Climate Science, Waves and PDEs for the Tropics:

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Abstract A readers guide to recent applied mathematics development in multi-scale modeling in the tropics is provided here including the mathematical theory of precipitation fronts as well as singular limits with variable coefficients in the fast variables.

1 Introduction

One of the grand challenges of contemporary science is a comprehensive predictive model for the atmosphere and coupled climate system. This is one of the most difficult multiscale problems in contemporary science because there is an incredible range of strongly interacting anisotropic nonlinear processes over many spatiotemporal scales; contemporary comprehensive computer models, GCMs, are currently incapable of adequately resolving or parameterizing many of these interactions on time scales appropriate for seasonal prediction as well as climate change projections. An overview for mathematicians can be found in the recent article by the author [33].

Basic questions which drive climate research are the prediction of the weather from 1 to 14 days, the prediction of climate variations on seasonal to yearly time scales and finally, climate change projections on decadal and centennial time scales as well as quantifying the uncertainty associated with these predictions. One of the striking recent observational discoveries is the profound impact of variations in the tropics on all of these problems. The primary issue in the influence of the tropics occurs through the interaction and organization of clouds into clusters, super clusters, and planetary scale dynamics, an inherently fully nonlinear multiscale process.

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For climate change, water vapor is the most important greenhouse gas and the microphysical processes in clouds are a key mechanism for radiative feedback. In fact, only a 4 % change in average cloudiness would overwhelm the effects of CO2 in climate change.

A new perspective on several of the issues discussed above for climate dynamics has been developed through the paradigm of modern applied mathematics, where rigorous and asymptotic multiscale mathematical theory, the development of prototype model problems and novel computational strategies all interact simultaneously in understanding these complex scientific problems.

2 Multi-scale Models in the Tropics and the Madden-Julian Oscillation

The dominant component of intraseasonal variability in the tropics is the 40- to 50-day tropical intraseasonal oscillation, often called the Madden-Julian oscillation (MJO) after its discoverers [30, 31]. In the troposphere, the MJO is an equatorial planetary-scale wave envelope of complex multiscale convective processes. It begins as a standing wave in the Indian Ocean and propagates eastward across the western Pacific Ocean at a speed of roughly ≈ 5 m/s [68]. The planetary scale circulation anomalies associated with the MJO significantly affect monsoon development, intraseasonal predictability in mid-latitudes, and the development of the El Nino Southern Oscillation (ENSO) in the Pacific Ocean, which is one of the most important components of seasonal prediction [68, 28].

Despite the widespread importance of the MJO, present day computer general circulation models (GCMs) typically have poor representations of it [29]. A growing body of evidence suggests that this poor performance of GCMs is due to the inadequate treatment of interactions of organized tropical convection on multiple spatiotemporal scales [29, 52]. Such hierarchical organized structures that generate the MJO as their envelope are the focus of current observational initiatives and modeling studies [52], and there is a general lack of theoretical understanding of these processes and the MJO itself.

A large number of theories attempting to explain the MJO through mechanisms such as evaporationwind feedback [8, 53], boundary layer frictional convective instability [66], stochastic linearized convection [61], radiation instability [58], and the planetary-scale linear response to moving heat sources [7]. While they all provide some insight into the mechanisms of the MJO, these theories are all at odds with the observational record in various crucial ways [68, 28], and it is therefore likely that none of them captures the fundamental physical mechanisms of the MJO. Nevertheless, they are all interesting theories that contribute to our understanding of certain aspects of the MJO. Other insight has been gained through the study of MJO-like waves in multi-cloud model simulations [46, 23] and in super-parameterization computer simulations [13, 14, 15, 51], which appear to capture many of the observed features of the MJO by accounting for smaller scale convective structures within the MJO envelope. The role of convective momentum transport from synoptic scale waves in producing key features of the MJOs planetary scale envelope has also been elucidated by multi-scale asymptotic models [38, 2, 3, 5, 48]. Despite all of the interesting contributions listed above, no theory for the MJO has yet been generally accepted, and the problem of explaining the MJO has recently been called the search for the Holy Grail of tropical atmospheric dynamics [58].

Although theory and simulation of the MJO remain difficult challenges, they are guided by the generally accepted, fundamental features of the MJO (i.e., the MJOs skeleton) on intraseasonal/planetary scales, which have been identified relatively clearly in observations:
I. peculiar dispersion relation of $d\omega/dk \approx 0$ [60, 67, 59],
II. slow phase speed of roughly 5 m/s [16, 17, 49], and
III. horizontal quadrupole vortex structure [16, 17, 49].

The goal of a recent article [47] is to design the simplest dynamical model that captures and predicts the intraseasonal/planetary scale features of the MJOs skeleton in I–III, and to recover these features robustly throughout the parameter space of the model.

Geophysical flows are a rich source of fascinating problems for applied mathematicians involving complex multi-scale nonlinear systems, where energy cascades upward from the small scales to the large scales through anisotropic processes involving vortices and gravity waves. On the other hand, the improved parameterization of unresolved features of moist tropical convection is a central challenge in current computer models for long range ensemble forecasting of weather and short term climate with large worldwide societal impact [52]. The reason for this is the observed multi-scale features of organized coherent tropical convection across a wide range of scales varying from tens of kilometers and a few hours to the planetary scale of order 40,000 km on intraseasonal time scales with significant energy transfer across these scales [19, 50, 54, 67]. Recent processing of observational data [50] suggests the statistical self-similarity of tropical convection from the smallest, shortest scales to organized mesoscale convective systems [18] to convective clusters to equatorial synoptic-scale superclusters to planetary/intraseasonal oscillations. For this reason, it is interesting to develop systematic multiscale asymptotic models [26, 25, 27, 34, 43] for the nonlinear cascade across scales in the tropics, and the author has done this recently for the self-similar behavior from the microscales to mesoscales to planetary/intraseasonal scales [35, 36]. Such quantitative models are useful for quantifying the observed multiscale behavior in, for example, tropical intraseasonal oscillations [2, 3, 4, 5, 38, 48].

3 The Dynamics of Equatorial Waves: Singular Limits with Fast Variable Coefficients

Geophysical flows are a rich source of novel problems for applied mathematics and the contemporary theory of partial differential equations (PDE) ([34] and references therein). The reason for this is that many physically important geophysical flows involve complex nonlinear interaction over multi-scales in both time and space so developing simplified reduced models which are simpler yet capture key physical phenomena is of central importance [12, 56, 57, 34, **?**]. In mid-latitudes, the fact that the rotational Coriolis terms are bounded away from zero leads to a strict temporal frequency scale separation between slow potential vorticity dynamics and fast gravity waves; this physical fact leads to new theorems justifying the quasi-geostrophic mid-latitude dynamics even with general unbalanced initial data for both rapidly rotating shallow water equations and completely stratified flows [6, 9, 10, 39, 34]. The strategy in the above proofs is to adapt the classical framework of Klainerman and Majda for singular limits [24, **?**, 34] together with the important generalizations by Schochet [63, 62], which allow for fast wave averaging, to the dispersive systems of geophysical flows; it is well known that these theories require constant symmetric hyperbolic coefficients for the fast wave dynamics in order to obtain higher derivative estimates on the solution.

At the equator, the tangential projection of the Coriolis force from rotation vanishes identically so that there is no longer a time scale separation between potential vortical flows and gravity waves. This has profound consequences physically that allow the tropics to behave as a waveguide with extremely warm surface temperatures. The resulting behavior profoundly influences longer term mid-latitude weather prediction and climate change through hurricanes, monsoons, El Nino, and global teleconnections with the mid-latitude atmosphere. How this happens through detailed physical mechanisms is one of the most important contemporary problems in the atmosphere-ocean science community with a central role played by nonlinear interactive heating involving the interaction of clouds, moisture, and convection [57, 64, 38, 2, 37, 1, 11, 44]. The variable coefficient degeneracy of the Coriolis term at the equator alluded to earlier leads to both important new physical effects as well as fascinating new mathematical phenomena and PDE's [43, 38, 2, 37, 1, 11, 44]. Chapter 9 of ref. [34] provides an introduction to these topics for mathematicians while ref. [11] introduces and studies the simplest physical equatorial models with moisture. In this equatorial context, the new multi-scale reduced dynamical PDE models are even relatively recent in origin [43]. Thus, the need for additional PDE theory is very important for these disciplinary problems and this is the main topic of recent research [40] [41] [42].

4 Precipitation Fronts: A Novel Hyperbolic Free Boundary Problem in Several Space Variables

Precipitation fronts are the boundaries between the zones of extremely moist air (with constant precipitation) such as over the Indonesian marine continent, the Indian ocean, and Western Pacific, and the zones of extremely dry air in the tropics and subtropics that occur over areas such as the Galapagos islands at the equator or the Arabian peninsula in the subtropics. An important practical question in contemporary meteorology for long range weather prediction and climate change projections is what determines the boundaries of the precipitating fronts as well as their evolution in time. Such assessments are performed, for example, by the Intergovernmental Panel for Climate Change (IPCC) by running extremely complex general computer models called GCM's. An important practical issue with the GCM's is how they treat moisture and what type of moisture waves do they produce at large scales compared with those in nature. This is very subtle. Although the GCM's have millions of variables and run on the largest supercomputers, they still have grid spacings of order 50 km to 200 km. In addition many complex physical processes need to be parametrized often by adhoc recipes guided by physical intuition. These issues are discussed in detail in [11], [65], [55], and the references there.

A novel mathematical theory of precipitating fronts was put forward in [11], [65], [55] to address the above issues in idealized tropical climate models consisting of a shallow water system for the temperature, T, and velocity, $u = (u;v)$, coupled with an equation for the moisture or humidity, q, through a relaxation source term P representing the depletion of moisture through precipitation and the condensational heating of the atmosphere in making clouds and depending on the type of moisture parametrization. (See [11] for a detailed derivation).

A novel point of view for atmospheric science developed in [11], [65], [55] is to formally take the zero relaxation limit, $\varepsilon \to 0$, and to study the type of the emerging precipitation fronts in order to get analytic insight into the behavior at positive ε. This procedure shows formally that, in the limit $\varepsilon \to 0$, precipitation fronts are free boundaries where $U = (u;T;q)$ is continuous across them but ∇U, which formally solves the hyperbolic system derived, has jumps satisfying the Rankine–Hugoniot–type shock conditions ([32]). These considerations were utilized in [11] to build three distinct wave families, namely drying, slow moistening, and fast moistening precipitating fronts, with the last two families violating Lax's shock conditions. Nevertheless, careful numerical experiments demonstrate (see [11], [20], [21]) and additional mathematical theory (see [65]) established, at positive ε, the robust reliability of all three wave types as well as interesting half smooth traveling waves. Finally numerical simulations (see [55]), again at ε positive, confirm a theory for reflection and transmission of waves impinging on precipitation zones.

Given all the above mentioned formal results it is extremely interesting to pass to the zero relaxation limit and to prove rigorously the existence and uniqueness of suitable weak solutions for the limiting problem. This is the main topic of the recent paper [45].

There is (see [11]) an interesting more complex version of moisture dynamics involving coupling with the barotropic model. Obtaining higher order energy estimates for this problem is a hard unsolved problem. The models discussed here are excellent ones for understanding the precipitation fronts at large scales in GCM's. The actual behavior as observed in nature is captured in a much more realistic fashion by more complex multi-cloud models [22, 23] with large scale instability.

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