\{k : \max\{|k_1|, |k_2|\} = 1\}$ as a subset, then Assumption 1.2 holds as long as $\mathcal{I}_0$ contains modes $(0, 1)$ and $(1, 1)$.*

Proof. By symmetry, $(0, -1), (-1, 1) \in \mathcal{I}_0$. Therefore $(1, 0) = (0, -1) + (1, 1)$ is in $\mathcal{I}_1$ and so is $(-1, 0)$. Likewise $(-1, 1) = (-1, 0) + (0, 1)$ and $(1, -1)$ are in $\mathcal{I}_2$. Because $\mathcal{I}$ is connected, so for any $k \in \mathcal{I}$, there is a path $(1, 0) = k_1, k_2, \dots, k_n = k$ where k_i and k_{i+1} are neighbors in $\mathbb{Z}^2$. One can then easily see $k_i \in \mathcal{I}_i + 1$, so our proof is complete. □

Lemma A.2. *The growth condition (1.7) holds for the truncated stochastic Navier Stokes. In particular, we show that $\mathcal{E}^{2^n}$ with all $n \in \mathbb{N}$, and $\exp(\lambda\mathcal{E})$ with $\lambda \in (0, 2d_0/\|\Sigma\|^2)$ are Lyapunov functions.*

Proof. As a matter of fact, because the Markov property, (2.2) implies that the Itô formula for $\mathcal{E}$ can be bounded by the following with $K_0 = \frac{1}{2d_0}|\mathbf{F}|^2 + \frac{1}{2}\mathrm{tr}(Q_\sigma)$

$$d\mathcal{E} = \frac{1}{2}\langle \mathbf{q}, d\mathbf{q}\rangle + \frac{1}{2}\langle d\mathbf{q}, \mathbf{q}\rangle \le -d_0\mathcal{E}dt + K_0 + \frac{1}{2}\langle \mathbf{q}, \Sigma d\mathbf{W}_t\rangle + \frac{1}{2}\langle \Sigma d\mathbf{W}_t, \mathbf{q}\rangle.$$

We apply the Itô formula to $\mathcal{E}^2$, by plugging the inequality above into

$$d\mathcal{E}^2 = 2\mathcal{E}d\mathcal{E} + \langle d\mathcal{E}, d\mathcal{E}\rangle,$$

we find that

$$d\mathcal{E}^2 \le -2d_0\mathcal{E}^2dt + 2K_0\mathcal{E} + 2\|\Sigma\|^2\mathcal{E} + \mathcal{E}[\langle \mathbf{q}, \Sigma d\mathbf{W}_t\rangle + \langle \Sigma d\mathbf{W}_t, \mathbf{q}\rangle].$$

Apply Young's inequality, there is a K_1 such that

$$d\mathcal{E}^2 \le -d_0\mathcal{E}^2dt + K_1 + \mathcal{E}[\langle \mathbf{q}, \Sigma d\mathbf{W}_t\rangle + \langle \Sigma d\mathbf{W}_t, \mathbf{q}\rangle].$$

By repeating this argument, there is a sequence of constant K_n such that

$$d\mathcal{E}^{2^n} \le -d_0\mathcal{E}^{2^n}dt + K_n + 2^{n-1}\mathcal{E}^{2^n-1}[\langle \mathbf{q}, \Sigma d\mathbf{W}_t\rangle + \langle \Sigma d\mathbf{W}_t, \mathbf{q}\rangle].$$

Then applying the Grönwall's inequality, we find that $\mathcal{E}^{2^n}$ is a Lyapunov function. As a consequence, $e^{d_0t}(\mathcal{E}^{2^n} - d_0^{-1}K_n)$ is a submartingale. So Doob's inequality and Jensen's inequality implies that $\mathbb{E}\sup_{t\le T}|\mathbf{q}|^p$ is bounded for any power p.

Likewise, applying Itô's formula to $\exp(\lambda\mathcal{E})$, we find that

$$\begin{aligned} d\exp(\lambda\mathcal{E}) &= \lambda\exp(\lambda\mathcal{E})d\mathcal{E} + \frac{1}{2}\lambda^2\exp(\lambda\mathcal{E})\langle d\mathcal{E}, d\mathcal{E}\rangle \\ &\leq (\frac{1}{2}\lambda^2\|\Sigma\|^2\mathcal{E} - d_0\lambda\mathcal{E} + K_0)\exp(\lambda\mathcal{E})dt + \frac{1}{2}\exp(\lambda\mathcal{E})\langle\mathbf{q}, \Sigma d\mathbf{W}_t\rangle + \frac{1}{2}\exp(\lambda\mathcal{E})\langle\Sigma d\mathbf{W}_t, \mathbf{q}\rangle. \end{aligned}$$

As a consequence, for any $\lambda \in (0, 2d_0/\|\Sigma\|^2)$, there is a $E_0 > 0$ and $d_0' > 0$ such that if $\mathcal{E} > E_0$, then

$$\frac{1}{2}\lambda^2\|\Sigma\|^2\mathcal{E} - d_0\lambda\mathcal{E} + K_0 < -d_0'.$$

So if we let

$$K_0' = \max_{0\leq x\leq E_0}(\frac{1}{2}\lambda^2\|\Sigma\|^2 x - d_0\lambda x + K_0)\exp(\lambda x)$$

we find the dissipative relation

$$d\exp(\lambda\mathcal{E}) \leq -d_0'\exp(\lambda\mathcal{E})dt + K_0' + \frac{1}{2}\exp(\lambda\mathcal{E})\langle\mathbf{q}, \Sigma d\mathbf{W}_t\rangle + \frac{1}{2}\exp(\lambda\mathcal{E})\langle\Sigma d\mathbf{W}_t, \mathbf{q}\rangle.$$

This with a localization argument can easily show that $\exp(\lambda\mathcal{E})$ is a Lyapunov function for $\lambda < 2d_0/\|\Sigma\|^2$. By turning it into a submartingale, it is easy to argue that

$$\mathbb{E}\sup_{t\leq T}\exp(\lambda\mathcal{E}) < \infty.$$

Note that $\lambda\mathcal{E} + \frac{1}{2\lambda}N^2 \geq N|\mathbf{q}|$, it is clear that

$$\mathbb{E}\sup_{t\leq T}\exp(N|\mathbf{q}|) < \infty, \tag{A.1}$$

for any $N > 0$.

We can now verify the moment bounds for derivative flows. Because in our case Σ are constant vector fields, we find that $J^\alpha_{0,t}$ follows a linear dynamics conditioned on the realization of $\mathbf{q}$

$$dJ_{0,t} = DY(\mathbf{q})J_{0,t}dt.$$

Likewise, the inverse derivative flow also follows a linear dynamics

$$dJ^{-1}_{0,t} = -J^{-1}_{0,t}DY(\mathbf{q})dt.$$

Therefore, $\mathbb{E}[\sup_{t\leq T}\|J_{0,t}\|^p]$ and $\mathbb{E}[\sup_{t\leq T}\|J^{-\mathbf{1}}_{0,t}\|^p]$ are both bounded by $\exp(Tp\sup_{t\leq T}\|DY(\mathbf{q})\|)$. Finally, we notice that $Y(\mathbf{q})$ depends quadratically in $\mathbf{q}$, so $\|DY((\mathbf{q})\|$ is bounded by $M|\mathbf{q}|$ for some M, so using (A.1) we can conclude our claim.

As for the higher order derivative, we denote the $J^\alpha_{0,t}$ as the higher order Frechet derivative in the iterative direction $\alpha = (\alpha_1, \ldots, \alpha_k)$, and $\beta_1 + \cdots \beta_n = \alpha$ if $\{\beta_j\}$ is a partition of $(\alpha_1, \ldots, a_n)$. Then using induction, and the fact that Y is quadratic so the third derivative is zero, we find there is a constant C_k such that

$$dJ^\alpha_{0,t} \leq DY(\mathbf{q})J^\alpha_{0,t}dt + C_k\sup\left\{\|D^2Y(\mathbf{q})\|\prod_{j\geq 1}\|J^{\beta_j}_{0,t}\|\right\},$$

while the supreme is taken over all $\sum \beta_j = \alpha$ with β_j being nonempty and not α itself. By Grönwall's inequality, there is a constant D_k such that

$$\sup_{t\leq T} \|J_{0,t}^{(k)}\| \leq D_k \exp\left(T \sup_{t\leq T} \|DY(\mathbf{q})\|\right) \sup_{t\leq T} \left\{ \|D^2 Y(\mathbf{q})\| \prod_{j\geq 1} \|J_{0,t}^{\beta_j}\| \right\}.$$

Notice that $D^2 Y(\mathbf{q})$ is simply a constant tensor, by Young's inequality, $\mathbb{E}\sup_{t\leq T} \|J_{0,t}^{(k)}\|^p$ can be bounded by combinations of

$$\mathbb{E}\exp\left(T \sup_{t\leq T} p' \|DY(\mathbf{q})\|\right) \quad \text{and} \quad \mathbb{E}\|J_{0,t}^{\beta_j}\|^{p_j},$$

of finite p' and p_j. Yet the quantities above are bounded by (A.1) or induction. □

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