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Prediction, State Estimation, and Uncertainty Quantification for Complex Systems

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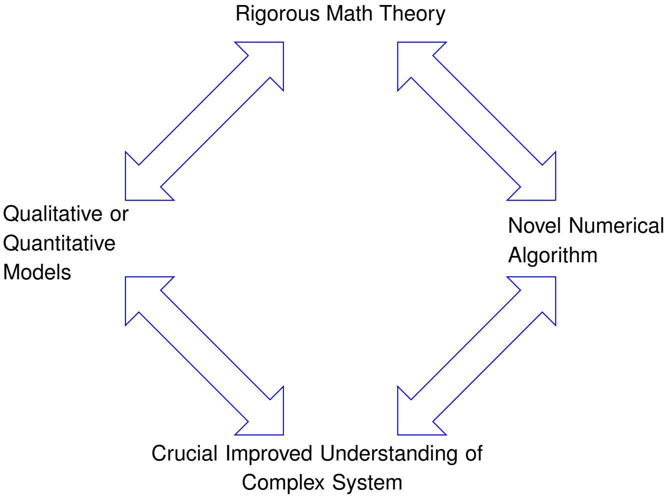
Grand Challenge in Climate Science as Extremely Complex System

- Important societal impacts: predicting long range weather forecasting (intraseasonal to interannual) and short term (decadal) climate change.
- Turbulent dynamical system: huge phase space and large dimension of instabilities.
- Other examples, engineering turbulence, neural science, material science.
- Need statistical, stochastic, thinking combined with nonlinear dynamics ideas.

Central Applied Math/Science Issues

1. Accurate prediction and representation of suitable statistics for observations from nature.
2. Model error: lack of physical understanding and inadequate resolution due to curse of ensemble size, computational overload in generating even small number of ensemble members is overwhelming.
3. Uncertainty quantification (UQ) accurate bounds for 1) and 2).
4. Low order models which achieve 1) and 3) while coping with 2) in an optimal fashion.
5. Rapid data assimilation or filtering to aid prediction.

Modern Applied Math Paradigm



Physics Constrained Nonlinear Regression Models for Time Series

Often we have a low-dimensional physical variable to observe and would like a low-order nonlinear stochastic model for its behavior.

Example: Low-frequency teleconnection patterns in the atmosphere.

1. Systematic Physical Derivation of Model.

   Physically constrained model by energy principles and exchange (Majda, Timofeyev & Vanden-Eijnden, 1999, *PNAS*).

   Problem: Requires sufficient separation of time scales, not always satisfied; need to know detailed models.

2. Ad hoc Quadratic Multi-Level Regression Model.

   Use data outcome to fit a quadratic regression model (Kravtsov, Ghil 2005; Wikle and Hooten 2010); allows nonlinearity and memory in time.

   Problem: No physical info in model. Majda and Yuan (*DCDS* 2012) show rigorously that such ad hoc nonlinear regression strategies can exhibit finite time blow-up and pathological invariant measures even though they fit the data with high precision.

1. Systematic Physical Derivation of Model.
2. Ad hoc Quadratic Multi-Level Regression Model.

Remedy: Systematic Physics Constrained Multi-level Quadratic Regression Models (Majda & Harlim, *Nonlinearity*, 2012).

Combine attractive features of (1) and (2) systematically to avoid pathology of (2) by physical constraints while allowing memory in effects.

Many more accessible mathematical issues and problems for systems!! blow-up and non-blow-up Thms.

A Hierarchy of Models for Predicting and Understanding

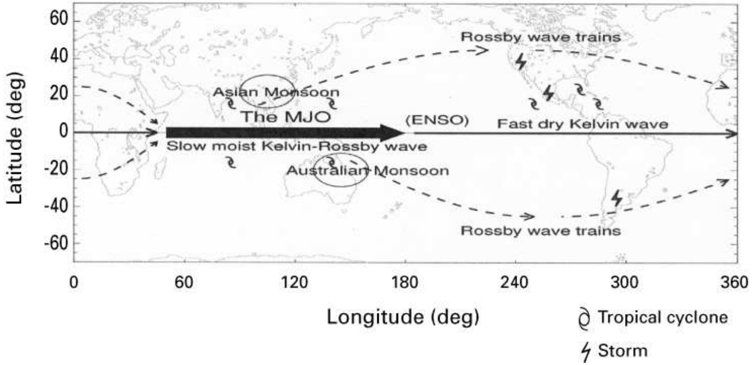
I. The Madden-Julian Oscillation (MJO)

the dominant component
of tropical intraseasonal variability

Global impact of MJO

The MJO affects

- El Niño-Southern Oscillation
- Monsoons
- Tropical cyclones
- Midlatitude predicability



from Moncrieff, Shapiro, Slingo, & Molteni, “Collaborative research at the intersection of weather and climate”, *WMO Bulletin*, 2007.

Novel Nonlinear Time-Series Techniques to Capture both Intermittency & Low-Frequency Variability in Massive Data Sets

Nonlinear Laplacian Spectral Analysis (NLSA)

(Giannakis and Majda, *PNAS* 2012)

NLSA combines:

- Lagged embedding
- Machine learning
- Adaptive weights
- Spectral entropy criteria

NLSA is applied to the data sets of dimensions $O(10^6)$!

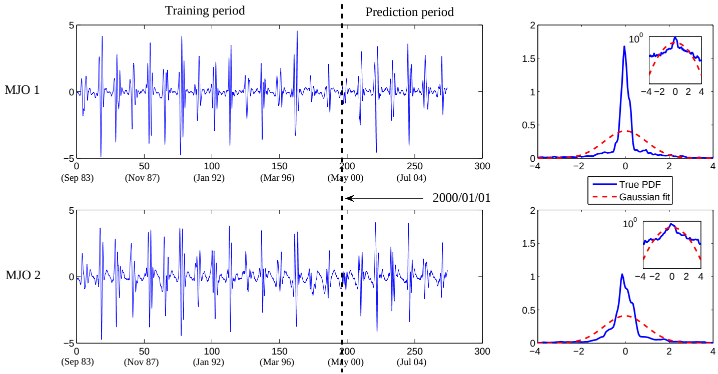
- Applications with W. Tung and E. Szekely to OLR for cloud patterns from tropics, MJO and Monsoon.
- Applications with M. Bushuk to Arctic sea ice reemergence.

Predicting Cloud Patterns of MJO through Low-Order Stochastic Models

(Nan Chen, Majda, Giannakis, *GRL* 2014)

(Nan Chen, Majda, *MWR* 2015)

NLSA Time-Series Techniques $\Longrightarrow$ 2 components of MJO Cloud Patterns



Intermittent bursts of MJO activity

Physics-Constrained Low-Order Nonlinear Stochastic Model for Predicting MJO Cloud Patterns (MJO1, MJO2)

Physics-Constrained Low-Order Stochastic Model

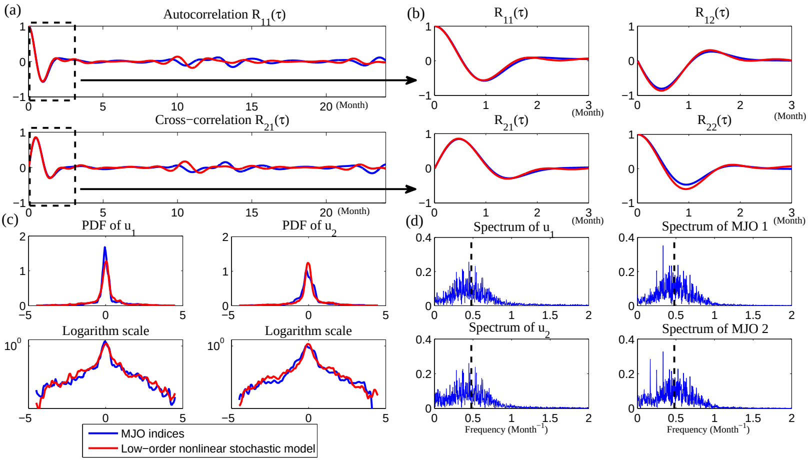
$$\begin{aligned}
du_1 &= (-d_u\, u_1 + \gamma\,(v + v_f(t))\, u_1 - (a + \omega_u)\, u_2)dt + \sigma_u\, dW_{u_1},\\
du_2 &= (-d_u\, u_2 + \gamma\,(v + v_f(t))\, u_2 + (a + \omega_u)\, u_1)dt + \sigma_u\, dW_{u_2},\\
dv &= (-d_v\, v - \gamma\,(u_1^2 + u_2^2))dt + \sigma_v\, dW_v,\\
d\omega_u &= (-d_\omega \omega_u + \hat{\omega}_u)dt + \sigma_\omega\, dW_\omega,
\end{aligned}$$

with

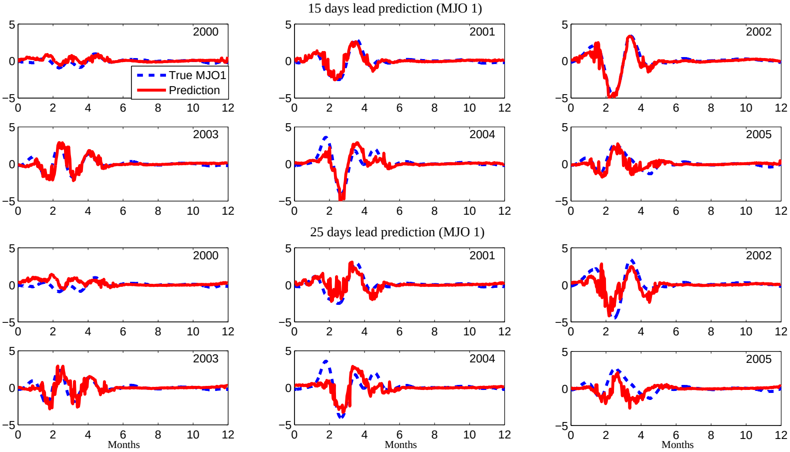
$$v_f(t) = f_0 + f_t \sin(\omega_f t + \phi).$$

- Observed variables u_1, u_2: MJO 1 and MJO 2 indices from NLSA.
- Hidden variables v, ω: stochastic damping and stochastic phase.
- Energy-conserving nonlinear interactions between (u_1, u_2) and (v, ω_u) (Majda and Harlim, Nonlinearity 2012).
- Effective data assimilation algorithm incorporating into prediction scheme.

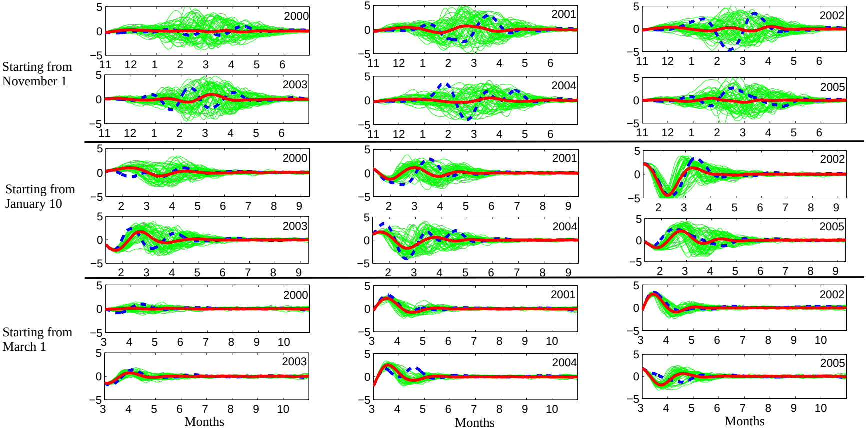
Calibration of parameters using *Information Theory* (Robust parameters)
Model vs. Observations: Non-Gaussian statistics match



Skillful prediction at 15- and 25-days lead times



Varying Start Date of Prediction



Ensemble spread $\Longleftrightarrow$ long-range forecast uncertainty is captured

II. Hierarchy of Models for MJO

A New Model for the MJO

Majda and Stechmann 2009 *PNAS*
“The Skeleton of Tropical Intraseasonal Oscillations”

Majda and Stechmann 2009 *JAS*
“Nonlinear Dynamics and Regional Variations in the MJO Skeleton”

Simultaneously captures all three fundamental features of the MJO skeleton:

1. Eastward propagation speed of ≈ 5 m/s
2. Peculiar dispersion relation of $\frac{d\omega}{dk} \approx 0$
3. Horizontal quadrupole vortex structure

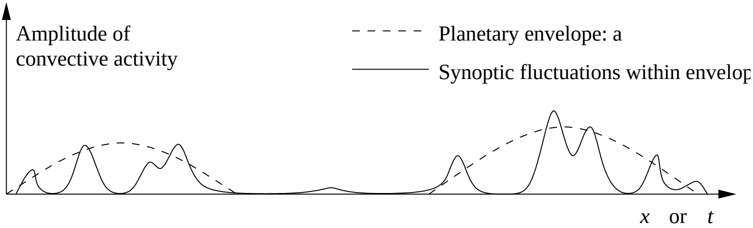
Fundamental mechanism proposed for MJO skeleton

Minimal, nonlinear oscillator model

Neutrally stable interactions between

1. planetary-scale, lower-tropospheric moisture: q
2. sub-planetary-scale, convection/wave activity: a

Based on multi-scale concepts



Tacit assumption: primary instabilities/damping occur on synoptic scales

Minimal nonlinear oscillator model

$$
\begin{aligned}
u_t - yv &= -p_x \\
yu &= -p_y \\
0 &= -p_z + \theta \\
u_x + v_y + w_z &= 0 \\
\theta_t + w &= \bar{H}a - s^\theta \\
q_t - \tilde{Q}w &= -\bar{H}a + s^q
\end{aligned}
$$

$$a_t = \Gamma q a$$

Linearized primitive equations

- Equatorial long-wave scaling
- Coriolis term: equatorial β-plane approx.

+

Dynamic equation for convective activity

- q: lower tropospheric moisture anomaly
- a: amplitude of convective activity envelope

Key mechanism: positive q creates a tendency to enhance convective activity a

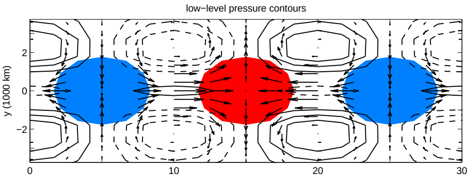
Minimal number of parameters: $s^\theta, \tilde{Q}, \Gamma$

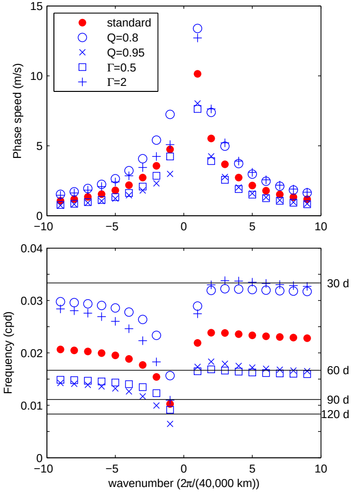
Observational evidence, Waliser 2003

Linear Theory

Simultaneously captures all three fundamental features of the MJO skeleton:

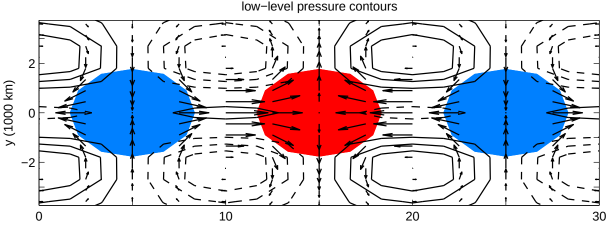
1. Eastward propagation speed of ≈ 5 m/s
2. Peculiar dispersion relation of $\frac{d\omega}{dk} \approx 0$
3. Horizontal quadrupole vortex structure





MJO Skeleton Index for identifying & monitoring MJO activity

Theoretical prediction of MJO structure



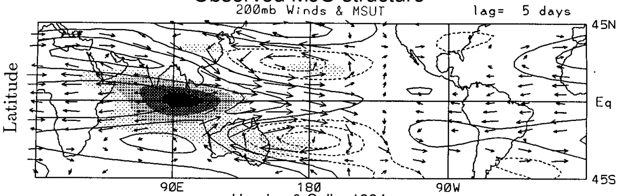
suppressed convection ($A < 0$)

enhanced convection ($A > 0$)

Ingredients:

- Kelvin wave structure on equator
- ***Rossby gyre structure off equator***

Observed MJO structure



Hendon & Salby 1994

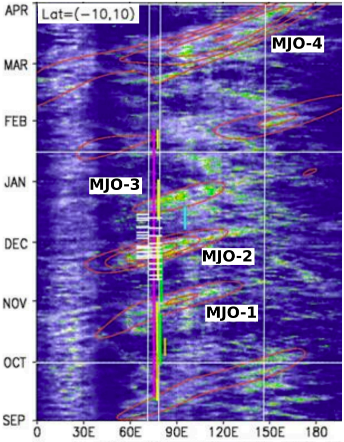
Motivation for Stochastic Skeleton Model

Need to capture:

1. intermittent generation of MJO events
2. organization of MJO events into wave trains (with growth and demise of wave trains)

Wave train of 2–3 MJO events ⟶

MJO events during DYNAMO/CINDY 2011–2012



Yoneyama et al 2014

A Stochastic Skeleton Model for the MJO

Thual, Majda, & Stechmann 2014 *JAS*

Replace $\partial_t a = \Gamma q a$

with *stochastic jump process* for growth/decay of a,

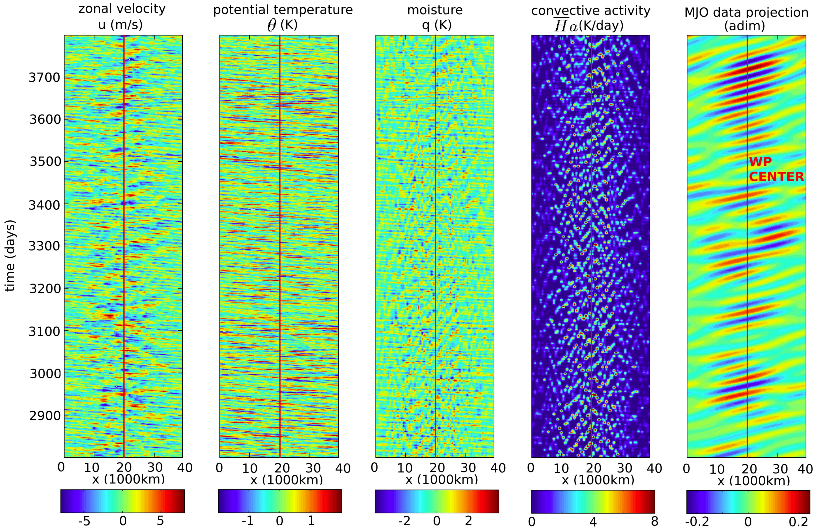
which satisfies $\partial_t \langle a \rangle = \Gamma \langle q a \rangle$ in the mean

Intuition:

Growth/decay of convective activity is *stochastic*,

due to unresolved synoptic/mesoscale fluctuations

Space-time variability



- intermittent generation of MJO events
- organization of MJO events into wave trains

(Geometric ergodicity, Majda and Xin Tong, CPAM 2015)

MJO event statistics in skeleton model and observations

(Stachnik, Waliser, Majda, Stechmann, Thual)

Number of MJO events:

Event Type	Observations 1979–2012	Stochastic Skeleton Model Idealized warm pool, 34 yrs
Primary	154	106
Continuing	330	381
Circumnavigating	15	27
Terminal	154	106

Average Duration of MJO events:

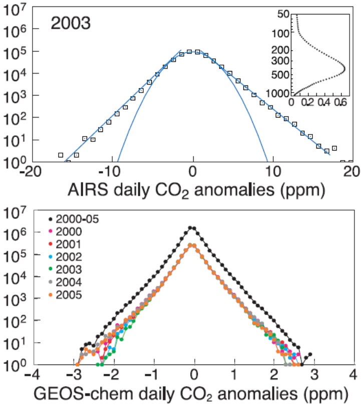
Observations: 39.7 days
Stochastic Skeleton Model: 34.8 days

Stochastic Skeleton Model reproduces *Observed MJO Statistics*

Rigorous Mathematical Models with Intermittency and Extreme Events

Neelin et al, *GRL*, 2011, CO and CO_2, probability distribution function (PDF) exhibit intermittency and extreme event in observations.

– Fat tails (nearly exponential) compared with Gaussian.



Model CO_2 as passive tracer with a mean gradient.

Exactly solvable test models with realistic features in climate change science

$$\frac{\partial T}{\partial t} + \vec{v}(\vec{x},t)\cdot\nabla T = \kappa \Delta T.$$

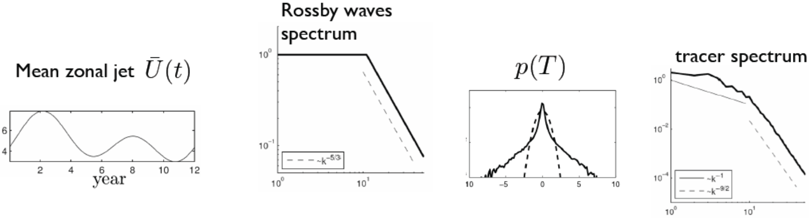
Turbulent velocity

$$\vec{v}(\vec{x},t) = (U(t), v(x,t))^T$$

$U(t), v(x,t)$ known random field.

Passive tracer with mean gradient

$$T = \alpha y + T'(x,t)$$



(Research expository: Majda and Gershgorin, *Phil. Roy. Soc.* 2013; Bourlioux and Majda, *Phys Fluids* 2002; Majda and Gershogorin, *PNAS* 2011, 2012)

- Model error and stochastic parameterization

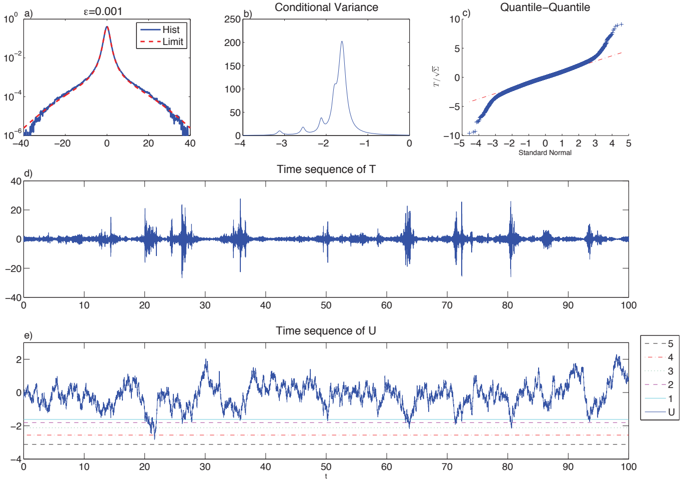
$$\frac{\partial T^M}{\partial t} + \vec{\bar{v}}^M\cdot\nabla T^M = (\kappa + \kappa_{eddy})\Delta T^M + \sigma_T \dot{W}.$$

- Extreme event prediction with model error (Di Qi and Majda, *Phys. D* 2015)

Rigorous analysis of extreme events

(Majda and Xin Tong, *Nonlinearity* 2015)

Rigorous PDF v.s. Simulation



Intermittent bursts occur when the random mean flow, $U(t)$, gets close to a certain resonant set, rigorous analysis.

Information-Theoretic Framework, Information Barrier and Improving Predictive Skill with Model Error

(Majda and Gershgorin, *PNAS*, 2011, 2012)

Information-theoretic framework is extensively applied in the study of model error, predictive skill and data assimilation. The following three information-theoretic measures are widely used,

1. The Shannon entropy of the residual $S(\mathbf{u} - \mathbf{u}^M)$ measures the uncertainty in the model $\mathbf{u}^M$ compared with the truth $\mathbf{u}$. It is the surrogate for the RMS error in the path-wise sense.

$$S(\mathbf{U}) := -\int p(\mathbf{U}) \ln p(\mathbf{U}) d\mathbf{U}, \qquad \mathbf{U} = \mathbf{u} - \mathbf{u}^M.$$

2. The mutual information $M(\mathbf{u}, \mathbf{u}^M)$ measures the dependence between $\mathbf{u}$ and $\mathbf{u}^M$. It is the surrogate for the anomaly pattern correlation in the path-wise sense.

$$M(\mathbf{u}, \mathbf{u}^M) := \int\int p(\mathbf{u}, \mathbf{u}^M) \ln \frac{p(\mathbf{u}, \mathbf{u}^M)}{\pi(\mathbf{u})\pi^M(\mathbf{u}^M)} d\mathbf{u} d\mathbf{u}^M.$$

3. The relative entropy $P(\pi, \pi^M)$ quantifies the lack of information or model error in the statistics of $\mathbf{u}^M$ relative to that of $\mathbf{u}$. It is also an indicator of assessing the disparity in the amplitudes and peaks between $\mathbf{u}^M$ and $\mathbf{u}$.

$$P(\pi, \pi^M) := \int \pi(\mathbf{u}) \ln \frac{\pi(\mathbf{u})}{\pi^M(\mathbf{u})} d\mathbf{u}.$$

A simple example with an intrinsic barrier for improving model sensitivity

Perfect model:
$$\frac{du}{dt} = au + v + F, \qquad \frac{dv}{dt} = qu + Av + \sigma \dot{W}.$$

Smooth Gaussian measure if $a + A < 0, \quad aA - q > 0.$

Imperfect model:
$$\frac{du_M}{dt} = -\gamma_M u_M + F_M + \sigma_M \dot{W}_M, \qquad \gamma_M > 0.$$

- Climate fidelity for imperfect model

$$\frac{F_M}{\gamma_M} = -\frac{AF}{aA - q}, \quad \frac{\sigma_M^2}{2\gamma_M} = \frac{\sigma^2}{2(a + A)(aA - q)} \equiv E.$$

- Response to change in forcing

$$\delta u = -\frac{A}{aA - q}\delta F, \quad \delta u_M = \frac{1}{\gamma_M}\delta F.$$

- Information model error in response to change in forcing

$$P(\pi_\delta, \pi_\delta^M) = \frac{1}{2}E^{-1}\left|-\frac{A}{aA - q} - \frac{1}{\gamma_M}\right|^2 |\delta F|^2 \qquad \text{for perfect model fidelity.}$$

With $A > 0$, the attempt to minimize the information theoretic model error is futile because no finite minimum over γ_M is achived and necessarily $\gamma_M \to \infty$ in the approach to the minimum – **intrinsic information barrier**.

Improving the predictive skill of imperfect models for complex systems in their response to external forcing

$$\text{Perfect system:} \quad u_t = F(u) + \sigma(u)\dot{W},$$
$$\text{Perturbed system:} \quad u_t^\delta = F(u^\delta) + \delta f(t) + \sigma(u^\delta)\dot{W}.$$

- Equilibrium statistical fidelity – a necessary condition.
- Combining the information theory with linear response theory in improving the predictive fidelity.
  - Leading order correction to the statistics of functional $A(u)$ for small δ,

$$\delta\langle A(u)\rangle = \int_0^t R_A(t-s)\delta f(s)ds,$$

    $R_A(t)$ – the linear response operator calculated through correlation functions in the unperturbed climate.
  - Improving the predictive skill by minimizing the model error to response,

$$P\left(\pi_\delta, \pi_\delta^M\right) = S(\pi_{G,\delta}) - S(\pi_\delta) + \frac{1}{2}\bar{\sigma}^{-2}\left(\int_0^t (R_{\bar{u}}(t-s) - R_{\bar{u}}^M(t-s))\delta f(s)ds\right)^2$$
$$+ \frac{1}{4}\bar{\sigma}^{-4}\left(\int_0^t (R_{\bar{\sigma}^2}(t-s) - R_{\bar{\sigma}^2}^M(t-s))\delta f(s)ds\right)^2 + O(\delta^3).$$

Examples and applications.

- Improving response in the turbulent tracer model: Majda and Gershgorin, *PNAS* 2011; Di Qi and Majda, 2015
- Low order models and climate change forcing: Majda and Di Qi, *JNLS*, 2015
- Intermittent models: Branicki and Majda, *Nonlinearity* 2012
- Model error in data assimilation: Branicki and Majda, *Comm. Math. Sci.*, 2014
- Low order model prediction: Nan Chen and Majda, *GRL* 2014, *MWR* 2015, *MCWF* 2015
- Prediction of Rogue waves: Cousins and Sapsis, *Phys D* 2014, *JFM* 2015.

UQ: Strategy Blending Info Theory with Statistical Response Theory and Statistical Energy Principle

Goal: Build a low order model for UQ for the change in response to external forcing.

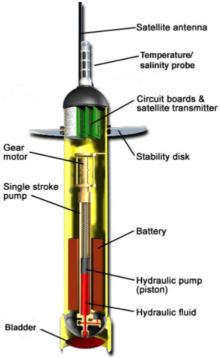
$$\text{Model} \Longleftrightarrow \text{Calibration phase} \Longleftrightarrow \text{Prediction.}$$

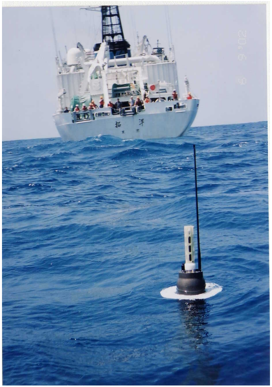
1. Model:
   - **A.** Low order stochastic model with coefficients depending on total energy of system
   - **B.** Utilize new statistical energy constrained principle (Majda, *PNAS* 2015)
2. Calibration phase: Combines info theory and kicked statistical response. One (or a few) expensive runs needed of full model.
3. Prediction for UQ: By solving low order stochastic model, achieve accurate UQ estimates for mean and variance in response of system to change in general forcing.

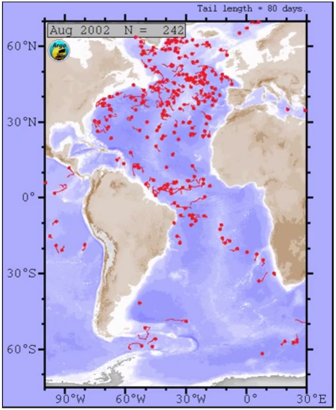
(Majda and Di Qi, *JNLS* 2015; Sapsis and Majda, *PNAS* 2013; Di Qi and Majda, Complex Geophysical Model, 2016)

Inverse Problems and Data Assimilation

Lagrangian Tracers: Oceanography







C. Jones, A. Apte, A. Stuart, ...

Inverse Problem: Noisy Lagrangian Tracers in Filtering Geophysical Flows

First rigorous math theory

(Nan Chen, Majda, Xin Tong, *Nonlinearity* 2014, *JNLS* 2015)

Observing L noisy trajectories $X_j(t)$,

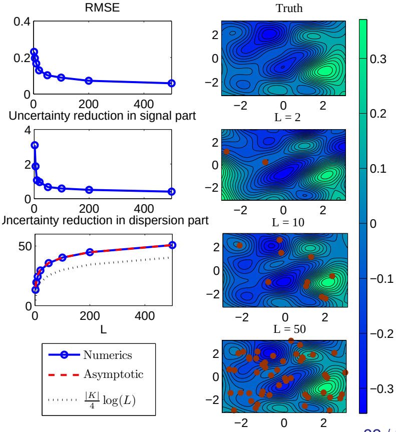
$$\frac{dX_j}{dt} = v(X_j(t), t) + \sigma_j \dot{W}_j.$$

Recover or estimate the velocity $\vec{v}$.

- Inherent nonlinearity in measurement.
- Build exact closed analytic formulas for the optimal filter for the velocity field.
- Prove a mean field limit at long times.

1. Recovering random incompressible flows

- Show an exponential increase in the number of tracers for reducing the uncertainty by a fixed amount – **a practical information barrier**.



2. Noisy Lagrangian tracers for filtering random rotating compressible flows

(Nan Chen, Majda, Xin Tong, *JNLS* 2015)

- Rotating shallow water models with multiscale features:
  - Slow modes – random incompressible geostrophically balanced (GB) flows.
  - Fast modes – random rotating compressible gravity waves.
- Highly nonlinear observations mixing GB and gravity modes.
- Proposing different filters.
  - Full filter – full forecast model & tracer observations.
  - Ideal reference GB filter – GB forecast model & GB observations.
  - Reduced filter – GB forecast model & mixed observations – a practical inexpensive imperfect filter.
- Rigorous math theory: Comparable high skill in recovering GB modes for all the filters in the geophysical scenario with small Rossby number.

Lessons for UQ and Failure of Polynomial Chaos

– Research expository: Majda and Branicki, *DCDS*, 2012.
– Exactly solvable test models for polynomial chaos: Branicki and Majda, *Comm. Math. Sci.*, 2013.

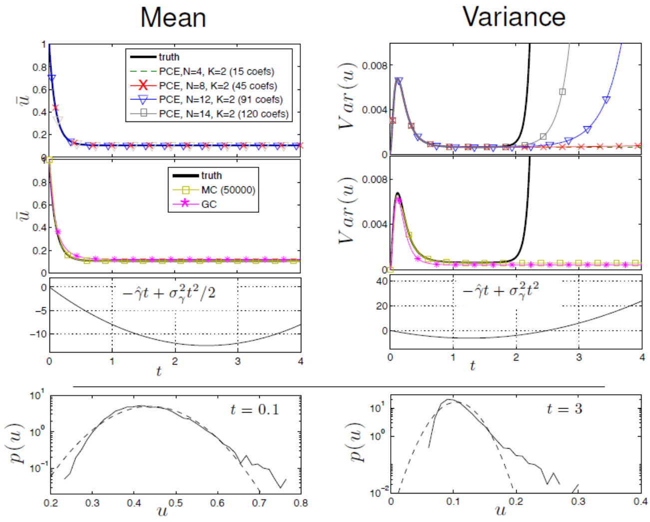
Failure of PC and even Monte Carlo with very large ensemble size.

- Simplest example: Linear ODE with parametric uncertainty

$$\dot{u} = -(\gamma + \sigma_\gamma \xi)u + f(t).$$

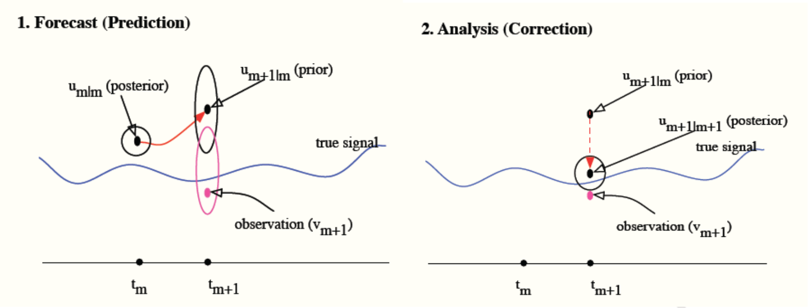
where parametric uncertainty is Gaussian random variable $\sigma_\gamma \xi$, ξ is $\mathcal{N}(0, 1)$.
Easy to exactly solve equations for mean, variance and any moment in time.

Failure of polynomial chaos and straightforward Monte Carlo.



Both PC with 120 coefficients and MC with 50,000 samples fail to predict the variance with any accuracy!

Filtering the Turbulent Signals



Filtering is a two-step process involving statistical prediction of the state variables through a forward operator followed by an analysis step at the next observation time which corrects this prediction on the basis of the statistical input of noisy observations of the system.

Practical Issue

- Turbulent dynamical system.
- Huge phase space, $N = O(10^6, 10^8, etc)$.
- Nonlinearity, small ensemble size $M = O(50, 100)$.

Applied algorithm

- Finite ensemble Kalman filter, (Evensen, 1995; C. Bishop, J. Anderson 2001; Kalnay, 2013). See M-H book.

Applied math

- Stuart, Reich,...

Central issues

- Why does EnKF often work well to estimate the mean with $M \leq N$?

Surprising pathology

- Catastrophic filter divergence. For filtering forced dissipative system with absorbing ball property such as L-96 model, EnKF can explode to machine infinity in finite time! (Harlim and Majda 2008; Gottwald and Majda, *NPG* 2013)

Well posedness of EnKF

- Kelly, Law, Stuart, *Nonlinearity* 2014.

Rigorous nonlinear stability for finite ensemble Kalman filter (EnKF)

(Xin Tong, Majda, Kelly, *Nonlinearity* 2015)

Filter divergence – a potential flaw for EnKF:

- Catastrophic filter divergence: the ensemble members diverging to infinity,
- Lack of stability: the ensemble members being trapped in locations far from the true process.

Finding practical conditions and modifications to rule out filter divergence with rigorous analysis:

- Ruling out catastrophic filter divergence by establishing an energy principle for the filter ensemble.
- Looking for energy principles inherited by the Kalman filtering scheme.
- Looking for modification schemes of EnKF that ensures an energy principle and preserving the original EnKF performance (Xin Tong, majda, Kelly, *Comm. Math. Sci.*, 2015).
- Verifying the nonlinear stability of EnKF through geometric ergodicity.

Rigorous example of catastrophic divergence:

- For filtering a nonlinear map with absorbing ball property (Kelly, Majda, Xin Tong, *PNAS* 2015).

Outstanding problem: Why and when is there accuracy in mean for $M \leq N$?

Need Statistically Accurate Inexpensive Forecast Models to Beat the Curse of Ensemble Size for Prediction, State Estimation and UQ

The MMT equation

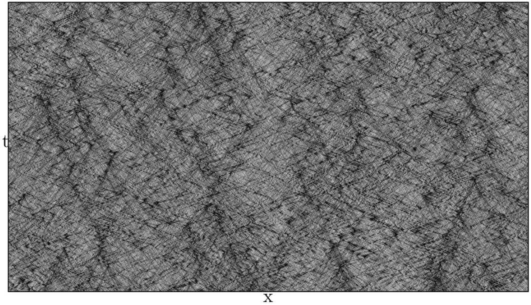
The MMT equation (Majda, McLaughlin and Tabak, 1997; Cai and M.M.T., *Phys. D* 2001)

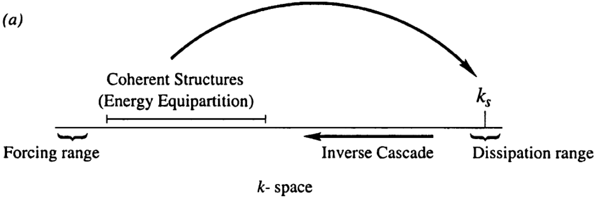
$$iu_t = |\partial_x|^{\frac{1}{2}} u + \lambda |u|^2 u - iAu + F.$$

Here we consider the case with the focusing nonlinearity, $\lambda = -1$, which induces spatially coherent 'solitonic' excitations at random spatial locations.

- The instability of collapsing solitons radiate energy to large scales producing direct and inverse turbulent cascades.
- In geophysical applications energy oftern flows from small scales to large scales (inverse cascade) creating a challenge for reduced modelling.
- Fractional dispersion are crucial with completely different behavior from NLS equation!

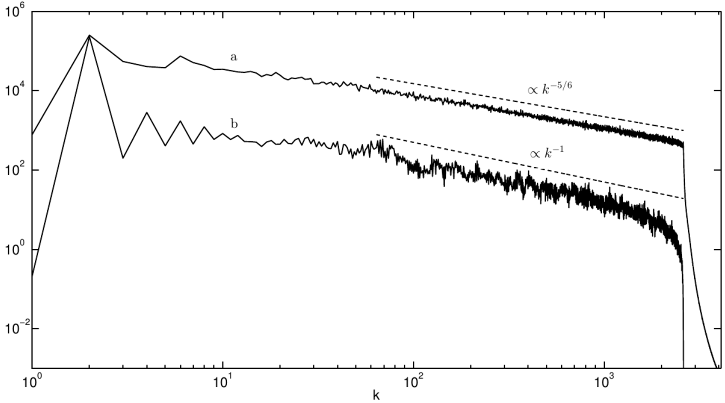
Visualization of $|\psi(x,t)|$ from simulation with $F_0 = 0.0163$; darker colors indicate higher amplitudes. Here the number of Fourier modes are $64^2 \approx 4000$.





From Cai etal, *Physica D* 2001.

High-resolution reference simulations

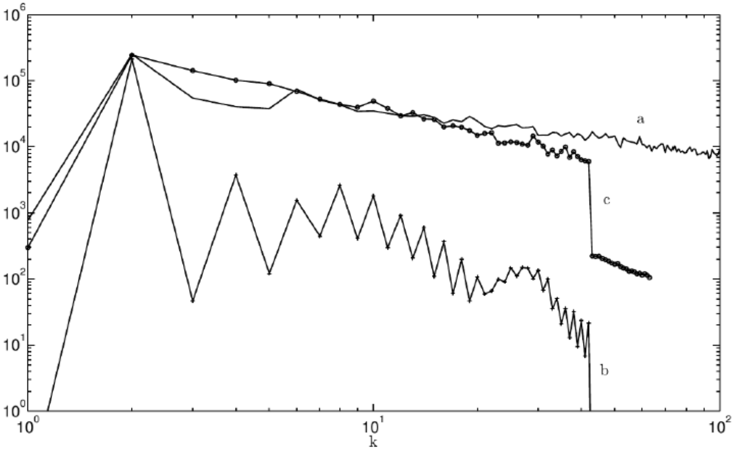


Simulation (a) uses $F_0 = 0.0163$; (b) uses $F_0 = 0.01625$. Both simulations are damped only for $2600 < |k| < 4096$ and $|k| = 1$.

Uses of MMT model:

1. Novel low order modelling: stochastic superparameterization (Majda and Grooms, *JCP* 2013; Grooms and Majda, *Comm. Math. Sci.* 2014).
2. Novel data assimilation (Branicki and Majda, *JCP* 2012; Grooms, Lee and Majda, *JCP* 2014)
3. Extreme event prediction (Cousins and Sapsis, *Phys. D.* 2014)

Stochastic Superparameterization in MMT



Spectra from simulations with $1/64$ as many points as the reference simulation (a), with no eddy terms (b) and with eddy terms (c).

Stochastic Superparameterization

1. A general framework for stochastic subgridscale modelling with no scale separation and no small-scale equilibration based on the Gaussian closure approximation and the point approximation.
2. Success in a difficult test problem with no scale separation ($k^{-5/6}$ spectra), coherent structures, dispersive waves, and an inverse cascade from unresolved scales into the large scales.
3. Overcome *curse of ensemble size* with judicious model error.

– See research expository article Majda and Grooms, *JCP* 2013; Grooms and Majda, PNAS, *JCP* 2013 for geophysical turbulence.

– See Khouider, Biello and Majda, *Comm. Math. Sci.* 2010; Deng, Khouider and Majda, *JAS* 2015 for stochastic multi-cloud model.

Multiscale Data Assimilation in Complex Turbulent System

Superparameterization (SP) and Multiscale Data Assimilation.

- Tremendously large dimension of turbulent signals requires cheap and robust coarse models for real prediction skills.
- SP is a cheap and robust under-resolved forecast model; approximates the large scale dynamics and provides small-scale statistics to estimate both the resolved and unresolved components of the true signal.
- Multiscale data assimilation framework – provides the estimate for the large-scale dynamics using SP as a coarse forecast model and partial observations of the true signal.
- Multiscale data assimilation shows robust filtering performance with a huge computational savings; better performance than other ad hoc approaches in the conventional (single-scale) data assimilation such as covariance inflation.

(Harlim & Majda, *SIAM. J. MMS*, 2013; Grooms, Lee & Majda, *JCP* 2014, *MWR* 2015; Lee & Majda, *SIAM. J. MMS*, 2015)

Blended particle filters for large dimensional chaotic dynamical systems.

Goal: Developing statistically accurate particle filters to capture non-Gaussian features in large dimensional chaotic dynamical systems.

- Space decomposition $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2),\ u_j \in \mathbb{R}^{N_j},\ N_1 + N_2 = N,\ N_1 \ll N.$
- Blended filters:
  - Particle filter – non-Gaussian statistics of $\mathbf{u}_1$.
  - Kalman filter – conditional Gaussian statistics $\mathbf{u}_2$ given $\mathbf{u}_1$.
- Attractive feature – adaptively change of the subspaces as time evolves in response to the uncertainty without a separation of time scales using nonlinear statistical forecast models.

Nonlinear statistical forecast models:

- QG-DO – quasilinear Gaussian dynamical orthogonality method.
- MQG-DO – more sophisticated modified QG-DO method.

(Majda, Di Qi & Sapsis, *PNAS* 2014; Di Qi & Majda, *Phys D.* 2015; Sapsis & Majda, *Phys D.* 2012; *PNAS* 2013)

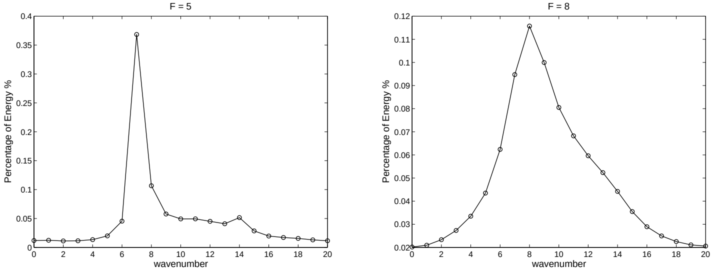
Lorenz 96 system

The Lorenz 96 system is a discrete periodic system described by the equations

$$\frac{du_j}{dt} = (u_{j+1} - u_{j-2})u_{j-1} - u_j + F, \qquad j = 0, \ldots, J-1,$$

with $j = 40$ the number of grids and F_i the deterministic forcing. See Majda & Harlim book (2012). The quadratic part conserves energy. We will study the case of weakly chaotic turbulence ($F = 5$), strongly chaotic turbulence ($F = 8$).

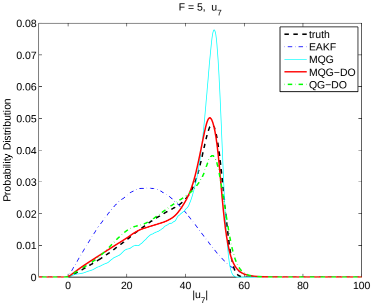
5 dim subspace of particles is used.

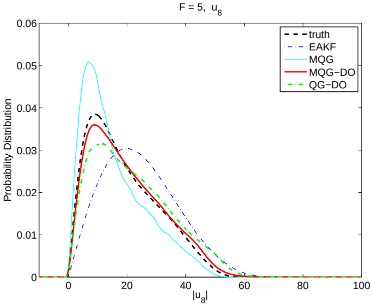


(a) $F = 5$ (b) $F = 8$

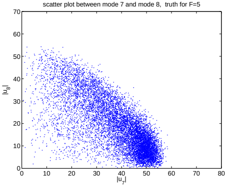
Figure: Energy Spectra

Capturing non-Gaussian statistics $F = 5, r_0 = 2, \Delta t = 1, p = 4$.

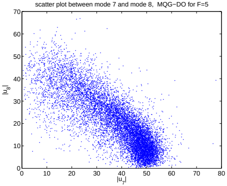




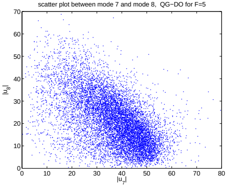
Regime scatter plot: mode u_7, u_8



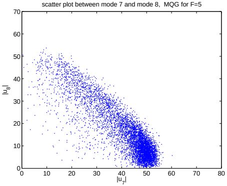
(c) truth



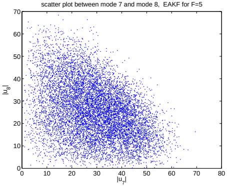
(d) MQG-DO



(e) QG-DO



(f) MQG



(g) EAKF

Stochastic Parameterized (Nonlinear) Extended Kalman Filter (SPEKF) and Dynamic Stochastic Superresolution (DSS)

- Cheap stochastic forecast models with judicious model error which are statistically exactly solvable and learn stochastic parameters “on the fly” from data
- DSS exploits SPEKF together with aliasing to achieve superresolution for subgrid scale filtering

References:

- Majda and Harlim, Filtering Complex Turbulent Systems (*Cambridge press* 2012)
- Keating, Majda and Smith, Ocean turbulence (*MWR* 2012)
- Branicki and Majda, Intermittency, black swans, wave turbulence (*JCP* 2012)
- Nan Chen, Giannakis, Majda and Herbei, MCMC algorithm for intermittency (*SIAM/ASA JUQ* 2014)
- Branicki and Majda, Turbulent Navier-Stokes (*JCP* 2016)

Thank you