

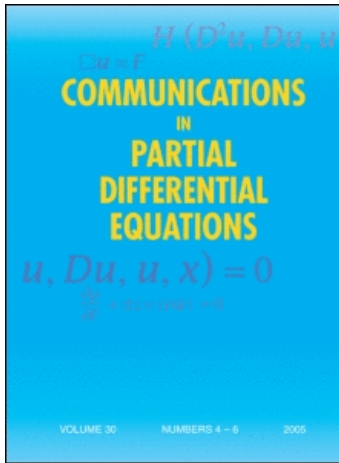
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Alexandre Dutrifoy^a; Andrew Majda^b

^a Chercheur Qualifié du F.N.R.S., à l'Université Libre de Bruxelles, Brussels, Belgium ^b Mathematics Department, Center for Atmosphere Ocean Science, Courant Institute of Mathematical Sciences, New York University, New York, New York, USA

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Fast Wave Averaging for the Equatorial Shallow Water Equations

ALEXANDRE DUTRIFOY¹ AND ANDREW MAJDA²

¹Chercheur Qualifié du F.N.R.S., à l'Université Libre de Bruxelles, Brussels, Belgium

²Mathematics Department, Center for Atmosphere Ocean Science, Courant Institute of Mathematical Sciences, New York University, New York, New York, USA

The equatorial shallow water equations in a suitable limit are shown to reduce to zonal jets as the Froude number tends to zero. This is a theorem of a singular limit with a fast variable coefficient due to the vanishing of the Coriolis force at the equator. Although it is not possible to get uniform estimates in classical Sobolev spaces (other than L^2) by differentiating the system, a new method exploiting the particular structure of the fast coefficient leads to uniform estimates in slightly different functional spaces. The computation of resonances shows that fast waves may interact with a strong external forcing, introduced to mimic the effects of moisture, to create zonal jets.

Keywords Equatorial shallow water equations; Fast averaging; Singular limit; Zonal jets.

AMS Subject Classification: 35Q35; 76N10; 76U05; 76M45.

1. Introduction

Geophysical equatorial flows are a rich source of novel problems both for applied mathematics and the theory of partial differential equations (see Majda, 2003 and references therein). The special feature at the equator is that the tangential projection of the Coriolis force from rotation vanishes identically there. Physically, this allows the tropics to behave as a waveguide with extremely warm surface temperatures, which influences the climate on a planetary scale through hurricanes, monsoons, El Niño, and global teleconnections with the mid-latitude atmosphere. The detailed physical mechanisms involved are the object of intensive studies in the atmosphere-ocean science community and also leads to new mathematical phenomena and PDE's (Biello and Majda, 2004, 2005; Frierson et al., 2004; Majda

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Address correspondence to Alexandre Dutrifoy, Chercheur Qualifié du F.N.R.S., à l'Université Libre de Bruxelles, Campus de la Plaine, CP 214, Boulevard du Triomphe, Brussels 1050, Belgium; E-mail: adutrif@ulb.ac.be

and Klein, 2003; Majda and Biello, 2003, 2004; Majda et al., 1999). Chapter 9 of Majda (2003) provides an introduction to these topics for mathematicians.

This is our second paper in a series about the rigorous derivation of reduced dynamics for flows in the equatorial region. A simple model for such flows is provided by the equatorial shallow water equations with dissipation and forcing

$$\begin{aligned} \partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} + \frac{1}{\epsilon} (y \vec{v}^\perp + \nabla h) &= -d \vec{v} + \frac{1}{\epsilon} (S_\epsilon^u, S_\epsilon^v) \\ \partial_t h + \vec{v} \cdot \nabla h + h \operatorname{div} \vec{v} + \frac{1}{\epsilon} \operatorname{div} \vec{v} &= -\tilde{d} h + \frac{1}{\epsilon} S_\epsilon^h, \end{aligned} \quad (1)$$

where $\vec{v} = (u, v)(t, x, y)$ is the horizontal velocity, $h = h(t, x, y)$ the height, x the longitude, y the distance to the equator, $\vec{v}^\perp = (-v, u)$, and, in the right-hand side, d, \tilde{d} are non-negative constant coefficients. The strong forcing terms $\epsilon^{-1} S_\epsilon^u, \epsilon^{-1} S_\epsilon^v, \epsilon^{-1} S_\epsilon^h$ are introduced to mimic the effects of convective heating. The equation (1) are written in non-dimensional variables under the assumption that both the Froude number (typical fluid velocity ratio to the gravity wave speed) and the height fluctuations are of order ϵ , which we regard here as a positive parameter tending to zero (its actual value is around 10^{-1}). (See Frierson et al., 2004; Majda and Klein, 2003.)

In our first paper (Dutrifoy and Majda, 2006) we studied the singular limit of the long-wave solutions of (1). That is, we first rescaled the system in the x -direction by setting $x' = \epsilon x$ and then showed that, under suitable assumptions, solutions converge as $\epsilon \rightarrow 0$ to solutions of the long-wave equations. In the present paper we go back to the original system (1) without rescaling. Instead of the long-wave equations, zonal jets constitute the slow limiting dynamics. Zonal jets are flows in the east-west direction alone ($v = 0$) and independent of the longitude x . Indeed, ignoring the forcing for the moment, we need

$$-yv + \partial_x h = 0 \quad (2)$$

$$yu + \partial_y h = 0 \quad (3)$$

$$\partial_x u + \partial_y v = 0 \quad (4)$$

if we want all terms of order ϵ^{-1} in (1) to vanish. Deriving (2) with respect to y , (3) with respect to x and subtracting gives $-v - y \partial_y v - y \partial_x u = 0$, hence $v = 0$ using (4). Then (2) gives $\partial_x h = 0$, so h must be independent of x , and so by (3) u must be independent of x too.

We do the fast averaging and obtain zonal jets as a singular limit (see Section 3 for more precise statements) by using the very same method that was introduced in Dutrifoy and Majda (2006) for the long-wave case. Our motivation is twofold.

From the mathematical point of view, as very few examples of singular limit of a symmetric hyperbolic system with fast variable coefficients have been treated previously (see Section 4 in Joly et al., 1995 for an example different from Dutrifoy and Majda, 2006), any new one probably deserves to be written in details. In mid-latitudes, the fact that the rotational Coriolis terms are bounded away from zero leads to a strict temporal frequency scale separation between slow potential vorticity dynamics and fast gravity waves; theorems justifying the quasi-geostrophic mid-latitude dynamics have been proved even with general unbalanced initial

data for both rapidly rotating shallow water equations and completely stratified flows (Bourgeois and Beale, 1994; Dutrifoy, 2004; Embid and Majda, 1996, 1998; Majda, 2003; Majda and Embid, 1998). However, the proofs require constant symmetric hyperbolic coefficients for the fast-wave dynamics in order to obtain higher derivative estimates on the solution. The rescaling $x' = \epsilon x$ in the long-wave scale has the technical advantage of removing the dependence on x in the fast coefficient operator: the terms of order ϵ^{-1} in the rescaled system only involve multiplication by y and derivation with respect to y . Still, even the dependence on y alone causes the serious difficulty that energy estimates in the usual Sobolev spaces blow up as $\epsilon \rightarrow 0$. Indeed, straightforwardly differentiating (SW_ϵ) with respect to y leads to terms with magnitude $O(\epsilon^{-1})$ from the commutators. This difficulty was overcome in Dutrifoy and Majda (2006) by exploiting the physical, particular structure of the fast operator to get a uniform estimate in a modified Sobolev space \tilde{W}^4 , denoting by \tilde{W}^m for any $m \in \mathbb{N}$ the space of functions $f \in L^2(\mathbb{T} \times \mathbb{R})$ such that

$$\sum_{\alpha+\beta+\gamma \leq m} \|y^\alpha \partial_x^\beta \partial_y^\gamma f\|_{L^2} < \infty.$$

With this estimate at hand, it is possible to follow the classical strategy for singular limits (Klainerman and Majda, 1981, 1982; Majda, 1984; Schochet, 1994a,b). We use the same method in the present paper. We wanted to gain some confidence in a setting only slightly different before attacking the fully stratified equations, which is our next goal and will be a more significant test of our method's generality. And we present here some novelty in the method as well, for we are now able to get uniform estimates in \tilde{W}^m for all $m \in \mathbb{N}$, in a way that would have also worked in the long-wave case (the presence of ∂_x in the operator happens to be harmless, simply because ∂_x commute with y and ∂_y).

From the geophysical point of view, the original scaling is completely different from the long-wave scaling and both have their own interest. Long waves and zonal jets are instances of those simplified reduced models which, being simpler and yet capturing qualitatively key physical phenomena, are so helpful in our understanding of the many physically important geophysical flows that involve complex nonlinear interactions over multiple scales, both in time and in space (Gill, 1982; Majda, 2003; Majda and Klein, 2003; Pedlosky, 1979; Philander, 1990). In the equatorial context, the new multi-scale reduced dynamical PDE models are relatively recent in origin (Majda and Klein, 2003) and additional PDE theory is needed for these disciplinary problems. Zonal jets are observed in the atmosphere as well as in the ocean but what brings them into existence is unclear. What we show here in our simple model is that although the interaction of fast waves between themselves has no influence on the mean flow (see Proposition 6.1 in Section 6.2), fast waves interact with an external forcing with fast oscillations to create slow waves (the zonal jets). Introducing that strong forcing (see Section 3) is an attempt to simulate the nonlinear interactive heating involving the interaction of clouds, moisture, and convection which plays a central role in equatorial dynamics (Biello and Majda, 2004, 2005; Frierson et al., 2004; Majda and Biello, 2003, 2004; Majda et al., 1999; Philander, 1990; Smith, 1997) (see Frierson et al., 2004 for the simplest physical equatorial models with moisture).

2. Reformulation

We do the same changes of variables as for the long-wave equations. The first one is

$$\tilde{h} = \frac{2h}{1 + \sqrt{1 + \epsilon h}}; \tag{5}$$

it transforms (1) into

$$\begin{aligned} \partial_t u + u \partial_x u + v \partial_y u + \frac{1}{2} \tilde{h} \partial_x \tilde{h} + \frac{1}{\epsilon} (-yv + \partial_x \tilde{h}) &= -du + \frac{1}{\epsilon} S_\epsilon^u \\ \partial_t v + u \partial_x v + v \partial_y v + \frac{1}{2} \tilde{h} \partial_y \tilde{h} + \frac{1}{\epsilon} (yu + \partial_y \tilde{h}) &= -dv + \frac{1}{\epsilon} S_\epsilon^v \\ \partial_t \tilde{h} + u \partial_x \tilde{h} + v \partial_y \tilde{h} + \frac{1}{2} \tilde{h} \partial_x u + \frac{1}{2} \tilde{h} \partial_y v + \frac{1}{\epsilon} (\partial_x u + \partial_y v) &= -\tilde{d}_\epsilon \tilde{h} + \frac{1}{\epsilon} S_\epsilon^{\tilde{h}} \end{aligned} \tag{6}$$

with

$$\tilde{d}_\epsilon = \tilde{d} - \frac{\epsilon \tilde{d} \tilde{h}}{4 + 2\epsilon \tilde{h}} \tag{7}$$

and

$$S_\epsilon^{\tilde{h}} = \frac{S_\epsilon^h}{1 + \frac{1}{2} \epsilon \tilde{h}}. \tag{8}$$

The second one is

$$r = \frac{1}{\sqrt{2}}(u + \tilde{h}), \quad l = \frac{1}{\sqrt{2}}(-u + \tilde{h}), \tag{9}$$

which gives

$$\partial_t \vec{U} + S_1^l(\vec{U}) \partial_x \vec{U} + S_2(\vec{U}) \partial_y \vec{U} + \frac{1}{\epsilon} \mathcal{L} \vec{U} = \mathcal{L}_{1,\epsilon} \vec{U} + \vec{F}_\epsilon \tag{10}$$

with the notation

$$\vec{U} = \begin{pmatrix} r \\ l \\ v \end{pmatrix}, \tag{11}$$

$$S_1^l = S_1^l(\vec{U}) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 3r - l & 0 & 0 \\ 0 & r - 3l & 0 \\ 0 & 0 & 2r - 2l \end{pmatrix} \tag{11}$$

$$S_2 = S_2(\vec{U}) = \frac{1}{4} \begin{pmatrix} 4v & 0 & r + l \\ 0 & 4v & r + l \\ r + l & r + l & 4v \end{pmatrix}, \tag{12}$$

$$\mathcal{L} = S_1^0 \partial_x + L \tag{13}$$

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where

$$S_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & L_- \\ 0 & 0 & L_+ \\ L_+ & L_- & 0 \end{pmatrix}, \tag{14}$$

in which L_+ and L_- denote the lowering and raising operators

$$L_{\pm} = \partial_y \pm y,$$

and finally

$$\mathcal{L}_{1,\epsilon} = -\frac{1}{2} \begin{pmatrix} d + \tilde{d}_\epsilon & -d + \tilde{d}_\epsilon & 0 \\ -d + \tilde{d}_\epsilon & d + \tilde{d}_\epsilon & 0 \\ 0 & 0 & 2d \end{pmatrix}$$

and

$$F_\epsilon^1 = \frac{1}{\epsilon\sqrt{2}}(S_\epsilon^u + S_\epsilon^{\tilde{h}}), \quad F_\epsilon^2 = \frac{1}{\epsilon\sqrt{2}}(-S_\epsilon^u + S_\epsilon^{\tilde{h}}), \quad F_\epsilon^3 = \frac{1}{\epsilon}S_\epsilon^v.$$

3. Initial Value Problems with Non-Resonant Forcing and Dissipation

We will use the notation $\underline{S} = (S_1^1, S_2)$ and write $\underline{S} \cdot \nabla$ as a shorthand for $S_1^1 \partial_x + S_2 \partial_y$.

We consider the initial value problems

$$\begin{cases} \partial_t \vec{U}_\epsilon + \underline{S} \cdot \nabla \vec{U}_\epsilon + \frac{1}{\epsilon} \mathcal{L} \vec{U}_\epsilon = \vec{F}_\epsilon + \mathcal{L}_{1,\epsilon} \vec{U}_\epsilon \\ \vec{U}|_{t=0} = \vec{U}_{0,\epsilon} \end{cases} \tag{15}$$

allowing the strong part of the forcing \vec{F}_ϵ to oscillate non-resonantly along the eigenspaces of \mathcal{L} . So we suppose that

$$\vec{F}_\epsilon(t) = \vec{F}_\epsilon^0(t) + \frac{1}{\epsilon} e^{-\frac{i}{\epsilon} \mathcal{L}_0} \vec{F}_\epsilon^1(t) \tag{16}$$

where $\vec{F}_\epsilon^0, \vec{F}_\epsilon^1$ are given, smooth, real-valued vector fields and where

$$\mathcal{L}_0 = \sum_{k,n,\alpha} c_\alpha^{k,n} \mathcal{P}_\alpha^{k,n},$$

$\mathcal{P}_\alpha^{k,n}$ denoting the projector on the eigenspace of \mathcal{L} corresponding to the eigenvalue $\lambda_\alpha^{k,n}$ (see Section 4). We assume all $c_\alpha^{k,n}$ to be pure imaginary numbers, like all $\lambda_\alpha^{k,n}$ are. We impose the condition

$$c_{-\alpha}^{-k,n} = \overline{c_\alpha^{k,n}} = -c_\alpha^{k,n} \tag{17}$$

to have \vec{F}_ϵ real-valued. The condition of non-resonance is

$$\inf_{k,n,\alpha} |c_\alpha^{k,n} - \lambda_\alpha^{k,n}| > 0. \tag{18}$$

Finally, we define $e^{\tau\mathcal{L}_0}$ and $e^{\tau\mathcal{L}}$ for $\tau \in \mathbb{R}$ by

$$e^{\tau\mathcal{L}_0} \vec{U} = \sum_{k,n,\alpha} e^{c_\alpha^{k,n}\tau} \mathcal{P}_\alpha^{k,n} \vec{U}$$

and

$$e^{\tau\mathcal{L}} \vec{U} = \sum_{k,n,\alpha} e^{\lambda_\alpha^{k,n}\tau} \mathcal{P}_\alpha^{k,n} \vec{U}.$$

Theorem 3.1 (Existence). *Let m be an integer ≥ 3 . If $\vec{F}_\epsilon^0, \vec{F}_\epsilon^1$, and $\partial_t \vec{F}_\epsilon^1$ are bounded in $C(\mathbb{R}^+; \tilde{W}^m)$, then for some $T > 0$ independent of ϵ , there is for each ϵ a solution \vec{U}_ϵ to (15) in $C^0([0, T]; \tilde{W}^m) \cap C^1([0, T]; \tilde{W}^{m-1})$.*

Theorem 3.2 (Fast Averaging). *If $m \geq 4$ and if we assume in addition that the forcing satisfies $\partial_t \vec{F}_\epsilon^1 \in Lip(\mathbb{R}^+; L^2)$ and $\vec{F}_\epsilon^0 \rightarrow \vec{F}_0^0, \vec{F}_\epsilon^1 \rightarrow \vec{F}_0^1$ and $\partial_t \vec{F}_\epsilon^1 \rightarrow \partial_t \vec{F}_0^1$ in L^2 , then*

$$\vec{U}_\epsilon = (\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\frac{1}{\epsilon}\mathcal{L}_0} \vec{F}_0^1 + e^{-\frac{1}{\epsilon}\mathcal{L}} \vec{u} + o(1)$$

in $C([0, T]; \tilde{W}^s)$ for all $s < m$, where $\vec{u} \in C([0, T]; \tilde{W}^m) \cap C^1([0, T]; \tilde{W}^{m-1})$ satisfies

$$\partial_t \vec{u} = \mathcal{P}_0 \vec{F}_0^0(t) \tag{19}$$

$$+ \sum_{\lambda_\alpha^{k,n} = c_\alpha^{k',n'}} \frac{\mathcal{P}_\alpha^{k,n} (\mathcal{L}_{1,0} \mathcal{P}_\alpha^{k',n'} \vec{F}_0^1(t))}{\lambda_\alpha^{k',n'} - c_\alpha^{k',n'}} \tag{20}$$

$$- \sum_{\lambda_\alpha^{k,n} = c_\alpha^{k',n'} + c_\alpha^{k'',n''}} \frac{\mathcal{P}_\alpha^{k,n} (\underline{\mathcal{L}} (\mathcal{P}_\alpha^{k',n'} \vec{F}_0^1(t)) \cdot \nabla (\mathcal{P}_\alpha^{k'',n''} \vec{F}_0^1(t)))}{(\lambda_\alpha^{k',n'} - c_\alpha^{k',n'}) (\lambda_\alpha^{k'',n''} - c_\alpha^{k'',n''})} \tag{21}$$

$$+ \sum_{\lambda_\alpha^{k,n} = \lambda_\alpha^{k',n'}} \mathcal{P}_\alpha^{k,n} (\mathcal{L}_{1,0} \mathcal{P}_\alpha^{k',n'} \vec{u}(t)) \tag{22}$$

$$- \sum_{\lambda_\alpha^{k,n} = \lambda_\alpha^{k',n'} + \lambda_\alpha^{k'',n''}} \mathcal{P}_\alpha^{k,n} (\underline{\mathcal{L}} (\mathcal{P}_\alpha^{k',n'} \vec{u}(t)) \cdot \nabla (\mathcal{P}_\alpha^{k'',n''} \vec{u}(t))) \tag{23}$$

$$- \sum_{\lambda_{\alpha}^{k,n} = \lambda_{\alpha'}^{k',n'} + c_{\alpha''}^{k'',n''}} \frac{\mathcal{P}_{\alpha}^{k,n}(\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{q}(t)) \cdot \nabla \mathcal{P}_{\alpha''}^{k'',n''} \vec{F}_0^1(t))}{\lambda_{\alpha'}^{k',n'} - c_{\alpha''}^{k'',n''}} \tag{24}$$

$$- \sum_{\lambda_{\alpha}^{k,n} = c_{\alpha'}^{k',n'} + \lambda_{\alpha''}^{k'',n''}} \frac{\mathcal{P}_{\alpha}^{k,n}(\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{F}_0^1(t)) \cdot \nabla(\mathcal{P}_{\alpha''}^{k'',n''} \vec{q}(t)))}{\lambda_{\alpha'}^{k',n'} - c_{\alpha''}^{k'',n''}}. \tag{25}$$

The sums are to be taken over all indices present in the formulas such that the condition under the sign of summation is satisfied. Moreover, only terms corresponding to $k = k' + k''$ (corresponding to $k = k'$ in (20) and (22)) are non-zero.

The term (19) is the simple contribution of the slow part of the forcing, while (20) is a contribution of the fast part which may be non-zero only because $\mathcal{L}_{1,0}$ is not the identity. The term (22) is similar: it means that fast waves may influence each other because of the dissipation. The other terms regroup resonances: between different modes of the forcing in (21), between different waves in (23), between forcing and waves in (24) and (25).

Corollary 3.3 (Zonal Jets Dynamics). *In particular, if $\mathcal{P}_0 \vec{U}$ denotes the projection on the zonal jets in (2), (3), (4), then*

$$\partial_t \mathcal{P}_0 \vec{q} = \mathcal{P}_0 \vec{F}_0^0 \tag{26}$$

$$+ \sum_{c_{\alpha'}^{0,n'} = 0} \frac{\mathcal{P}_0^{0,n}(\mathcal{L}_{1,0} \mathcal{P}_{\alpha'}^{0,n'} \vec{F}_0^1)}{\lambda_{\alpha'}^{0,n'} - c_{\alpha'}^{0,n'}} \tag{27}$$

$$+ \sum_{c_{\alpha'}^{k',n'} = c_{\alpha''}^{k'',n''}} \frac{\mathcal{P}_0^{0,n}(\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{F}_0^1) \cdot \nabla(\mathcal{P}_{-\alpha''}^{-k'',n''} \vec{F}_0^1))}{(\lambda_{\alpha'}^{k',n'} - c_{\alpha'}^{k',n'}) (\lambda_{\alpha''}^{k'',n''} - c_{\alpha''}^{k'',n''})} \tag{28}$$

$$+ \sum \mathcal{P}_0^{0,n}(\mathcal{L}_{1,0} \mathcal{P}_0^{0,n'} \vec{q}) \tag{29}$$

$$+ \sum_{\lambda_{\alpha'}^{k',n'} = c_{\alpha''}^{k'',n''}} \frac{\mathcal{P}_0^{0,n}(\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{q}) \cdot \nabla(\mathcal{P}_{-\alpha''}^{-k'',n''} \vec{F}_0^1))}{\lambda_{\alpha''}^{k'',n''} - \lambda_{\alpha'}^{k',n'}} \tag{30}$$

$$+ \sum_{\lambda_{\alpha'}^{k',n'} = c_{\alpha''}^{k'',n''}} \frac{\mathcal{P}_0^{0,n}(\underline{\mathcal{S}}(\mathcal{P}_{-\alpha''}^{-k'',n''} \vec{F}_0^1) \cdot \nabla(\mathcal{P}_{\alpha'}^{k',n'} \vec{q}))}{\lambda_{\alpha''}^{k'',n''} - \lambda_{\alpha'}^{k',n'}}. \tag{31}$$

The physical interpretation of (30) and (31) is that fast waves may interact with an external forcing to create zonal jets. We check that the sum of (30) and (31) is not always zero on an example of forcing having a single mode in Section 7.

4. Eigenvalues and Eigenfunctions of \mathcal{L}

The content of this section is essentially a rephrasing of Ripa’s discussion about the eigenvalues and eigenfunctions of \mathcal{L} (see Ripa, 1982, 1983a,b).

Suppose that $x \in l\mathbb{T}$ and $y \in \mathbb{R}$ (we keep the size of the periodic domain in x as a parameter as it has an influence on the resonances).

Let us recall the definition of the parabolic cylinder functions ϕ_n : for $n \in \mathbb{N}$,

$$\phi_n(y) = (2^n n! \sqrt{\pi})^{-1/2} H_n(y) e^{-y^2/2}$$

where

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$$

is the Hermite polynomial of degree n .

The functions $e^{\frac{2\pi}{T} ikx} \phi_n(y)$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ form an orthonormal basis of $L^2(I\mathbb{T} \times \mathbb{R})$. Hence

$$(L^2(I\mathbb{T} \times \mathbb{R}))^3 = E^{k,-2} \oplus E^{k,-1} \bigoplus_{\substack{k \in \mathbb{Z} \\ n \in \mathbb{N}}} E^{k,n} \tag{32}$$

if we note for each $k \in \mathbb{Z}$

$$E^{k,-2} = \left\langle e^{\frac{2\pi}{T} ikx} \begin{pmatrix} \phi_0 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$E^{k,-1} = \left\langle e^{\frac{2\pi}{T} ikx} \begin{pmatrix} \phi_1 \\ 0 \\ 0 \end{pmatrix}, e^{\frac{2\pi}{T} ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_0 \end{pmatrix} \right\rangle$$

and

$$E^{k,n} = \left\langle e^{\frac{2\pi}{T} ikx} \begin{pmatrix} \phi_{n+2} \\ 0 \\ 0 \end{pmatrix}, e^{\frac{2\pi}{T} ikx} \begin{pmatrix} 0 \\ \phi_n \\ 0 \end{pmatrix}, e^{\frac{2\pi}{T} ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_{n+1} \end{pmatrix} \right\rangle$$

for $n \geq 0$.

We will see, using the raising and lowering properties of L_{\pm}

$$\frac{1}{\sqrt{2}} L_- \phi_n = -\sqrt{n+1} \phi_{n+1} \quad \text{for } n \geq 0, \tag{33}$$

$$\frac{1}{\sqrt{2}} L_+ \phi_n = \sqrt{n} \phi_{n-1} \quad \text{for } n \geq 1, \tag{34}$$

$$L_+ \phi_0 = 0, \tag{35}$$

that the operator \mathcal{L} has a disjoint action in these subspaces (see Sections 4.2 and 4.3); since $i\mathcal{L}$ is self-adjoint in $(L^2(I\mathbb{R} \times \mathbb{T}))^3$ equipped with the usual scalar (hermitian) product, $i\mathcal{L}|_{E^{k,n}}$ is self-adjoint for all $k \in \mathbb{Z}$ and all $n \geq -2$. This has two consequences. First, \mathcal{L} has mutually orthogonal eigenvectors forming a basis of $E^{k,n}$. Since the decomposition (32) is orthogonal, there is actually an orthonormal basis of $(L^2(I\mathbb{R} \times \mathbb{T}))^3$ formed by the eigenvectors of \mathcal{L} . Second, the eigenvalues of $i\mathcal{L}|_{E^{k,n}}$ are real. It also turns out that there are all simple (see (38), (40) and Proposition 4.2 below), so there are always three of them for each $(k, n) \in \mathbb{Z} \times \mathbb{N}$.

4.1. Action of \mathcal{L} in $E^{k,-2}$

We have

$$\mathcal{L}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_0 \\ 0 \\ 0 \end{pmatrix} = \frac{2\pi}{l}ike^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_0 \\ 0 \\ 0 \end{pmatrix}. \quad (36)$$

We note

$$\vec{f}_0^{k,-2} = e^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_0 \\ 0 \\ 0 \end{pmatrix} \quad (37)$$

for each k . In view of (36), $\vec{f}_0^{k,-2}$ is an eigenvector of \mathcal{L} corresponding to the eigenvalue

$$\lambda_0^{k,-2} = \frac{2\pi}{l}ik. \quad (38)$$

4.2. Action of \mathcal{L} in $E^{k,-1}$

We have

$$\mathcal{L}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_1 \\ 0 \\ 0 \end{pmatrix} = \frac{2\pi}{l}ike^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_1 \\ 0 \\ 0 \end{pmatrix} + e^{\frac{2\pi}{l}ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_0 \end{pmatrix}$$

and

$$\mathcal{L}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_0 \end{pmatrix} = -e^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_1 \\ 0 \\ 0 \end{pmatrix}.$$

An easy computation on the matrix

$$\begin{pmatrix} \frac{2\pi}{l}ik & -1 \\ 1 & 0 \end{pmatrix}$$

shows that \mathcal{L} has the eigenvectors

$$\vec{f}_{\pm 1}^{k,-1} = \left(\frac{\pi}{l}ik \pm i\sqrt{\frac{\pi^2}{l^2}k^2 + 1} \right) e^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_1 \\ 0 \\ 0 \end{pmatrix} + e^{\frac{2\pi}{l}ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_0 \end{pmatrix} \quad (39)$$

corresponding to the eigenvalues

$$\lambda_{\pm 1}^{k,-1} = \frac{\pi}{l}ik \pm i\sqrt{\frac{\pi^2}{l^2}k^2 + 1}. \quad (40)$$

4.3. Action of \mathcal{L} in $E^{k,n}$ for $n \geq 0$

We have

$$\mathcal{L}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_{n+2} \\ 0 \\ 0 \end{pmatrix} = \frac{2\pi}{l}ike^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_{n+2} \\ 0 \\ 0 \end{pmatrix} + \sqrt{n+2}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_{n+1} \end{pmatrix}$$

$$\mathcal{L}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} 0 \\ \phi_n \\ 0 \end{pmatrix} = -\frac{2\pi}{l}ike^{\frac{2\pi}{l}ikx} \begin{pmatrix} 0 \\ \phi_n \\ 0 \end{pmatrix} - \sqrt{n+1}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_{n+1} \end{pmatrix}$$

and

$$\mathcal{L}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_{n+1} \end{pmatrix} = -\sqrt{n+2}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} \phi_{n+2} \\ 0 \\ 0 \end{pmatrix} + \sqrt{n+1}e^{\frac{2\pi}{l}ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_n \end{pmatrix}.$$

The equation for the eigenvalues of the matrix

$$\begin{pmatrix} \frac{2\pi}{l}ik & 0 & -\sqrt{n+2} \\ 0 & -\frac{2\pi}{l}ik & \sqrt{n+1} \\ \sqrt{n+2} & -\sqrt{n+1} & 0 \end{pmatrix} \tag{41}$$

is

$$-\lambda^3 - \left(\frac{4\pi^2}{l^2}k^2 + 2n + 3\right)\lambda - \frac{2\pi}{l}ik = 0. \tag{42}$$

If we set $\lambda = i\mu$, this equation is equivalent to

$$-\mu^3 + \left(\frac{4\pi^2}{l^2}k^2 + 2n + 3\right)\mu + \frac{2\pi}{l}k = 0. \tag{43}$$

Remark 4.1. Since i times matrix (41) is hermitian, the solutions of (42) are pure imaginary numbers (in other words, the solutions of (43) are real).

Proposition 4.2. For each $(l, k, n) \in \mathbb{R}_0^+ \times \mathbb{Z} \times \mathbb{N}$, the solutions of (42) are distinct.

Proof. Let $\beta = 2\pi k/l$ and $\gamma = \beta^2 + 2n + 3$. Then (42) is just $f(\lambda) = 0$ with $f(\lambda) \stackrel{\text{def}}{=} -\lambda^3 - \gamma\lambda - i\beta$. As $f'(\lambda) = -3\lambda^2 - \gamma$, a multiple root of f has to satisfy

$$\gamma = -3\lambda^2 \tag{44}$$

and therefore, plugging this in $f(\lambda) = 0$,

$$2\lambda^3 - i\beta = 0. \tag{45}$$

But (44) and (45) imply $(\gamma/3)^3 = \beta^2/4$, which is not possible because $\gamma/3 = \beta^2/3 + (2n + 3)/3 \geq \beta^2/4 + 1$. \square

Notation 4.3. We denote the eigenvalues of $\mathcal{L}|_{E^{k,n}}$ by $\lambda_\alpha^{k,n}$ ($=i\mu_\alpha^{k,n}$), with $\alpha = -1, 0, 1$, following the convention

$$\mu_{-1}^{k,n} < \mu_0^{k,n} < \mu_1^{k,n}. \tag{46}$$

Corollary 4.4. For any $(k, n) \in \mathbb{Z} \times \mathbb{N}$, three eigenvectors of \mathcal{L} in E_n^k corresponding to three different eigenvalues always form an orthogonal basis of E_n^k .

Proposition 4.5. If $n \geq 0$ and either

$$k \neq 0$$

or

$$k = 0 \text{ and } \alpha \neq 0,$$

then

$$\lambda_\alpha^{k,n} \neq \begin{cases} \frac{2\pi}{l} ik \\ -\frac{2\pi}{l} ik \end{cases}.$$

Proof. In the first case, with β, γ , and f as in the proof of Proposition 4.2, we have to check that $f(\pm i\beta) \neq 0$. But $f(\pm i\beta) = \pm i\beta^3 \mp i\beta\gamma - i\beta = -i\beta(\mp\beta^2 + \gamma + 1)$; now $\beta = 2\pi k/l \neq 0$ as $k \neq 0$ by assumption, and $\mp\beta^2 + \gamma + 1 \geq 2n + 4 > 0$.

For $k = 0$, the eigenvalues and eigenfunctions of \mathcal{L} are simply those of L . In particular (Dutrifoy and Majda, 2006), $\lambda_\alpha^{0,n} = \alpha i\sqrt{2n + 3} \neq 0$ and so $\lambda_\alpha^{0,n} \neq 0$ if $\alpha \neq 0$. \square

Hence

$$\vec{f}_\alpha^{k,n} = \frac{\sqrt{n+2}}{\frac{2\pi}{l} ik - \lambda_\alpha^{k,n}} e^{\frac{2\pi}{l} ikx} \begin{pmatrix} \phi_{n+2} \\ 0 \\ 0 \end{pmatrix} + \frac{\sqrt{n+1}}{\frac{2\pi}{l} ik + \lambda_\alpha^{k,n}} e^{\frac{2\pi}{l} ikx} \begin{pmatrix} 0 \\ \phi_n \\ 0 \end{pmatrix} + e^{\frac{2\pi}{l} ikx} \begin{pmatrix} 0 \\ 0 \\ \phi_{n+1} \end{pmatrix} \tag{47}$$

are eigenvectors corresponding to $\lambda_\alpha^{k,n}$ for $n \geq 0$ if k and α are not simultaneously zero.

It is easy to check (Dutrifoy and Majda, 2006) that

$$\vec{f}_0^{0,n} = \sqrt{n+1} \begin{pmatrix} \phi_{n+2} \\ 0 \\ 0 \end{pmatrix} + \sqrt{n+2} \begin{pmatrix} 0 \\ \phi_n \\ 0 \end{pmatrix} \tag{48}$$

are eigenvectors corresponding to $\lambda_0^{0,n}$ for $n \geq 0$.

4.4. Synthesis

4.4.1. The Orthonormal Basis.

Definition 4.6. Let $\vec{g}_\alpha^{k,n} = \vec{f}_\alpha^{k,n} / \|\vec{f}_\alpha^{k,n}\|_{(L^2(I\mathbb{T} \times \mathbb{R}))^3}$, with $\vec{f}_\alpha^{k,n}$ defined by

$$(37) \quad \text{for } n = -2, \alpha = 0 \text{ and } k \in \mathbb{Z}$$

$$(39) \quad \text{for } n = -1, \alpha = \pm 1 \text{ and } k \in \mathbb{Z}$$

$$(48) \quad \text{for } n \geq 0 \text{ and } \alpha = k = 0$$

$$(47) \quad \text{for all other values of } k, n \text{ and } \alpha.$$

Proposition 4.7. For each $(k, n) \in \mathbb{Z} \times \mathbb{N}$, $\{\vec{g}_{-1}^{k,n}, \vec{g}_{-0}^{k,n}, \vec{g}_1^{k,n}\}$ is an orthonormal basis of $E^{k,n}$.

Corollary 4.8. The vectors $\vec{g}_0^{k,-2}, \vec{g}_{-1}^{k,-1}, \vec{g}_1^{k,-1}$ and $\vec{g}_\alpha^{k,n}$ for $k \in \mathbb{Z}, n \in \mathbb{N}$ and $\alpha \in \{-1, 0, 1\}$ form an orthonormal basis of $(L^2(I\mathbb{T} \times \mathbb{R}))^3$.

Corollary 4.9. For any $m \in \mathbb{N}$, the norm of $\vec{U} \in (\tilde{W}^m(I\mathbb{T} \times \mathbb{R}))^3$ is equivalent to

$$\|\partial_x^m \vec{U}\|_{L^2} + \left(\sum_{k,n,\alpha} (n+3)^m |\langle \vec{U}, \vec{g}_\alpha^{k,n} \rangle|^2 \right)^{1/2}.$$

(For a proof of the last corollary, see Proposition 2.2 in Dutrifoy and Majda, 2006.)

4.4.2. Symmetry Properties.

Proposition 4.10. For all k, n and $\alpha, \lambda_{-\alpha}^{-k,n} = -\lambda_\alpha^{k,n}$.

Proof. The property is immediate to check for $n = -2$ and $n = -1$ from the explicit expressions (38) and (40).

For $n \geq 0$, using Notation 4.3, $\mu_{-1}^{-k,n} < \mu_0^{-k,n} < \mu_1^{-k,n}$ are the solutions of

$$-\mu^3 + \left(\frac{4\pi^2}{l^2} k^2 + 2n + 3 \right) \mu - \frac{2\pi}{l} k = 0.$$

But so are $-\mu_{-1}^{k,n} < -\mu_0^{k,n} < -\mu_{-1}^{k,n}$, for (43) is equivalent to

$$-(-\mu)^3 + \left(\frac{4\pi^2}{l^2} k^2 + 2n + 3 \right) (-\mu) - \frac{2\pi}{l} k = 0.$$

Hence we must have $\mu_{-1}^{-k,n} = -\mu_1^{k,n}, \mu_0^{-k,n} = -\mu_0^{k,n}$, and $\mu_1^{-k,n} = -\mu_{-1}^{k,n}$. □

Proposition 4.11. For all k, n , and $\alpha, \vec{g}_{-\alpha}^{-k,n} = \overline{\vec{g}_\alpha^{k,n}}$.

Proof. It is sufficient to check the property on the $\vec{f}_\alpha^{k,n}$. And indeed the property is obvious in (37), (39) and (48), while it is true in (47) by Proposition 4.10. □

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4.4.3. Projectors.

Definition 4.12. Let $\mathcal{P}_\alpha^{k,n}$ denote the projector on $\langle \vec{g}_\alpha^{k,n} \rangle$.

Remark 4.13. Note that $\lambda_\alpha^{k,n} = 0$ if and only if $k = \alpha = 0$.

Proposition 4.14. For each n and α , the projection on $\langle \vec{g}_\alpha^{0,n} \rangle$ is equal to the mean on Γ of the projection on $\langle \vec{e}_\alpha^n \rangle$.

See Sections 2.4 and 2.5 in Dutrifoy and Majda (2006) for the precise definitions of \vec{e}_α^n (normed eigenvectors of L) and $P_\alpha^{(n)}$ (corresponding projectors).

Proof. If $k = 0$, $\vec{f}_\alpha^{k,n} = \vec{e}_\alpha^n$ (or $i\vec{e}_\alpha^n$) is actually a function of y alone. Therefore

$$\begin{aligned} \mathcal{P}_\alpha^{0,n} \vec{U} &= (\mathcal{P}_\alpha^{0,n} \vec{U})(y) = \frac{1}{l} \iint_{\Gamma \times \mathbb{R}} \vec{U}(x', y') \overline{\vec{g}_\alpha^{0,n}(x', y')} dx' dy' \vec{g}_\alpha^{0,n}(x, y) \\ &= \frac{1}{l} \int_{\Gamma} \left(\int_{\mathbb{R}} \vec{U}(x', y') \overline{\vec{e}_\alpha^n(y')} dy' \right) \vec{e}_\alpha^n(y) dx' \\ &= \frac{1}{l} \int_{\Gamma} (P_\alpha^{(n)} \vec{U})(x', y) dx'. \quad \square \end{aligned}$$

Corollary 4.15. The projector on the kernel of \mathcal{L} is related to the projector on the kernel of L by

$$\mathcal{P}_0 \vec{U} = \frac{1}{l} \int_{\Gamma} (P_0 \vec{U})(x', y) dx'. \quad (49)$$

Proof.

$$\begin{aligned} \mathcal{P}_0 \vec{U} &= \sum_{n=-1}^{\infty} \mathcal{P}_0^{n,0} \vec{U} = \frac{1}{l} \sum_{n=-1}^{\infty} \int_{\Gamma} P_0^{(n)} \vec{U}(x', y) dx' \\ &= \frac{1}{l} \int_{\Gamma} (P_0 \vec{U})(x', y) dx'. \quad \square \end{aligned}$$

Note that with (49) as the zonal average, $\mathcal{P}_0 \vec{U}$ is precisely the projection on (2), (3), (4) defining the zonal jets.

5. A Priori Estimates

Let \vec{U} be a smooth, real-valued solution of

$$\partial_t \vec{U} + \underline{S} \cdot \nabla \vec{U} + \frac{1}{\epsilon} \mathcal{L} \vec{U} = \vec{F}, \quad (50)$$

where $\underline{S} = (S_1^1, S_2)$ is a couple of symmetric, real-valued matrices and \vec{F} is an unspecified forcing, all of them continuous in time with values in \widetilde{W}^m for some $m \in \mathbb{N}$.

In Section 5.2 we prove the following estimate.

Proposition 5.1. *The solutions of (50) satisfy*

$$\begin{aligned} \|\vec{U}(t)\|_{\tilde{W}^m} &\leq C\|\vec{U}(0)\|_{\tilde{W}^m} + C\int_0^t \|F(t')\|_{\tilde{W}^m} dt' \\ &\quad + C\int_0^t (\|\underline{S}(t')\|_{L^\infty} + \|\nabla\underline{S}(t')\|_{L^\infty})\|\vec{U}(t')\|_{\tilde{W}^m} dt' \\ &\quad + C\int_0^t \|\nabla\vec{U}(t')\|_{L^\infty}\|\underline{S}(t')\|_{\tilde{W}^m} dt'. \end{aligned} \tag{51}$$

This actually also provides uniform a priori estimates for the solutions of

$$\partial_t\vec{U} + \underline{S}(\vec{V}) \cdot \nabla\vec{U} + \frac{1}{\epsilon}\mathcal{L}\vec{U} = \vec{F}_\epsilon + \mathcal{L}_{1,\epsilon}\vec{U}, \tag{52}$$

where \vec{V} is given and \vec{F}_ϵ is defined by (16). Indeed, if

$$\begin{aligned} \vec{G}_\epsilon &= (\mathcal{L} - \mathcal{L}_0)^{-1}e^{-\frac{t}{\epsilon}\mathcal{L}_0}\vec{F}_\epsilon^1, \\ \tilde{U} &= \vec{U} - \vec{G}_\epsilon, \\ \tilde{V} &= \vec{V} - \vec{G}_\epsilon, \end{aligned} \tag{53}$$

then

$$\begin{aligned} \partial_t\tilde{U} + \frac{1}{\epsilon}\mathcal{L}\tilde{U} &= \partial_t\vec{U} + \frac{1}{\epsilon}\mathcal{L}\vec{U} + \frac{1}{\epsilon}(\mathcal{L} - \mathcal{L}_0)^{-1}\mathcal{L}_0e^{-\frac{t}{\epsilon}\mathcal{L}_0}\vec{F}_\epsilon^1 \\ &\quad - (\mathcal{L} - \mathcal{L}_0)^{-1}e^{-\frac{t}{\epsilon}\mathcal{L}_0}\partial_t\vec{F}_\epsilon^1 - \frac{1}{\epsilon}\mathcal{L}(\mathcal{L} - \mathcal{L}_0)^{-1}e^{-\frac{t}{\epsilon}\mathcal{L}_0}\vec{F}_\epsilon^1 \\ &= \partial_t\vec{U} + \frac{1}{\epsilon}\mathcal{L}\vec{U} - (\mathcal{L} - \mathcal{L}_0)^{-1}e^{-\frac{t}{\epsilon}\mathcal{L}_0}\partial_t\vec{F}_\epsilon^1 - \frac{1}{\epsilon}e^{-\frac{t}{\epsilon}\mathcal{L}_0}\vec{F}_\epsilon^1 \end{aligned}$$

and so (52) is equivalent to

$$\partial_t\tilde{U} + \underline{S}(\tilde{V} + \vec{G}_\epsilon) \cdot \nabla\tilde{U} + \frac{1}{\epsilon}\mathcal{L}\tilde{U} = \tilde{F}_\epsilon + \mathcal{L}_{1,\epsilon}\tilde{U} - \underline{S}(\tilde{V}) \cdot \nabla\vec{G}_\epsilon \tag{54}$$

with

$$\tilde{F}_\epsilon = \vec{F}_\epsilon^0 + \mathcal{L}_{1,\epsilon}\vec{G}_\epsilon - \underline{S}(\vec{G}_\epsilon) \cdot \nabla\vec{G}_\epsilon - (\mathcal{L} - \mathcal{L}_0)^{-1}e^{-\frac{t}{\epsilon}\mathcal{L}_0}\partial_t\vec{F}_\epsilon^1.$$

Thanks to the assumption of non-resonance (18),

$$(\mathcal{L} - \mathcal{L}_0)^{-1} = \sum_{k,n,\alpha} \frac{1}{\lambda_\alpha^{k,n} - c_\alpha^{k,n}} \mathcal{P}_\alpha^{k,n}$$

is bounded on \tilde{W}^m for all m . Hence we get a uniform estimate on \tilde{U} (and therefore also on \vec{U}) by substituting $\tilde{F}_\epsilon + \mathcal{L}_{1,\epsilon}\tilde{U} - \underline{S}(\tilde{V}) \cdot \nabla\vec{G}_\epsilon$ to \vec{F} and $\tilde{V} + \vec{G}_\epsilon$ to \underline{S} in (51).

5.1. Multiplication in \tilde{W}^m

Proposition 5.2. *Let $f \in \tilde{W}^m \cap L^\infty(I\mathbb{T} \times \mathbb{R})$ and $g \in W^m \cap L^\infty(I\mathbb{T} \times \mathbb{R})$. Then*

$$\|fg\|_{\tilde{W}^m} \leq C(\|f\|_{L^\infty}\|g\|_{W^m} + \|g\|_{L^\infty}\|f\|_{\tilde{W}^m}). \tag{55}$$

Proof. We reproduce the proof of the estimate of fg in the classical Sobolev space W^m as given in Klainerman and Majda (1981) (where it is in turn credited to Moser, 1966); the only twist is that a variant of the Gagliardo–Nirenberg inequalities will be needed.

We have to estimate $y^\alpha \partial_x^\beta \partial_y^\gamma (fg)$ in L^2 for all α, β , and γ such that

$$N = \alpha + \beta + \gamma \leq m.$$

By Leibnitz’s formula,

$$\partial_x^\beta \partial_y^\gamma (fg) = \sum_{\substack{\beta' + \beta'' = \beta \\ \gamma' + \gamma'' = \gamma}} c_{\beta'\gamma'} \partial_x^{\beta'} \partial_y^{\gamma'} f \partial_x^{\beta''} \partial_y^{\gamma''} g$$

for some constants $c_{\beta'\gamma'}$. So, by Hölder’s inequality,

$$\|y^\alpha \partial_x^\beta \partial_y^\gamma (fg)\|_{L^2} \leq C \sum_{\substack{\beta' + \beta'' = \beta \\ \gamma' + \gamma'' = \gamma}} \|y^\alpha \partial_x^{\beta'} \partial_y^{\gamma'} f\|_{L^{\frac{2N}{\alpha + \beta' + \gamma'}}} \|\partial_x^{\beta''} \partial_y^{\gamma''} g\|_{L^{\frac{2N}{\beta'' + \gamma''}}}. \tag{56}$$

Lemma 5.3.

$$\|y^\alpha \partial_x^{\beta'} \partial_y^{\gamma'} f\|_{L^{\frac{2N}{\alpha + \beta' + \gamma'}}} \leq C \|f\|_{L^\infty}^{1 - \frac{\alpha + \beta' + \gamma'}{N}} \|f\|_{\tilde{W}^N}^{\frac{\alpha + \beta' + \gamma'}{N}}, \tag{57}$$

$$\|\partial_x^{\beta''} \partial_y^{\gamma''} g\|_{L^{\frac{2N}{\beta'' + \gamma''}}} \leq C \|g\|_{L^\infty}^{1 - \frac{\beta'' + \gamma''}{N}} \|g\|_{W^N}^{\frac{\beta'' + \gamma''}{N}}. \tag{58}$$

The lemma is proved just below. When we plug (57) and (58) into (56), we get

$$\|y^\alpha \partial_x^\beta \partial_y^\gamma (fg)\|_{L^2} \leq C \sum_{\substack{\beta' + \beta'' = \beta \\ \gamma' + \gamma'' = \gamma}} (\|f\|_{L^\infty} \|g\|_{W^N})^{1 - \frac{\alpha + \beta' + \gamma'}{N}} (\|g\|_{L^\infty} \|f\|_{\tilde{W}^N})^{\frac{\alpha + \beta' + \gamma'}{N}}.$$

Then (55) follows by Young’s inequalities. □

Proof of Lemma 5.3. The inequalities (58) are nothing but the classical, well-known Gagliardo–Nirenberg inequalities. To prove (57), let us set $F = \partial_x^{\beta'} \partial_y^{\gamma'} f$ and $\delta = \beta' + \gamma'$. The left-hand side of (57) is

$$\begin{aligned} \|y^\alpha F\|_{L^{\frac{2N}{\alpha + \delta}}} &= \left(\int |y|^{2N\frac{\alpha + \delta}{\alpha + \delta}} |F|^{2N\frac{\alpha + \delta}{\alpha + \delta}} \right)^{\frac{\alpha + \delta}{2N}} \\ &= \left(\int |y|^{2N\frac{\alpha + \delta}{\alpha + \delta}} |F|^{\frac{2N\alpha}{(\alpha + \delta)(N - \delta)}} |F|^{\frac{2N(N - \alpha - \delta)}{(\alpha + \delta)(N - \delta)}} \right)^{\frac{\alpha + \delta}{2N}} \\ &\leq \left(\int |y|^{2(N - \delta)} |F|^2 \right)^{\frac{\alpha}{2(N - \delta)}} \left(\int |F|^{\frac{2N}{\delta}} \right)^{\frac{\delta}{2N} (1 - \frac{\alpha}{N - \delta})} \end{aligned}$$

by the Hölder inequality with conjugate exponents $(\alpha + \delta)(N - \delta)/N\alpha$ and $(\alpha + \delta)(N - \delta)/(\delta(N - \alpha - \delta))$. So

$$\|y^\alpha F\|_{L^{\frac{2N}{\alpha+\delta}}} \leq \|F\|_{L^{\frac{2N}{\delta}}}^{1-\frac{\alpha}{N-\delta}} \|y^{N-\delta} F\|_{L^2}^{\frac{\alpha}{N-\delta}},$$

that is,

$$\|y^\alpha \partial_x^{\beta'} \partial_y^{\gamma'} f\|_{L^{\frac{2N}{\alpha+\beta'+\gamma'}}} \leq \|\partial_x^{\beta'} \partial_y^{\gamma'} f\|_{L^{\frac{2N}{\beta'+\gamma'}}}^{1-\frac{\alpha}{N-\beta'-\gamma'}} \|y^{N-\beta'-\gamma'} \partial_x^{\beta'} \partial_y^{\gamma'} f\|_{L^2}^{\frac{\alpha}{N-\beta'-\gamma'}}.$$

Now

$$\|\partial_x^{\beta'} \partial_y^{\gamma'} f\|_{L^{\frac{2N}{\beta'+\gamma'}}} \leq \|f\|_{L^\infty}^{1-\frac{\beta'+\gamma'}{N}} \|f\|_{\tilde{W}^N}^{\frac{\beta'+\gamma'}{N}}$$

by (58). As both $\|y^{N-\beta'-\gamma'} \partial_x^{\beta'} \partial_y^{\gamma'} f\|_{L^2}$ and $\|f\|_{W^N}$ are bounded by $\|f\|_{\tilde{W}^N}$, we have

$$\|y^\alpha \partial_x^{\beta'} \partial_y^{\gamma'} f\|_{L^{\frac{2N}{\alpha+\beta'+\gamma'}}} \leq \|f\|_{L^\infty}^{(1-\frac{\beta'+\gamma'}{N})(1-\frac{\alpha}{N-\beta'-\gamma'})} \|f\|_{\tilde{W}^N}^{\frac{\alpha}{N-\beta'-\gamma'} + \frac{\beta'+\gamma'}{N}(1-\frac{\alpha}{N-\beta'-\gamma'})},$$

which is exactly (57). □

5.2. Estimate in \tilde{W}^m for $m \geq 1$

We focus on the estimate of

$$\left(\sum_{k,n,\alpha} (n+3)^m |\langle \tilde{U}, \tilde{g}_\alpha^{k,n} \rangle|^2 \right)^{1/2}$$

since differentiating the system with respect to x yields an estimate on $\partial_x^m \tilde{U}$ in L^2 without any special difficulty (see Corollary 4.9).

Let us apply $\mathcal{P}_\alpha^{k,n}$ to (50). This gives

$$\partial_t \mathcal{P}_\alpha^{k,n} \tilde{U} + \mathcal{P}_\alpha^{k,n} (\underline{S} \cdot \nabla \tilde{U}) + \frac{1}{\epsilon} \lambda_\alpha^{k,n} \mathcal{P}_\alpha^{k,n} \tilde{U} = \mathcal{P}_\alpha^{k,n} \tilde{F},$$

for all k, n , and α . Then we take the scalar product of both sides with $(n+3)^m \mathcal{P}_\alpha^{k,n} \tilde{U}$, sum over k, n and α , add and subtract $\frac{1}{2^m} \langle L_-^m \tilde{U}, L_-^m (\underline{S} \cdot \nabla \tilde{U}) \rangle$ and retain only the real part. This gives

$$\begin{aligned} & \frac{1}{2} \partial_t \left(\sum_{k,n,\alpha} (n+3)^m \|\mathcal{P}_\alpha^{k,n} \tilde{U}\|_{L^2}^2 \right) + \frac{1}{2^m} \langle L_-^m \tilde{U}, L_-^m (\underline{S} \cdot \nabla \tilde{U}) \rangle \\ &= \Re \left(\frac{1}{2^m} \langle L_-^m \tilde{U}, L_-^m (\underline{S} \cdot \nabla \tilde{U}) \rangle - \sum_{k,n,\alpha} (n+3)^m \langle \mathcal{P}_\alpha^{k,n} \tilde{U}, \mathcal{P}_\alpha^{k,n} (\underline{S} \cdot \nabla \tilde{U}) \rangle \right) \\ & \quad + \Re \left(\sum_{k,n,\alpha} (n+3)^m \langle \mathcal{P}_\alpha^{k,n} \tilde{U}, \mathcal{P}_\alpha^{k,n} \tilde{F} \rangle \right) \end{aligned} \tag{59}$$

because the real part of $\lambda_\alpha^{k,n}$ is always zero.

Lemma 5.4. *Let $\underline{S}, \vec{U} \in \tilde{W}^m \cap \text{Lip}(\Gamma \times \mathbb{R})$. Then*

$$|\langle L_-^m \vec{U}, L_-^m(\underline{S} \cdot \nabla \vec{U}) \rangle| \leq C \|\vec{U}\|_{\tilde{W}^m} (\|\underline{S}\|_{L^\infty} + \|\nabla \underline{S}\|_{L^\infty}) \|\vec{U}\|_{\tilde{W}^m} + \|\nabla \vec{U}\|_{L^\infty} \|\underline{S}\|_{\tilde{W}^m}. \quad (60)$$

Proof. We write

$$L_-^m(\underline{S} \cdot \nabla \vec{U}) = \underline{S} \cdot \nabla L_-^m \vec{U} + [L_-^m, \underline{S} \cdot \nabla] \vec{U}.$$

The scalar product $\langle L_-^m \vec{U}, \underline{S} \cdot \nabla L_-^m \vec{U} \rangle$ is estimated by integration by parts, exploiting the symmetry of \underline{S} :

$$\begin{aligned} & \iint L_-^m \vec{U} \cdot (\underline{S} \cdot \nabla L_-^m \vec{U}) dx dy \\ &= \sum_{i,j=1}^3 \left(\iint L_-^m U^i (S_1^1)_{ij} \partial_x (L_-^m U^j) dx dy + \iint L_-^m U^i (S_2)_{ij} \partial_y (L_-^m U^j) dx dy \right) \\ &= \frac{1}{2} \sum_{i,j=1}^3 \left(\iint (S_1^1)_{ij} \partial_x (L_-^m U^i L_-^m U^j) dx dy + \iint (S_2)_{ij} \partial_y (L_-^m U^i L_-^m U^j) dx dy \right) \\ &= -\frac{1}{2} \sum_{i,j=1}^3 \iint (\partial_x (S_1^1)_{ij} + \partial_y (S_2)_{ij}) L_-^m U^i L_-^m U^j dx dy. \end{aligned}$$

Thus

$$\langle L_-^m \vec{U}, \underline{S} \cdot \nabla L_-^m \vec{U} \rangle \leq C \|\nabla \underline{S}\|_{L^\infty} \|L_-^m \vec{U}\|_{L^2}^2,$$

which is bounded by the term $C \|\nabla \underline{S}\|_{L^\infty} \|\vec{U}\|_{\tilde{W}^m}^2$ in the right-hand side of (60).

The commutator $[L_-^m, \underline{S} \cdot \nabla] \vec{U}$ is treated as follows. Remark that

$$L_-^m = (\partial_y - y)^m = \hat{\partial}_y^m + (-1)^m y^m + R_{m-1}$$

where

$$R_{m-1} = \sum_{k+l \leq m-1} c_{kl} y^k \partial_y^l$$

for some constants c_{kl} .

- Since R_{m-1} is order $m - 1$, we can estimate $\underline{S} \cdot \nabla R_{m-1} \vec{U}$ and $R_{m-1}(\underline{S} \cdot \nabla \vec{U})$ in L^2 separately:

$$\|\underline{S} \cdot \nabla R_{m-1} \vec{U}\|_{L^2} \leq \|\underline{S}\|_{L^\infty} \|\vec{U}\|_{\tilde{W}^m},$$

and

$$\|R_{m-1}(\underline{S} \cdot \nabla \vec{U})\|_{L^2} \leq C \|\underline{S} \cdot \nabla \vec{U}\|_{\tilde{W}^{m-1}} \leq C (\|\underline{S}\|_{L^\infty} \|\vec{U}\|_{\tilde{W}^m} + \|\nabla \vec{U}\|_{L^\infty} \|\underline{S}\|_{\tilde{W}^{m-1}})$$

thanks to Proposition 5.2.

- The multiplication by y^m commutes with \underline{S} and ∂_x , so

$$\begin{aligned} \|[y^m, \underline{S} \cdot \nabla] \vec{U}\|_{L^2} &= \|S_2[y^m, \partial_y] \vec{U}\|_{L^2} \\ &= m \|S_2 y^{m-1} \vec{U}\|_{L^2} \leq C \|S_2\|_{L^\infty} \|\vec{U}\|_{\tilde{W}^{m-1}}. \end{aligned}$$

- Finally, the estimate

$$\|[\partial_y^m, \underline{S} \cdot \nabla] \vec{U}\|_{L^2} \leq C (\|\nabla \underline{S}\|_{L^\infty} \|\vec{U}\|_{W^m} + \|\nabla \vec{U}\|_{L^\infty} \|\underline{S}\|_{W^m})$$

is classical (Klainerman and Majda, 1981).

Thus

$$|\langle L_-^m \vec{U}, [L_-^m, \underline{S} \cdot \nabla] \vec{U} \rangle| \leq \|L_-^m \vec{U}\|_{L^2} \|[L_-^m, \underline{S} \cdot \nabla] \vec{U}\|_{L^2}$$

is also bounded by the right-hand side of (60). □

Lemma 5.5. *Let $\vec{U}_1, \vec{U}_2 \in (\tilde{W}^m(l\mathbb{T} \times \mathbb{R}))^3$. Then*

$$\left| \frac{1}{2^m} \langle L_-^m \vec{U}_1, L_-^m \vec{U}_2 \rangle - \sum_{k,n,\alpha} (n+3)^m \langle \mathcal{P}_\alpha^{k,n} \vec{U}_1, \mathcal{P}_\alpha^{k,n} \vec{U}_2 \rangle \right| \leq C \|\vec{U}_1\|_{\tilde{W}^m} \|\vec{U}_2\|_{\tilde{W}^{m-2}}. \tag{61}$$

Proof. For $n \geq 0$, since

$$\vec{g}_\alpha^{k,n}(x, y) = e^{\frac{2\pi}{l} ikx} \begin{pmatrix} a\phi_{n+2}(y) \\ b\phi_n(y) \\ c\phi_{n+1}(y) \end{pmatrix} \tag{62}$$

for constants $a = a_\alpha^{k,n}$, $b = b_\alpha^{k,n}$, $c = c_\alpha^{k,n}$, we have

$$\begin{aligned} \left(\frac{1}{\sqrt{2}}L_-\right)^m \vec{g}_\alpha^{k,n} &= e^{\frac{2\pi}{l} ikx} \left(a \left(\frac{1}{\sqrt{2}}L_-\right)^m \begin{pmatrix} \phi_{n+2} \\ 0 \\ 0 \end{pmatrix} + b \left(\frac{1}{\sqrt{2}}L_-\right)^m \begin{pmatrix} 0 \\ \phi_n \\ 0 \end{pmatrix} \right. \\ &\quad \left. + c \left(\frac{1}{\sqrt{2}}L_-\right)^m \begin{pmatrix} 0 \\ 0 \\ \phi_{n+1} \end{pmatrix} \right) \\ &= (-1)^m e^{\frac{2\pi}{l} ikx} \left(a \frac{\sqrt{(n+2+m)!}}{\sqrt{(n+2)!}} \begin{pmatrix} \phi_{n+2+m} \\ 0 \\ 0 \end{pmatrix} + b \frac{\sqrt{(n+m)!}}{\sqrt{n!}} \begin{pmatrix} 0 \\ \phi_{n+m} \\ 0 \end{pmatrix} \right. \\ &\quad \left. + c \frac{\sqrt{(n+1+m)!}}{\sqrt{(n+1)!}} \begin{pmatrix} 0 \\ 0 \\ \phi_{n+1+m} \end{pmatrix} \right). \end{aligned} \tag{63}$$

If we set by convention $\phi_{-2} = \phi_{-1} = 0$, the formulas (62) and (63) are also valid for $n = -2$ and $n = -1$ (with $b_0^{k,-2} = c_0^{k,-2} = 0$ and $b_0^{k,-1} = 0$). So

$$\begin{aligned} \frac{1}{2^m} \langle L_-^m \vec{U}_1, L_-^m \vec{U}_2 \rangle &= \sum_{k,n,\alpha} \sum_{k',n',\alpha'} \left\langle \left(\frac{1}{\sqrt{2}} L_- \right)^m \mathcal{P}_\alpha^{k,n} \vec{U}_1, \left(\frac{1}{\sqrt{2}} L_- \right)^m \mathcal{P}_{\alpha'}^{k',n'} \vec{U}_2 \right\rangle \\ &= \sum_{k,n,\alpha,\alpha'} \langle \vec{U}_1, \vec{g}_\alpha^{k,n} \rangle \overline{\langle \vec{U}_2, \vec{g}_{\alpha'}^{k,n} \rangle} \left(a_\alpha^{k,n} \overline{a_{\alpha'}^{k,n}} \frac{(n+2+m)!}{(n+2)!} \right. \\ &\quad \left. + b_\alpha^{k,n} \overline{b_{\alpha'}^{k,n}} \frac{(n+m)!}{n!} + c_\alpha^{k,n} \overline{c_{\alpha'}^{k,n}} \frac{(n+1+m)!}{(n+1)!} \right). \end{aligned}$$

We have also

$$\begin{aligned} \sum_{k,n,\alpha} (n+3)^m \langle \mathcal{P}_\alpha^{k,n} \vec{U}_1, \mathcal{P}_\alpha^{k,n} \vec{U}_2 \rangle &= (n+3)^m \sum_{k,n,\alpha} \langle \vec{U}_1, \vec{g}_\alpha^{k,n} \rangle \overline{\langle \vec{U}_2, \vec{g}_\alpha^{k,n} \rangle} \\ &= \sum_{k,n,\alpha,\alpha'} \langle \vec{U}_1, \vec{g}_\alpha^{k,n} \rangle \overline{\langle \vec{U}_2, \vec{g}_{\alpha'}^{k,n} \rangle} \\ &\quad \times \left(a_\alpha^{k,n} \overline{a_{\alpha'}^{k,n}} + b_\alpha^{k,n} \overline{b_{\alpha'}^{k,n}} + c_\alpha^{k,n} \overline{c_{\alpha'}^{k,n}} \right) (n+3)^m \end{aligned}$$

because $\langle \vec{g}_\alpha^{k,n}, \vec{g}_{\alpha'}^{k,n} \rangle = a_\alpha^{k,n} \overline{a_{\alpha'}^{k,n}} + b_\alpha^{k,n} \overline{b_{\alpha'}^{k,n}} + c_\alpha^{k,n} \overline{c_{\alpha'}^{k,n}} = \delta_{\alpha\alpha'}$. Since the polynomials $(n+2+m)!/(n+2)!$, $(n+m)!/n!$ and $(n+1+m)!/(n+1)!$ have n^m as leading-order term, the difference of each of them with $(n+3)^m$ is only of order $m-1$. Therefore

$$\begin{aligned} &\left| \frac{1}{2^m} \langle L_-^m \vec{U}_1, L_-^m \vec{U}_2 \rangle - \sum_{k,n,\alpha} (n+3)^m \langle \mathcal{P}_\alpha^{k,n} \vec{U}_1, \mathcal{P}_\alpha^{k,n} \vec{U}_2 \rangle \right| \\ &\leq C \sum_{k,n,\alpha,\alpha'} (n+3)^{m-1} \left| \langle \vec{U}_1, \vec{g}_\alpha^{k,n} \rangle \overline{\langle \vec{U}_2, \vec{g}_{\alpha'}^{k,n} \rangle} \right| \\ &\leq C \sum_{\alpha,\alpha'} \sum_{k,n} (n+3)^{m/2} \left| \langle \vec{U}_1, \vec{g}_\alpha^{k,n} \rangle \right| (n+3)^{m/2-1} \left| \langle \vec{U}_2, \vec{g}_{\alpha'}^{k,n} \rangle \right| \\ &\leq C \sum_{\alpha,\alpha'} \left(\sum_{k,n} (n+3)^m \left| \langle \vec{U}_1, \vec{g}_\alpha^{k,n} \rangle \right|^2 \right)^{1/2} \left(\sum_{k,n} (n+3)^{m-2} \left| \langle \vec{U}_2, \vec{g}_{\alpha'}^{k,n} \rangle \right|^2 \right)^{1/2} \\ &\leq 3C \|\vec{U}_1\|_{\tilde{W}^m} \|\vec{U}_2\|_{\tilde{W}^{m-2}}. \quad \square \end{aligned}$$

Applying Lemmas 5.4 and 5.5 with $\vec{U}_1 = \vec{U}$ and $\vec{U}_2 = \underline{S} \cdot \nabla \vec{U}$, and the Cauchy-Schwarz inequality, we get from (59)

$$\begin{aligned} &\frac{1}{2} \partial_t \left(\sum_{k,n,\alpha} (n+3)^m \|\mathcal{P}_\alpha^{k,n} \vec{U}\|_{L^2}^2 \right) \\ &\leq C \|\vec{U}\|_{\tilde{W}^m} (\|\underline{S}\|_{L^\infty} + \|\nabla \underline{S}\|_{L^\infty}) \|\vec{U}\|_{\tilde{W}^m} + \|\nabla \vec{U}\|_{L^\infty} \|\underline{S}\|_{\tilde{W}^m} + C \|\vec{U}\|_{\tilde{W}^m} \|\underline{S} \cdot \nabla \vec{U}\|_{\tilde{W}^{m-2}} \\ &\quad + \left(\sum_{k,n,\alpha} (n+3)^m \|\mathcal{P}_\alpha^{k,n} \vec{U}\|_{L^2}^2 \right)^{1/2} \left(\sum_{k,n,\alpha} (n+3)^m \|\mathcal{P}_\alpha^{k,n} \vec{F}\|_{L^2}^2 \right)^{1/2}. \end{aligned}$$

As

$$\|\underline{S} \cdot \nabla \vec{U}\|_{\tilde{W}^{m-2}} \leq C(\|\underline{S}\|_{L^\infty} \|\vec{U}\|_{\tilde{W}^{m-1}} + \|\nabla \vec{U}\|_{L^\infty}) \|\underline{S}\|_{\tilde{W}^{m-2}}$$

by Proposition 5.2 and

$$\begin{aligned} \left(\sum_{k,n,\alpha} (n+3)^m \|\mathcal{P}_\alpha^{k,n} \vec{U}\|_{L^2}^2 \right)^{1/2} &\leq C \|\vec{U}\|_{\tilde{W}^m} \\ \left(\sum_{k,n,\alpha} (n+3)^m \|\mathcal{P}_\alpha^{k,n} \vec{F}\|_{L^2}^2 \right)^{1/2} &\leq C \|\vec{F}\|_{\tilde{W}^m} \end{aligned}$$

by Corollary 4.9, we conclude that

$$\begin{aligned} &\partial_t \left(\sum_{k,n,\alpha} (n+3)^m \|\mathcal{P}_\alpha^{k,n} \vec{U}\|_{L^2}^2 \right)^{1/2} \\ &\leq C(\|\underline{S}\|_{L^\infty} + \|\nabla \underline{S}\|_{L^\infty}) \|\vec{U}\|_{\tilde{W}^m} + C \|\nabla \vec{U}\|_{L^\infty} \|\underline{S}\|_{\tilde{W}^m} + C \|\vec{F}\|_{\tilde{W}^m}. \end{aligned} \tag{64}$$

6. Proofs of the Theorems

6.1. Existence of Solutions

Since our a priori estimates are uniform in ϵ , the classical proof of existence for symmetric hyperbolic system (Majda, 1984) gives for all $m \geq 3$ solutions \vec{U}_ϵ to (15) in $C^0([0, T]; \tilde{W}^m) \cap C^1([0, T]; \tilde{W}^{m-1})$ for some $T > 0$ independent of ϵ with a bound $\|\vec{U}_\epsilon(t)\|_{\tilde{W}^m} \leq C$ uniform in ϵ and $t \in [0, T]$.

6.2. Fast Averaging

We have $\|e^{\tau \mathcal{L}} \vec{U}\|_{\tilde{W}^{m-1}} \leq C \|\vec{U}\|_{\tilde{W}^{m-1}}$ for some constant C independent of τ and \vec{U} . Let $\tilde{U}_\epsilon = \vec{U}_\epsilon - \vec{G}_\epsilon$, with \vec{G}_ϵ defined by (53). Making $\tilde{U} = \tilde{V} = \tilde{U}_\epsilon$ in (54) gives

$$\partial_t \tilde{U}_\epsilon + \underline{S}(\tilde{U}_\epsilon + \vec{G}_\epsilon) \cdot \nabla \tilde{U}_\epsilon + \frac{1}{\epsilon} \mathcal{L} \tilde{U}_\epsilon = \tilde{F}_\epsilon + \mathcal{L}_{1,\epsilon} \tilde{U}_\epsilon - \underline{S}(\tilde{U}_\epsilon) \cdot \nabla \vec{G}_\epsilon,$$

so

$$\partial_t (e^{\frac{t}{\epsilon} \mathcal{L}} \tilde{U}_\epsilon) = e^{\frac{t}{\epsilon} \mathcal{L}} \left(\tilde{F}_\epsilon + \mathcal{L}_{1,\epsilon} \tilde{U}_\epsilon - \underline{S}(\tilde{U}_\epsilon) \cdot \nabla \vec{G}_\epsilon - \underline{S}(\tilde{U}_\epsilon + \vec{G}_\epsilon) \cdot \nabla \tilde{U}_\epsilon \right)$$

is bounded in $C([0, T]; \tilde{W}^{m-1})$.

By the Lions–Aubin compactness lemma, $e^{\frac{t}{\epsilon} \mathcal{L}} \tilde{U}_\epsilon \rightarrow \vec{u}$ in $C([0, T]; \tilde{W}^{m-1})$ where $\vec{u} \in \text{Lip}([0, T]; \tilde{W}^{m-1})$ satisfies, following Schochet’s theory (Schochet, 1994a),

$$\partial_t \vec{u} = \lim_{T_1 \rightarrow \infty} \frac{1}{T_1} \int_{T_0}^{T_0+T_1} h(t, \tau) d\tau \tag{65}$$

with

$$\begin{aligned}
 h(t, \tau) = & e^{\tau \mathcal{L}} (\vec{F}_0^0(t) + \mathcal{L}_{1,0}(\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1 \\
 & - \underline{\mathcal{S}}((\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1) \cdot \nabla((\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1)) \\
 & - (\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \partial_t \vec{F}_0^1 + \mathcal{L}_{1,0} e^{-\tau \mathcal{L}} \vec{u}(t) \\
 & - \underline{\mathcal{S}}(e^{-\tau \mathcal{L}} \vec{u}(t)) \cdot \nabla((\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1(t)) \\
 & - \underline{\mathcal{S}}(e^{-\tau \mathcal{L}} \vec{u}(t)) \cdot \nabla(e^{-\tau \mathcal{L}} \vec{u}(t)) \\
 & - \underline{\mathcal{S}}((\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1(t)) \cdot \nabla(e^{-\tau \mathcal{L}} \vec{u}(t))),
 \end{aligned}$$

if the limit in (65) exists in L^2 , uniformly in T_0 (in Dutrifoy and Majda, 2006 we give, using a lemma also due to Schochet, 1994b, a self-contained justification of a similar assertion and it can easily be adapted here). And indeed we have the following convergences:

$$\begin{aligned}
 & \frac{1}{T_1} \int_{T_0}^{T_0+T_1} e^{\tau \mathcal{L}} \vec{F}_0^0(t) d\tau \rightarrow \mathcal{P}_0 \vec{F}_0^0(t) \\
 & \frac{1}{T_1} \int_{T_0}^{T_0+T_1} e^{\tau \mathcal{L}} (\mathcal{L}_{1,0}(\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1) d\tau \rightarrow \sum_{\lambda_x^{k,n} = c_{x'}^{k',n'}} \frac{\mathcal{P}_\alpha^{k,n} (\mathcal{L}_{1,0} \mathcal{P}_{\alpha'}^{k',n'} \vec{F}_0^1(t))}{\lambda_{\alpha'}^{k',n'} - c_{\alpha'}^{k',n'}} \\
 & \frac{1}{T_1} \int_{T_0}^{T_0+T_1} e^{\tau \mathcal{L}} (\underline{\mathcal{S}}((\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1) \cdot \nabla((\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1)) d\tau \\
 & \rightarrow \sum_{\lambda_x^{k,n} = c_{\alpha'}^{k',n'} + c_{\alpha''}^{k'',n''}} \frac{\mathcal{P}_\alpha^{k,n} (\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{F}_0^1(t)) \cdot \nabla(\mathcal{P}_{\alpha''}^{k'',n''} \vec{F}_0^1(t)))}{(\lambda_{\alpha'}^{k',n'} - c_{\alpha'}^{k',n'}) (\lambda_{\alpha''}^{k'',n''} - c_{\alpha''}^{k'',n''})} \\
 & \frac{1}{T_1} \int_{T_0}^{T_0+T_1} e^{\tau \mathcal{L}} ((\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \partial_t \vec{F}_0^1) d\tau \rightarrow \sum_{\lambda_x^{k,n} = c_{\alpha'}^{k',n'}} \frac{\mathcal{P}_\alpha^{k,n} (\mathcal{P}_{\alpha'}^{k',n'} \partial_t \vec{F}_0^1(t))}{\lambda_{\alpha'}^{k',n'} - c_{\alpha'}^{k',n'}} = 0
 \end{aligned}$$

(due to the orthogonality of the eigenspaces and the non-resonance assumption)

$$\begin{aligned}
 & \frac{1}{T_1} \int_{T_0}^{T_0+T_1} e^{\tau \mathcal{L}} (\mathcal{L}_{1,0} e^{-\tau \mathcal{L}} \vec{u}(t)) d\tau \rightarrow \sum_{\lambda_x^{k,n} = \lambda_{\alpha'}^{k',n'}} \mathcal{P}_\alpha^{k,n} (\mathcal{L}_{1,0} \mathcal{P}_{\alpha'}^{k',n'} \vec{u}(t)) \\
 & \frac{1}{T_1} \int_{T_0}^{T_0+T_1} e^{\tau \mathcal{L}} (\underline{\mathcal{S}}(e^{-\tau \mathcal{L}} \vec{u}(t)) \cdot \nabla((\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1(t))) d\tau \\
 & \rightarrow \sum_{\lambda_x^{k,n} = \lambda_{\alpha'}^{k',n'} + c_{\alpha''}^{k'',n''}} \frac{\mathcal{P}_\alpha^{k,n} (\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{u}(t)) \cdot \nabla \mathcal{P}_{\alpha''}^{k'',n''} \vec{F}_0^1(t))}{\lambda_{\alpha''}^{k'',n''} - c_{\alpha''}^{k'',n''}} \\
 & \frac{1}{T_1} \int_{T_0}^{T_0+T_1} e^{\tau \mathcal{L}} (\underline{\mathcal{S}}(e^{-\tau \mathcal{L}} \vec{u}(t)) \cdot \nabla(e^{-\tau \mathcal{L}} \vec{u}(t))) d\tau \\
 & \rightarrow \sum_{\lambda_x^{k,n} = \lambda_{\alpha'}^{k',n'} + \lambda_{\alpha''}^{k'',n''}} \mathcal{P}_\alpha^{k,n} (\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{u}(t)) \cdot \nabla(\mathcal{P}_{\alpha''}^{k'',n''} \vec{u}(t)))
 \end{aligned}$$

$$\begin{aligned} & \frac{1}{T_1} \int_{T_0}^{T_0+T_1} e^{\tau \mathcal{L}} (\underline{\mathcal{S}}((\mathcal{L} - \mathcal{L}_0)^{-1} e^{-\tau \mathcal{L}_0} \vec{F}_0^1(t)) \cdot \nabla(e^{-\tau \mathcal{L}} \vec{u}(t))) d\tau \\ & \rightarrow \sum_{\lambda_{\alpha'}^{k,n} = c_{\alpha'}^{k',n'} + \lambda_{\alpha''}^{k'',n''}} \frac{\mathcal{P}_\alpha^{k,n} (\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{F}_0^1(t)) \cdot \nabla(\mathcal{P}_{\alpha''}^{k'',n''} \vec{u}(t)))}{\lambda_{\alpha'}^{k',n'} - c_{\alpha'}^{k',n'}}. \end{aligned}$$

This proves Theorem 3.2. We get Corollary 3.3 by imposing the restriction $k = \alpha = 0$ in the summations:

$$\begin{aligned} \partial_t \mathcal{P}_0 \vec{u} &= \mathcal{P}_0 \vec{F}_0 + \sum_{c_{\alpha'}^{0,n'} = 0} \frac{\mathcal{P}_0^{0,n} (\mathcal{L}_{1,0} \mathcal{P}_{\alpha'}^{0,n'} \vec{F}_0^1)}{\lambda_{\alpha'}^{0,n'} - c_{\alpha'}^{0,n'}} \\ & - \sum_{c_{\alpha'}^{k',n'} + c_{\alpha''}^{-k',n''} = 0} \frac{\mathcal{P}_0^{0,n} (\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{F}_0^1) \cdot \nabla(\mathcal{P}_{\alpha''}^{-k',n''} \vec{F}_0^1))}{(\lambda_{\alpha'}^{k',n'} - c_{\alpha'}^{k',n'}) (\lambda_{\alpha''}^{-k',n''} - c_{\alpha''}^{-k',n''})} \end{aligned} \tag{66}$$

$$+ \sum \mathcal{P}_0^{0,n} (\mathcal{L}_{1,0} \mathcal{P}_0^{0,n'} \vec{u}) - \sum_{\lambda_{\alpha'}^{k',n'} + c_{\alpha''}^{-k',n''} = 0} \frac{\mathcal{P}_0^{0,n} (\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{u}) \cdot \nabla(\mathcal{P}_{\alpha''}^{-k',n''} \vec{F}_0^1))}{\lambda_{\alpha''}^{-k',n''} - c_{\alpha''}^{-k',n''}} \tag{67}$$

$$- \sum_{c_{\alpha'}^{k',n'} + \lambda_{\alpha''}^{k'',n''} = 0} \frac{\mathcal{P}_0^{0,n} (\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{F}_0^1) \cdot \nabla(\mathcal{P}_{\alpha''}^{-k',n''} \vec{u}))}{\lambda_{\alpha'}^{k',n'} - c_{\alpha'}^{k',n'}} \tag{68}$$

$$- \sum_{\lambda_{\alpha'}^{k',n'} + \lambda_{\alpha''}^{k'',n''} = 0} P_0 (\underline{\mathcal{S}}(\mathcal{P}_{\alpha'}^{k',n'} \vec{u}) \cdot \nabla(\mathcal{P}_{\alpha''}^{k'',n''} \vec{u})). \tag{69}$$

We get (28) from (66) replacing α'' by $-\alpha''$, thanks to the conditions (17) and Proposition 4.10. We get (30) from (67) in the very same way. We get (31) from (68) interchanging (n', α') and (n'', α'') , substituting $-k'$ to k' , and then again replacing α'' by $-\alpha''$. Corollary 3.3 finally follows because the interaction of fast waves has no influence on the slow dynamics:

Proposition 6.1. *The sum (69) is zero.*

Proof. That sum must be taken over all $n', k', \alpha', n'', k'',$ and α'' such that

$$k' + k'' = 0 \tag{70}$$

and

$$\lambda_{\alpha'}^{k',n'} + \lambda_{\alpha''}^{k'',n''} = 0. \tag{71}$$

If (70) and (71) are satisfied, then $\lambda = \lambda_{\alpha'}^{k',n'}$ is solution of both

$$\lambda^3 + \lambda \left(\frac{4\pi^2}{l^2} k'^2 + 2n' + 3 \right) + \frac{2\pi}{l} ik' = 0$$

and

$$-\lambda^3 - \lambda \left(\frac{4\pi^2}{l^2} k'^2 + 2n'' + 3 \right) - \frac{2\pi}{l} ik' = 0.$$

By addition, either $\lambda = 0$ —which implies $k' = k'' = \alpha' = \alpha'' = 0$ (see Remark 4.13)—or $n' = n''$, and so the sum can be split in two:

- a sum over all n' and n'' with $k' = k'' = \alpha' = \alpha'' = 0$
- a sum over all n', k' and α' with $n'' = n', k'' = -k', \alpha'' = -\alpha'$ (see Proposition 4.10) and either $k' \neq 0$ or $\alpha' \neq 0$.

The first sum is zero because each term is zero:

$$\begin{aligned} & P_0(\underline{S}(\mathcal{P}_0^{n',0}\vec{\varrho}\vec{u}(t)) \cdot \nabla(\mathcal{P}_0^{n'',0}\vec{\varrho}\vec{u}(t))) \\ &= P_0\left(S_2\left(\frac{1}{l} \int_{\mathbb{T}} P_0^{(n')} \vec{\varrho}\vec{u}(t, x) dx\right) \partial_y \left(\frac{1}{l} \int_{\mathbb{T}} P_0^{(n'')} \vec{\varrho}\vec{u}(t, x) dx\right)\right) = 0 \end{aligned}$$

for all n' and n'' (see the end of Section 3 in Dutrifoy and Majda, 2006). The second sum is zero because the terms corresponding to (k', α') and $(-k', -\alpha')$ cancel each other for all n' . Indeed,

$$\begin{aligned} P_0(\underline{S}(\mathcal{P}_{\alpha'}^{k',n'}\vec{\varrho}\vec{u}(t)) \cdot \nabla(\mathcal{P}_{-\alpha'}^{-k',n'}\vec{\varrho}\vec{u}(t))) &= P_0(\underline{S}(\langle \vec{\varrho}\vec{u}, \vec{g}_{\alpha'}^{k',n'} \rangle \vec{g}_{\alpha'}^{k',n'}) \cdot \nabla(\langle \vec{\varrho}\vec{u}, \vec{g}_{-\alpha'}^{-k',n'} \rangle \vec{g}_{-\alpha'}^{-k',n'})) \\ &= \langle \vec{\varrho}\vec{u}, \vec{g}_{\alpha'}^{k',n'} \rangle \langle \vec{\varrho}\vec{u}, \vec{g}_{-\alpha'}^{-k',n'} \rangle P_0(\underline{S}(\vec{g}_{\alpha'}^{k',n'}) \cdot \nabla \vec{g}_{-\alpha'}^{-k',n'}), \end{aligned}$$

so

$$\begin{aligned} & P_0(\underline{S}(\mathcal{P}_{\alpha'}^{k',n'}\vec{\varrho}\vec{u}(t)) \cdot \nabla(\mathcal{P}_{-\alpha'}^{-k',n'}\vec{\varrho}\vec{u}(t))) + P_0(\underline{S}(\mathcal{P}_{-\alpha'}^{-k',n'}\vec{\varrho}\vec{u}(t)) \cdot \nabla(\mathcal{P}_{\alpha'}^{k',n'}\vec{\varrho}\vec{u}(t))) \\ &= \langle \vec{\varrho}\vec{u}, \vec{g}_{\alpha'}^{k',n'} \rangle \langle \vec{\varrho}\vec{u}, \vec{g}_{-\alpha'}^{-k',n'} \rangle P_0(\underline{S}(\vec{g}_{\alpha'}^{k',n'}) \cdot \nabla \vec{g}_{-\alpha'}^{-k',n'} + \underline{S}(\vec{g}_{-\alpha'}^{-k',n'}) \cdot \nabla \vec{g}_{\alpha'}^{k',n'}); \end{aligned}$$

but since $\vec{g}_{-\alpha'}^{-k',n'} = \overline{\vec{g}_{\alpha'}^{k',n'}}$ (see Proposition 4.11),

$$P_0(\underline{S}(\vec{g}_{\alpha'}^{k',n'}) \cdot \nabla \overline{\vec{g}_{-\alpha'}^{-k',n'}} + \underline{S}(\overline{\vec{g}_{-\alpha'}^{-k',n'}}) \cdot \nabla \vec{g}_{\alpha'}^{k',n'}) = P_0(2 \Re(\underline{S}(\vec{g}_{\alpha'}^{k',n'}) \cdot \nabla \overline{\vec{g}_{-\alpha'}^{-k',n'}})),$$

which is zero in view of (11), (12), (37), (39) and (47). □

7. Example

For any $a \in \mathbb{C}$, let

$$\vec{F}_0^1(t) = a \vec{f}^{k_0,-2} + \bar{a} \vec{f}^{-k_0,-2} = ae^{\frac{2\pi}{l} ik_0 x} \begin{pmatrix} \phi_0(y) \\ 0 \\ 0 \end{pmatrix} + \text{c.c.}$$

and

$$c_0^{k_0,-2} = \lambda_1^{k_0,-1}.$$

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Then the sum of (30) and (31) reduces to

$$\begin{aligned} & \sum_{n=-2}^{+\infty} \frac{\mathcal{P}_0^{0,n}(\underline{S}(\mathcal{P}_1^{k_0,-1}\vec{u}) \cdot \nabla(\mathcal{P}_0^{-k_0,-2}\vec{F}_0^1))}{\lambda_0^{k_0,-2} - \lambda_1^{k_0,-1}} + \text{c.c.} \\ & + \sum_{n=-2}^{+\infty} \frac{\mathcal{P}_0^{0,n}(\underline{S}(\mathcal{P}_0^{-k_0,-2}\vec{F}_0^1) \cdot \nabla(\mathcal{P}_1^{k_0,-1}\vec{u}))}{\lambda_0^{k_0,-2} - \lambda_1^{k_0,-1}} + \text{c.c.} \end{aligned} \tag{72}$$

As

$$\mathcal{P}_1^{k_0,-1}\vec{u} = ue^{\frac{2\pi}{l}ik_0x} \begin{pmatrix} \lambda_1^{k_0,-1}\phi_1 \\ 0 \\ \phi_0 \end{pmatrix},$$

for some scalar function $u = u(t)$, we have

$$S_1^l(\mathcal{P}_1^{k_0,-1}\vec{u}) = \frac{u}{2\sqrt{2}} e^{\frac{2\pi}{l}ik_0x} \lambda_1^{k_0,-1}\phi_1 \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$S_2(\mathcal{P}_1^{k_0,-1}\vec{u}) = \frac{u}{4} e^{\frac{2\pi}{l}ik_0x} \begin{pmatrix} 4\phi_0 & 0 & \lambda_1^{k_0,-1}\phi_1 \\ 0 & 4\phi_0 & \lambda_1^{k_0,-1}\phi_1 \\ \lambda_1^{k_0,-1}\phi_1 & \lambda_1^{k_0,-1}\phi_1 & 4\phi_0 \end{pmatrix},$$

while

$$\mathcal{P}_0^{-k_0,-2}\vec{F}_0^1 = \bar{a} e^{-\frac{2\pi}{l}ik_0x} \phi_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and so

$$S_1^l(\mathcal{P}_0^{-k_0,-2}\vec{F}_0^1) = \frac{\bar{a}}{2\sqrt{2}} e^{-\frac{2\pi}{l}ik_0x} \phi_0 \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$S_2(\mathcal{P}_0^{-k_0,-2}\vec{F}_0^1) = \frac{\bar{a}}{4} e^{-\frac{2\pi}{l}ik_0x} \phi_0 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The explicit expressions of $\lambda_0^{k_0,-2}$ (38) and $\lambda_1^{k_0,-1}$ (40) yield

$$\lambda_0^{k_0,-2} - \lambda_1^{k_0,-1} = \frac{2\pi}{l}ik_0 - \left(\frac{\pi}{l}ik_0 + i\sqrt{\frac{\pi^2}{l^2}k_0^2 + 1} \right) = \lambda_{-1}^{k_0,-1}.$$

From (33) and (35), it follows that

$$\partial_y \phi_0 = \frac{1}{2}(L_+ + L_-)\phi_0 = -\frac{1}{\sqrt{2}}\phi_1.$$

Hence

$$\begin{aligned} (72) &= \frac{1}{\lambda_{-1}^{k_0-1}} \sum_{n=-2}^{+\infty} \mathcal{P}_0^{0,n} \left(-\frac{u}{2\sqrt{2}} \lambda_1^{k_0-1} \phi_1 \bar{a} \frac{2\pi}{l} ik_0 \phi_0 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - u \phi_0 \frac{\bar{a}}{\sqrt{2}} \phi_1 \begin{pmatrix} 1 \\ 0 \\ * \end{pmatrix} \right. \\ &\quad \left. + \frac{\bar{a}}{2\sqrt{2}} \phi_0 u \frac{2\pi}{l} ik_0 \begin{pmatrix} 3\lambda_1^{k_0-1} \phi_1 \\ 0 \\ * \end{pmatrix} - \frac{\bar{a}}{4} \phi_0 \frac{u}{\sqrt{2}} \phi_1 \begin{pmatrix} 1 \\ 1 \\ * \end{pmatrix} \right) + \text{c.c.} \\ &= -\frac{\bar{a}u}{4\sqrt{2} \lambda_{-1}^{k_0-1}} \sum_{n=-2}^{+\infty} \mathcal{P}_0^{0,n} \left(\phi_0 \phi_1 \begin{pmatrix} 5 \\ 4 \\ * \end{pmatrix} \right) + \text{c.c.} \end{aligned}$$

By (48), the n th term is non-zero for general a and u if

$$5\sqrt{n+1} \langle \phi_0 \phi_1, \phi_{n+2} \rangle + 4\sqrt{n+2} \langle \phi_0 \phi_1, \phi_n \rangle \neq 0. \quad (73)$$

The scalar products $\langle \phi_0 \phi_1, \phi_{n+2} \rangle$ and $\langle \phi_0 \phi_1, \phi_n \rangle$ are zero if n is even, but

$$\langle \phi_0 \phi_1, \phi_n \rangle = -\frac{4\sqrt{2} \pi^{3/4}}{l\sqrt{3}} (-3)^{-(n+1)/2} \frac{n!!}{\sqrt{n!}}$$

if n is odd (Busbridge, 1948; Ripa, 1982). So

$$\frac{\sqrt{n+1} \langle \phi_0 \phi_1, \phi_{n+2} \rangle}{\sqrt{n+2} \langle \phi_0 \phi_1, \phi_n \rangle} = -\frac{1}{3} \neq -\frac{4}{5}$$

and (73) is satisfied if and only if n is odd.

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