Real World Turbulence and Modern Applied Mathematics

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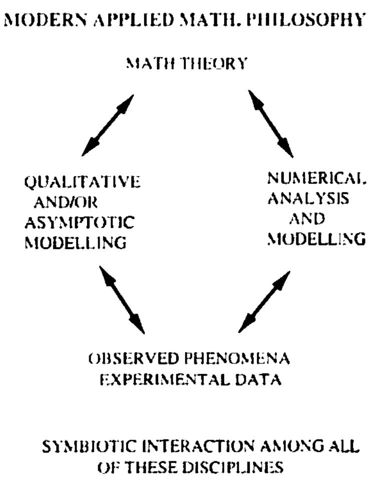
1. Introduction

The topic for my own view of mathematics at the millenium involves real world turbulence, centered around turbulence in the atmosphere and the ocean, some of the current central scientific issues, and the prospects for significant contributions in the future through modern applied mathematics. Modern applied mathematics has flourished as a discipline in roughly the last twenty years where practitioners are at ease and flexibly utilize the many facets of applied mathematics in the following diagram:

Why do I discuss the topic of turbulence in atmosphere/ocean science? There is no doubt that one of the grand challenges for science in the next century is to achieve a detailed understanding of the atmosphere/ocean/ and land, including their interaction through fluid mechanical, biological, and chemical processes so that detailed reliable predictions of short term



and longer term climate can be made. These problems are so complex because different nonlinear physical processes intervene and couple throughout an amazingly wide range of physical scales, ranging from millimeters to the order of ten thousand kilometers, in a highly anisotropic fashion. No supercomputer in the forseeable future will be able to represent all of the physical processes occurring on all of these scales, and detailed observations of all of these scales simultaneously will not be possible. The current scientific progress that has been made thus far in predicting weather and climate involves computer models called general circulation models (GCM's) where the gross features of the atmosphere and ocean are modelled accurately, but the detailed interaction with physical processes on small length scales (usually smaller than the order of a few hundred kilometers) is parametrized. The way these effects are parametrized involves theories for crude modelling of inherently random stochastic turbulence from scales and processes that are not represented in detail. There is a great need for understanding and improving these parametrizations, especially for climate prediction where the spatial resolution of the GCM's is even more strongly limited due to the long interval of time integration.

Understanding such issues necessarily involves a fascinating and novel interplay among such mathematical topics as nonlinear partial differential equations, probability theory and statistics, chaotic dynamics, etc., all interacting in the symbiotic mode of modern applied mathematics mentioned above as well as through close collaboration with atmosphere/ocean scientists.

In my essay, I attempt to outline several of these scientific issues, some of the current directions of interest for mathematicians, and some of the future prospects for multi-disciplinary interaction for applied mathematicians. I also will try to convey some of the flavor of the modus operandi of modern applied mathematics.

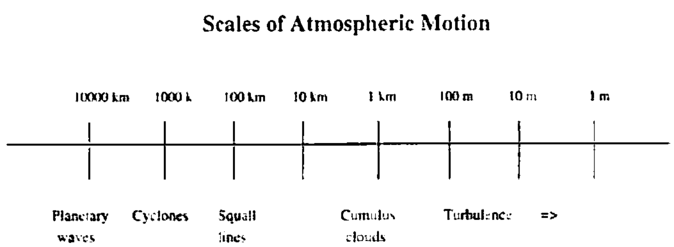


FIGURE 1

Tropical Western Pacific

Mesoscale

• Precipitating Clouds	1-10 km	< 1 hr
• Cloud Ensembles	10-100 km	> hrs
• Mesoscale Cloud Systems	~ 500 km	hrs-days
• "Super" Cloud Clusters	1000-2000 km	days-weeks

Large Scale

• Tropical Waves (Kelvin, Rossby-Gravity, see [1])	Synoptic	week
• Mean Circulations (Hadley, Walker, see [1])	Planetary	interseasonal
• El Nino Southern Oscillation	Planetary	interannual

FIGURE 2. Spatio-Temporal Scales of Motion

2. The scales of atmospheric motion and the strong effect of rotation and stratification

In order to give an example of the wide range of spatial scales involved in predicting the weather and climate, I have charted in Figure 1 the scales of atmospheric motion.

In Figure 2, I have listed the spatio-temporal scales of motion in an important part of the world for predicting short term climate, the Tropical Western Pacific. It should be evident to the reader that distinct physical phenomena on a very wide range of spatio-temporal scales in the atmosphere participate in the evolution of short term climate which involves their coupling to a similar broad range of scales in the ocean.

In atmosphere/ocean science there is little doubt about the equations governing the fluid motion; all of the uncertainty arises from scales of motion that are unresolved and must be treated statistically. The simplest illustrative model for the dynamics of the atmosphere is given by rotating stratified Boussinesq equations for the velocity and density variations,

$$
\begin{aligned}
&\frac{D\vec{v}}{Dt} + (Ro)^{-1}\vec{e}_3 \times \vec{v} + (Fr)^{-1}\rho\vec{e}_3 = -\nabla p \\
&\frac{D\rho}{Dt} - (Fr)^{-1}v_3 = 0 \qquad (1)\\
&div\ \vec{v} = 0.
\end{aligned}
$$

Here $\vec{v} = (v_1, v_2, v_3)$ is the fluid velocity, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}.\nabla$ is the convective derivative, $\vec{e}_3 = (0,0,1), p$ is the pressure, and ρ involves the variation of density created from interaction through the force of gravity. The term, $(Ro)^{-1}\vec{e}_3 \times \vec{v}$ represents the tangent plane approximation to the earth's rotation while the two terms multiplying Fr^{-1} represent the effect of buoyancy forcing due to gravity on the dynamics.

The nondimensional coefficient representing the strength of rotation is the Rossby number,

$$Ro = \frac{V}{Lf} \tag{2}$$

while the strength of (stable) stratification is measured by the Froude number

$$Fr = \frac{V}{HN}. \tag{3}$$

In (2), f is the rotation frequency while V is the typical fluid velocity and L is the typical horizontal length scale; for (3), H is the typical vertical height and N is the buoyancy frequency.

For the moment, let's compare these equations with the more familiar incompressible Euler equations in 3-D

$$\begin{aligned} \frac{D\vec{v}}{Dt} &= -\nabla p \\ div\, \vec{v} &= 0. \end{aligned} \tag{4}$$

Clearly the dynamics in these two equations in (1) and (4) differ through the terms involving the Rossby number, Ro, and Froude number, Fr. At first glance, these quantities in (1) seem like harmless lower order terms compared with (4). Indeed on the very small spatial scales of ordinary human motion, the effects of the earth's rotation, for example, are negligible in agreement with our collective experience; however, for the very large planetary scales on the order of 10,000 km, the atmospheric motions are strongly affected by rotation, and rotational effects are important in the polar oceans even over scales of only a few kilometers. In fact, most of the fluid motion of interest for climate modelling is characterized by

(5)
either strong stratification, $Fr << 1$
or strong rotation, $Ro << 1$
or both.

The interested reader can consult the basic texts ([1], [2], [3]) for extensive discussions.

Thus, the "lower order" terms in (1) from rotation and stratification in fact play a dominant role in the dynamics of the atmosphere/ocean as compared with (4). The result is striking new phenomena in (1) involving

(6)
$$\begin{gathered}\text{inertio-gravity waves}\\ \text{and}\\ \text{scalar potential vorticity}\end{gathered}$$

(see [1], [2], [3]). This strongly contrasts with the dynamics for ordinary fluid flow in (4) which involves evolution of the vector vorticity, $\vec{\omega}$, through self-stretching alone ([4],[5]). This prominent effect of self-stretching in ordinary homogeneous fluid flows in fact is usually lower order for geophysical flows except at the smallest length scales. There has been a significant amount of recent activity in the P.D.E. community, revealing subtle behavior in the dynamics for (1) in the limiting regimes described in (5) (see [6], [7], [8], [9], [10], [11] and the references there).

Controlled laboratory experiments in strongly stratified and/or rotating flows provide great insight into the differences between homogeneous flows satisfying (4) and the highly anisotropic inhomogeneous flows with strong stratification and/or rotation satisfying (1) and (5); such experiments also provide wonderful opportunities for modern applied mathematics. Next, I briefly describe one such example of this interaction.

Embid and the author ([7], [8], [9]) have developed mathematical theories for averaging over fast gravity waves in strongly stratified flows for general unbalanced initial data, as occurs on mesoscales in both the atmosphere and ocean. The author and Grote ([12]) have utilized the reduced dynamics in the low Froude number limit given by these theories to develop a novel theoretical model which captures several salient features of the remarkable recent laboratory experiments of Fincham, Maxworthy, and Spedding ([13]) on turbulent decaying strongly stratified flows. Both the theory and the experiments yield families of initial columnar dipole vortices with dominant vertical vorticity which collapse into pancaked vortex sheets with dominant horizontal vorticity while simultaneously, vertical dissipation of energy strongly dominates horizontal dissipation. The actual experiments are fully turbulent while the analytical models of Majda and Grote ([12]) involve a wide range of vertical scales but a fixed horizontal scale of motion.

3. Turbulence, waves, and universal scaling laws in the atmosphere and ocean

The universal scaling law of stochastic turbulence theory, which is probably familiar to the reader, arises for the homogenous incompressible fluid equations from (4) with dissipation added. This is Kolmogorov's famous $\frac{5}{3}$ law and states ([14]) that the energy spectrum for velocity fluctuations at intermediate scales, larger than the dissipation scale but smaller than the large scale motions, behaves universally according to the $-\frac{5}{3}$ power, i.e.,

(7)
$$E(k) = C|k|^{-\frac{5}{3}}.$$

The derivation is based on a postulated cascade of energy from large to small scales and dimensional analysis. This scaling law has been confirmed in a variety of engineering flows although a fundamental mathematical derivation remains elusive ([14]). The significance of such a universal law for energy is that it connects properties of the fluid flow on scales that can be understood (large scale engineering flows) to those that have large uncertainty (the turbulent scales).

For the atmosphere and the ocean, are there such universal scaling laws over appropriate length scales? From the discussion in section 2, such laws would necessarily involve a fascinating highly anisotropic interplay between rotation and stratification, manifested as turbulence from the different physical structures involving inertial-gravity waves and scalar potential vorticity. It is remarkable that such universal laws abound for different scaling regimes in the atmosphere and the ocean. Here is a partial list of some of these universal laws with their appropriate spatial scaling regime.

(8)

1) The Gage-Nastrom spectrum ([15]) for the upper troposphere/lower stratosphere on scales from $0(10km)$ to $0(1.000km)$.
2) The Garrett-Munk Spectrum ([16]) for internal gravity waves in the ocean on scales from $0(.1m)$ to $(1km)$.
3) The Charney Spectrum for planetary waves and baroclinic turbulence in the atmosphere on scales from $0(10^3km)$ to $0(10^4km)$.
4) The Phillips Spectrum for ocean surface waves over deep water ([17]) on scales from $0(10^{-2}m)$ to $0(1m)$.
5) The Kolmogorov spectrum ([14]) in the atmospheric boundary layer at heights of roughly 100 meters on scales from $0(10^{-3}m)$ to $0(1m)$.

The perceptive reader will note that the famous Kolmogorov spectrum of engineering turbulence applies in a very limited regime of small scales in the atmosphere as listed in 5) above. A very active and hotly debated contemporary research area involves the physical origins of the universal laws from 1) - 4). In particular, does energy flow from the large scales to the small scales or, more significantly for stochastic modelling, is there an inverse energy cascade from the small unresolvable scales to the larger scales? The special volume ([18]) contains contemporary discussion of these issues including contributions from several applied mathematicians besides atmosphere/ocean scientists. Modern applied mathematics has much to offer in understanding these problems which mix both waves and vortices. An excellent contemporary contribution blending sophisticated mathematical theory and careful numerics in studying the inverse cascade for geophysical flows is the recent study by Smith and Waleffe ([19]). Next, I describe another recent contribution ([20]) demonstrating the role of modern applied mathematics in these problems.

3.1. Dispersive wave turbulence. Are there formal turbulence theories generalizing Kolmogorov's theory to predict the universal spectra in (8)? The spectra listed in 1), 2), 3), 4) above all have features of dispersive wave turbulence. There are very elegant theories ([21], [22]) for the spectra of wave turbulence which retain features of Kolmogorov's theory and applying to dispersive waves in a Hamiltonian setting. These theories have been applied to prototype physical flows and involve uncontrolled perturbation expansions utilizing Feynman diagrams with regimes of success and also failure for the predictions. How can one assess these theories in an unambiguous fashion?

Tabak, McLaughlin, and the author ([20]) have introduced recently a family of elementary models with rich features for testing dispersive wave turbulence. The model equations have the form

$$i\psi_t = |\partial_x|^{\alpha}\psi + |\partial_x|^{-\frac{\beta}{4}}\left(\left||\partial_x|^{-\frac{\beta}{4}}\psi\right|^2|\partial_x|^{-\frac{\beta}{4}}\psi\right) \tag{9}$$

for a complex scalar ψ where α and β are fixed parameters with $0 < \alpha < I$ and $-\infty < \beta < \infty$. Here the operator $|\partial_x|^{\gamma}$ is the pseudo-differential operator defined by

$$|\partial_x|^{\gamma}\psi = \int_{-\infty}^{\infty} e^{ixk}|k|^{\gamma}\hat{\psi}(k)dk \tag{10}$$

where $\hat{}$ denotes the Fourier transform. The family of equations in (9) are the simplest one-dimensional models involving dispersive waves and nontrivial (quartric) resonances. These models have unambiguous and truly remarkable turbulent energy spectra (see Figs. 14a), b), c), d) from ([20]) for a wide range of parameter values in β with α fixed at $\alpha = \frac{1}{2}$. These spectra typically extend over two clean scaling decades and have quasi-Gaussian statistical structure. The models also have an explicitly solvable weak turbulence theory ([21], [22]) which is presented pedagogically in ref. ([20]) and exhibits rich predicted behavior as the parameter β is varied. In an unambiguous fashion, the predictions of weak turbulence theory can be checked against the calculated spectra from numerical simulations for this model. There is a large discrepancy between weak turbulence theory and the calculated spectra; a new closure theory is proposed in ref. ([20]) which agrees with the calculated spectra.

The above results suggest that one-dimensional dispersive wave turbulence is nontrivial and structurally much simpler than vortical turbulence. Furthermore, such models are intriguing elementary ones for features of geophysical flows. It would be an exciting and accessible mathematical development if the turbulence theory in such models could be understood rigorously.

4. Turbulent reaction diffusion equations

The simplest model for rotating stratified flow as occurs in the atmosphere or ocean is given by the Boussinesq equations in (1). There are other very important physical, chemical and biological processes in the atmosphere and ocean, such as the production of greenhouse gases and the depletion of ozone in the stratosphere, the condensation, evaporation, ice formation, and precipitation of cloud water in the lower atmosphere, and the tracking of anthropogenic chemical tracers in the ocean. All of these problems involve turbulent reaction diffusion equations for the concentrations, mass fractions, or mixing ratio of these scalar fields. The effect of the change in time of such scalar fields on the velocity field, itself, is often extremely small. Thus, the prototypical example of a turbulent reaction diffusion equation is given by the scalar field,

$$\frac{\partial Z}{\partial t} + \left(\vec{V}(\vec{x},t) + \vec{v}(\vec{x},t)\right) \cdot \nabla Z = K\Delta Z + f(Z) \tag{11}$$

with a prescribed velocity field. In (11), $f(Z)$ is a nonlinear reactive source term, $K > 0$ is the molecular diffusivity, and the velocity field has two parts

(12)
A) $\vec{V}(x,,t)$, the known deterministic large scale velocity field with $\operatorname{div} \vec{V} = 0$
B) $\vec{v}(x,t)$, the inherently statistical velocity field, arising from fluctuations on the unresolved scales with $\operatorname{div} \vec{v} = 0$

In more realistic models, $\vec{Z}$ is often a vector with many interacting components; here for simplicity in exposition, I assume that Z is a scalar field.

4.1. Turbulent diffusion. The simplest important situation for turbulent reaction diffusion equations involves a passive tracer where the nonlinear source terms vanish identically and the equations in (11) have the simpler form,

$$\frac{\partial T}{\partial t} + \left(\vec{V}(\vec{x},t) + \vec{v}(\vec{x},t)\right) \cdot \nabla T = K\Delta T. \tag{13}$$

The notation T stands for tracer here, and not necessarily temperature. The reader should not be fooled by the simplicity in (13); while this equation is linear for T, it is statistically nonlinear. To see this, I let $\langle \cdot \rangle$ denote the average over the random fluctuations. The simplest useful statistic in a tracer field is the mean passive scalar density, $\langle T(\vec{x},t)\rangle$. Averaging (13) yields the equation

(14)
$$\frac{\partial\left\langle T(\vec{x},t)\right\rangle}{\partial t} + \vec{V}(\vec{x},t)\cdot\nabla\left\langle T(\vec{x},t)\right\rangle = K\Delta\left\langle T(\vec{x},t)\right\rangle - \left\langle \vec{v}(\vec{x},t)\cdot\nabla T(\vec{x},t)\right\rangle.$$

Note that the equation in (14) is not a closed equation for $\left\langle T(\vec{x},t)\right\rangle$ because the average of the advective term, $\langle \vec{v}\cdot\nabla T\rangle$, cannot be related simply to a functional of $\left\langle T(\vec{x},t)\right\rangle$. This is the simplest version of the famous closure problem of turbulence theory.

On the other hand, atmosphere/ocean scientists as well as engineers are driven by pragmatic needs to assess the effect of unresolved scales a priori. The standard way which they achieve this is by postulating a relationship,

$$-\left\langle \vec{v}(\vec{x},t) \cdot \nabla T(\vec{x},t) \right\rangle = \nabla \cdot \left(\bar{K}_T \cdot \nabla T(\vec{x},t) \right) \tag{15}$$

where $\bar{K}_T$ is some constant "eddy diffusivity" matrix which can be estimated roughly from auxilliary physical considerations. The ad hoc formula in (15) is an example of a closure principle. Turbulent diffusion in GCM's for the atmosphere and ocean is treated in this manner.

4.1.1. *The Mathematics of Closure for Turbulent Diffusion.* It is obviously a very interesting mathematical problem to decide the validity of such closure hypotheses as in (15) given the nature of the unresolved velocity scales. There has been a major effort among applied mathematicians and physicists in developing simplified models where closure theory can be understood rigorously for equations for both the mean, $\langle T \rangle$, and higher order statistics such as tracer correlations. The models of this type that have been completely analyzed fall into the following special cases:

(16)

1) *Periodic Velocity Models:*
$\vec{V} = \vec{V}_0$, constant, and $\vec{v}(\vec{x},t)$ space-time periodic or random with short-range correlations ([23],[24],[25],[26],[27],[28]).

2) *Simple Shear Layer Models:*
In two space variables, (x,y), $\vec{v} = \left(w(t), v(x,t) \right)$ where $w(t), v(x,t)$ have arbitrary and even long-range correlated statistics ([29],[30],[31],[32],[33],[34], [35]).

3) *Rapid Decorrelation in Time Models:*
The velocity statistics are essentially white noise in time ([36],[37],[38],[39],[40],[41], [42]).

In the situation from 1), the mathematical theories for turbulent diffusion apply at large scales and long times and yield the equation in (15) together with formulas for the effective diffusivity tensor, $\bar{K}_T$; these formulas ([26],[27],[35]) reveal a very subtle relationship between the small scale flow geometry, $\vec{v}(\vec{x},t)$, and the mean flow, $\vec{V}_0$. Examples with the simple shear layer models from 2) rigorously demonstrate ([29],[30],[31],[32],[35]) that nonlocal space-time equations are needed for the averages with suitable long-range correlated velocity fields rather than the simple local diffusion equation postulated through (15). Also, the subtle issues of behavior at finite times can be understood with full mathematical rigor in the models from 2). In the situation from 3), the closure postulate in (15) is exact for all times but subtle issues in analyzing the resulting variable coefficient P.D.E.'s emerge. Once one has rigorously analyzed models like those in (15), it becomes very interesting to check the capabilities of numerical Monte Carlo methods ([43],[44],[45],[27]) or analytical renormalization methods ([46],[47],[48],[49]) to understand their capabilities in recovering the predictions of these models.

There is a great need for further mathematical understanding of turbulent diffusion with applications to atmosphere/ocean science. An extensive research/expository article on turbulent diffusion with much more detailed discussion of all of the above issues has been written very recently by Kramer and the author ([35]).

4.1.2. *Closure Procedures for Turbulent Reaction Diffusion Equations.* Much less is known rigorously regarding closure procedures for the turbulent reaction diffusion equations in (11) with a nonzero source term, $f(Z)$, despite their practical importance in both atmosphere/ocean science and other disciplines such as combustion engineering. Souganidis, Embid, and the author have discussed renormalized front propagation with both periodic ([50],[51],[52]) and fractal ([53],[54]) small scale velocity fields. They have also compared ([55]) the predictions of popular closure theories in the applied community with rigorous exactly solvable answers for simple turbulent flow geometries. Pope has written two excellent review papers on formal closure procedures for reactive flows ([56],[57]) which are strongly recommended by the author.

One of the major uncertainties in predicting climate involves clouds and successfully parametrizing cloud water content. Krueger ([58] and private communication) has successfully modelled the turbulent droplet spectrum for clouds by utilizing a novel closure procedure for turbulent reaction diffusion equations due to Kerstein ([59],[60],[61]). The linear eddy models of Kerstein are stochastic models involving one-dimensional stochastic rearrangement maps which mimic the process of turbulent deformation and diffusion. Given their success and potential for improved modelling in cloud physics, a very interesting research topic involves a definitive quantitative mathematical understanding of this novel closure procedure in some well-designed idealized models. A first step in this direction is due to Childress and Klapper ([62]).

5. Stochastic modelling for climate prediction

An area with great importance for future developments in climate prediction involves simplified stochastic modelling of nonlinear features of the coupled atmosphere/ocean system. This area also holds great promise for future mathematical developments involving novel stochastic modelling for nonlinear P.D.E.s with both turbulence and large scale chaotic dynamics. The practical reasons for such needs are easy to understand. In the forseeable future, it will be impossible to resolve the effects of the coupled atmosphere/ocean system through computer models with detailed time integration of the atmosphere on decadal scales. However, the questions of interest also change. For example, for climate prediction, one is not interested in whether there is a significant deflection of the storm track northward in the Atlantic during a specific week in January of a given year, but rather whether the mean and variance of the storm track are large during one or several years of winter seasons and what is the impact of this trend on the ocean, for example.

The idea of simplified stochastic modelling for unresolved space-time scales in climate modelling is over twenty years old and emerged from fundamental papers by Hasselmann ([63]) and Leith ([64]), who also asked the important related question of whether there is an appropriate flucuation-dissipation relation valid for the climate on large spatio-temporal scales. In the atmosphere/ocean science community, there is a recent flourishing of ideas utilizing simple linear stochastic equations to model and predict short term and decadal climate changes such as the atmospheric influence on El Nino ([65],[66]), the North Atlantic Oscillation (NAO) ([67],[68]), and related issues such as stochastic models for mid-latitude storm tracks ([69],[70],[71]). In contrast to this use of linear stochastic models in atmosphere/ocean science, stochastic modelling of homogeneous isotropic turbulence naturally leads to nonlinear stochastic models ([14],[72]) which attempt to yield self-consistent low order statistics. A very natural and potentially very rich area for both mathematical analysis and applications in atmosphere/ocean science involves assessing the need for nonlinear stochastic modelling and the validity of approximate linear Markov models in prototype problems in atmosphere/ocean science. For coupling the atmosphere and ocean, one would like to develop stochastic models which incorporate nonlinear feedback but exploit the great disparity in time scales of the response between the atmosphere and the ocean. It is especially natural to attempt to use and refine asymptotic mathematical techniques of stochastic averaging ([73]) and stochastic resonance in intermediate models for coupling the atmosphere and the ocean in the tropics ([65]).

In a completely different direction, there is great current interest in the locations in the Labrador and Greenland Seas where open ocean convection occurs in response to cold air outbreaks from the polar atmosphere. At these discrete locations, as well as the Weddell Sea, the atmosphere directly influences the ocean and the thermohaline circulation, governing the poleward transport of heat in the ocean, locally overturns and profoundly influences the present climate. A very recent review article by Marshall and Schott ([74]) surveys observations, theory and models. Legg, Marshall, Visbeck, and Jones ([75],[76],[77]) have utilized Heton models in a two layer setting to model the spreading phase of open-ocean convection. These calculations reveal the emergence of coherent tilted dipole clusters which transport heat from the open convection site. In order to make predictions and for crude parametrization of open-ocean convection events in GCM's, it is desirable to develop a statistical theory for such propagating clusters of Hetons that depends only on the bulk features of energy, circulation, and angular momentum and not the detailed dynamics. Such a theory necessarily involves a fascinating interplay between long range barotropic organization and short range baroclinic organization, since the effect of rotation near the sites of open convection is important on small scales on the order of 1 to 1.5 k.m. while the sites of open convection span hundreds of kilometers. A first paper on this topic involving novel statistical theories in a closed basin has just been developed by DiBattista and the author ([78]).

6. Concluding remarks

Clearly, I believe that there are exciting prospects for future developments involving modern applied mathematics and atmosphere/ocean science. Several different directions with great promise involve inherently random or stochastic processes interacting with nonlinear P.D.E.s. Understanding climate is a world-wide scientific challenge so this special millenium volume sponsored by the International Mathematical Union is an especially appropriate forum for this article. I hope that it will inspire young mathematicians to understand and work in these exciting future directions for mathematics.

References

1. Gill, A.E., (1982), *Atmosphere-Ocean Dynamics*, Academic Press, New York.
2. Cushman-Roisin, B., (1994), *Introduction to Geophysical Fluid Dynamics*, Prentice Hall, New Jersey.
3. Pedlosky, J., (1987), *Geophysical Fluid Dynamics* (2nd ed.), Springer-Verlag, New York, New York (1987).
4. Majda, A., (1986), "Vorticity and the mathematical theory of incompressible flow," *Comm. Pure Appl. Math.* 39, 5187-5209.
5. Majda, A., and Bertozzi, A., (1999), *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, UK.
6. Bourgeois, A., and Beale, J.T., (1994), "Validity of the quasigeostrophic model for large scale flow in the atmosphere and ocean," *SIAM J. Math. Anal.* 25, 1023-1068.
7. Embid, P., and Majda, A., (1998), "Averaging over fast waves for geophysical flows with unbalanced initial data", *Theor. Comp. Fluid Dynamics* 11, 155-169.
8. Embid, P., and Majda, A., (1998), "Low Froude Number Limiting Dynamics for Stably Stratified Flow with Small or Fixed Rossby Numbers", *Geophys. Astrophys. Fluid Dynamics* 87 (1-2), 1-50.
9. Majda, A., and Embid, P., (1996), "Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity", *Comm. P.D.E.*, 21 (3&4), pp. 619-658.
10. Grenier, E., and Masmoudi, N., (1997), "Ekman layers of rotating fluids, the case of well-prepared initial data", *Comm. Partial Differential Equations*, 22, 963-975.
11. Masmoudi, N., (1998), "Ekman layers of rotating fluids, the case of general initial data", (to appear in *Comm. Pure. Appl. Math.*)
12. Majda, A., and Grote, M., (1997) "Model dynamics and vertical collapse in decaying strongly stratified flows", *Phys. Fluids*, 9 (10), 2932-2940.
13. Fincham, A.M., Maxworthy, T., and Spedding, G.R., (1996), "The horizontal and vertical structure of the vorticity field in freely-decaying stratified grid turbulence", *Dyn. Atmosph. Oceans* 23, 155-169.
14. Chorin, A., (1995), *Vorticity and Turbulence*, Springer-Verlag, New York.

15. Gage, K.S., and Nastrom, G., (1986). "Theoretical interpretations of atmospheric wave number spectra of wind and temperature observed by commercial aircraft during GASP". *J. Atmos. Sci.* **43**. 729-740.
16. Garrett. C., and Munk, W., (1979). "Internal waves in the ocean". *Ann. Rev. Fluid Mech.* **11**. 339-352.
17. Phillips, O.M., (1977), *The Dynamics of the Upper Ocean*, Cambridge University Press, Cambridge, UK.
18. Charbonneau, P., Herring, J., and Milliff, R., editors, (1998), Special Issue on: *Stratified and Rotating Turbulence: Parts I and II, Theoretical and Computational Fluid Dynamics*. **11** (3/4).
19. Smith, L., and Waleffe, F., (1999). "Transfer of energy to 2-D large scales in forced rotating 3-D turbulence", *Physics Fluids* **11**, no. 6, 1608–1622.
20. Majda, A., McLaughlin, D., and Tabak, E., (1997). "A one-dimensional model for dispersive wave turbulence"', *J. Nonlinear Sci.*, 6, 9-44.
21. Zakharov, V.E., (1984), "Kolmogorov spectra in weak turbulence problems", *Handbook Plasma Phys.*, **2**.
22. Zakharov, V.E., Lvov, V., and Falkovich, G., (1992), *Wave Turbulence*, Springer-Verlag, New York.
23. McLaughlin, D.W., Papanicolaou, G.C., and Pironneau, O.R., (1985). "Convection of microstructure and related problems", *SIAM J. Appl. Math.*, **45(5)**, 780-797.
24. Avellaneda, M., and Majda, A.J., (1989), "Stieltjes integral representation and effective diffusivity bounds for turbulent transport", *Phys. Rev. Lett.*, **62(7)**, 753-755.
25. Avellaneda, M., and Majda, A.J., "An integral representation and bounds on the effective diffusivity in passive advection by laminar and turbulent flows", *Comm. Math. Phys.*, **138**, 339-391.
26. Majda, A.J., and McLaughlin, R.M., (1993), "The effect of mean flows on enhanced diffusivity in transport by incompressible periodic velocity fields", *Stud. Appl. Math.*, **89(3)**, 245-279.
27. McLaughlin, R., (1998), "Numerical averaging and fast homogenization", *J. Statist. Phys.*, **90(3-4)**, 597-626.
28. Fannjiang, A., and Papanicolaou, G., (1994), "Convection enhanced diffusion for periodic flows", *SIAM J. Appl. Math.*, **54(2)**, 333-408.
29. Avellaneda, M., and Majda, A., (1990), "Mathematical models with exact renormalization for turbulent transport", *Commun. Math. Phys.*, **131**, 381-429.
30. Avellaneda,M., and Majda, A., (1992), "Mathematical models with exact renormalization for turbulent transport, II: fractal interfaces, non-Gaussian statistics and the sweeping effect", *Comm. Math. Phys.*, **146**, 139-204.
31. Avellaneda,M., and Majda, A., (1992), "Renormalization theory for eddy-diffusivity in turbulent transport", *Phys. Rev. Letters*, **68(20)**, 3028-3031.
32. Avellaneda, M., and Majda, A., (1992), "Superdiffusion in nearly stratified flows", *Journal of Statistical Physics*, **69**, 689-729.
33. Avellaneda, M., and Majda, A., (1994), "Simple examples with features of renormalization for turbulent transport", *Philos. Trans. Roy. Soc. Ser. A*, **346**, 205-233.
34. Horntrop, D.J., and Majda, A.J., (1994), "Subtle statistical behavior in simple models for random advection-diffusion", *J. Math. Sci. Univ. Tokyo*. **1**. 1-48.
35. Majda, A., and Kramer, P., (1999), "Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena", *Physics Reports* **314(4-5)**. 237–574.
36. Kraichnan, R.H., (1968), "Small-scale structure of a scalar field convected by turbulence", *Phys. Fluids*, **11(5)**, 945-953.
37. Kraichnan, R.H., (1970), "Diffusion by a random velocity field", *Phys. Fluids*, **13(1)**, 22-31.
38. Majda, A.J., (1993), "Explicit inertial range renormalization theory in a model for turbulent diffusion", *Journal of Statistical Physics*, **73**, 515-542.

39. Majda, A.J., (1994), "Random shearing direction models for isotropic turbulent diffusion", *Journal of Statistical Physics*, **75**, 1153-1165.
40. Molchanov, S.A., (1991), "Ideas in the theory of random media", *Acta Applicandae Math.*, **22**, 139-282.
41. Molchanov, S.A., Ruzmaikin, A.A., and Sokoloff, D.D., (1983), "Dynamo equations in a random short-term correlated velocity field", *Magnitnaja Gidrodinamika*, 4, 67-73.
42. Klyatskin, V.I., Woyczynski, W.A., and Gurarie, D., (1996), "Short-time correlation approximations for diffusing tracers in random velocity fields: a functional approach", In: *Stochastic Modelling in Physical Oceanography*, vol. 39 of *Progr. Probabl.* 221-269, Birkhäuser, Boston.
43. Elliott, F.W., Jr., Horntrop, D.J., and Majda, A., (1997), "Monte Carlo Methods for turbulent tracers with long range and fractal random velocity fields", *Chaos*, **7** **(11)**, 39-44.
44. Elliott, F.W., Jr., Horntrop, D.J., and Majda, A., (1997), "A Fourier-Wavelet Monte Carlo Method for fractal random fields", *J. Comp. Physics*, **132**, 384-408.
45. Elliott, F., and Majda, A., (1994), "A wavelet Monte Carlo method for turbulent diffusion with many spatial scales", *Journal of Computational Physics*, **113**, 82-111.
46. Avellaneda, M., and Majda, A., (1992), "Approximate and exact renormalization theories for a model for turbulent transport", *Phys. Fluids A*, 4 (1), 41-57.
47. Avellaneda, M., and Majda, A., (1993), "Application of an approximate R-N-G theory to a model for turbulent transport with exact renormalization", IMA vol. 55, *Turbulence in Fluid Flows: A Dynamical System Approach*, Edited by Foias, Temam, and Sell, 1-32.
48. Smith, L.M., and Woodruff, S.L., (1998), "Renormalization-group analysis of turbulence. In: *Annual review of fluid mechanics*, vol. 30 of *Annu. Rev. Fluid Mech.*, 275-310, Annual Reviews, Palo Alto, CA.
49. Vanden Eijnden, E., Kramer, P.R., and Majda, A.J., (1999), "Testing approximate closures for turbulent diffusion on some model flows", (in preparation, to be submitted to *J. Statist. Phys.*.
50. Majda, A.J., and Souganidis, P.E., (1994), "Large-scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales", *Nonlinearity*, **7(1)**, 1-30.
51. Embid, P.F., Majda, A.J., and Souganidis, P.E., (1994), "Effective geometric front dynamics for premixed turbulent combustion with separated velocity scales", *Comb. Sci. and Tech.*, **103**, 85-115.
52. Embid, P.F., Majda, A.J., and Souganidis, P.E., (1996), "Examples and counterexamples for Huygens Principle in premixed combustion", *Comb. Sci. and Tech.*, **120(1-6)**, 273-303.
53. Majda, A.J., and Souganidis, P.E., (1996), "Bounds on enhanced turbulent flame speeds for combustion with fractal velocity fields", *J. Statist. Phys.*, **83(5-6)**, 933-954.
54. Majda, A.J., and Souganidis, P.E., (1998), "Flame fronts in a turbulent combustion model with fractal velocity fields", *Comm. Pur Appl. Math.*, **LI**, 1337-1348.
55. Embid, P.F., Majda, A.J., and Souganidis, P.E., (1995), "Comparison of turbulent flame speeds from complete averaging and the G-equation", *Phys. Fluids*, **7(8)**, 2052-2060.
56. Pope, S.B., (1994), "Lagrangian PDF methods for turbulent flows", In: *Annual review of fluid mechanics*, vol. 26, 23-63, Annual Reviews, Palo Alto, CA.
57. Pope, S.B., (1991), "Mapping closures for turbulent mixing and reaction", *Theoret. Comput. Fluid Dynamics*, 2, 255-270.
58. Krueger, S., (1993), "Linear eddy modeling of entrainment and mixing in stratus clouds", *J. Atmos. Sci.*, **50**, 3078-3090.
59. Kerstein, A.R., (1991), "Linear-eddy modelling of turbulent transport. Part 6. Microstructure of diffusive scalar mixing fields", *J. Fluid Mech.*, **231**, 361-394.

60. Kerstein, A.R., (1988). "Linear-eddy model of turbulent scalar transport and mixing", *Combust. Sci. and Technology*. 60, 391-421.
61. McMurtry, P.A., Gansauge, T.C., Kerstein, A.R., and Krueger, S.K., (1993), "Linear eddy simulations of mixing in a homogenous turbulent flow", *Phys. Fluids A*. 5(4). 1023-1034.
62. Childress, S., and Klapper, I., (1991). "On some transport properties of Baker maps", *J. Stat. Phys.*, 897-914.
63. Hasselman, K., (1976), "Stochastic climate models. Part I: Theory". *Tellus*, 28. 473-485.
64. Leith, C.E., (1975), "Climate response and fluctuation dissipation", *J. Atmos. Sci.*, 32, 2022-2025.
65. Kleeman, R., and Power, S., (1994), "Limits to predictability in a coupled ocean-atmosphere model due to atmospheric noise", *Tellus*, 529-540.
66. Kleeman, R., and Moore, A., (1997). "A theory for the limitation of ENSO predictability due to stochastic atmospheric transients", *J. Atmos. Sci.*, 54, 753-767.
67. Griffies, S., and Tziperman, E., (1995), "A linear thermohaline oscillator driven by stochastic atmospheric forcing", *J. Climate*, 8. 2440-2453.
68. Saravanan, R., and McWilliams, J., (1997), "Advective ocean-atmosphere interaction: An analytical stochastic model with implications for decadal variability", *J. Climate*.
69. DelSole, T., (1996). "Can quasigeostrophic turbulence be modeled stochastically?", *J. Atmos. Sci.*, 53, 1617-1633.
70. DelSole, T., and Farrell, B.F., (1996). "The quasilinear equilibration of a thermally maintained, stochastically excited jet in a two-layer model", *J. Atmos. Sci.*, 53, 1781-1797.
71. Whitaker, J.A., and Sardeshmukh, P.D., (1998). "A linear theory of extratropical synoptic eddy statistics", *J. Atmos. Sci.*, 55, 237-258.
72. Kraichnan, R.H., (1959), "The structure of isotropic turbulence at very high Reynolds numbers", *J. Fluid Mech.*, 5, 497-543.
73. Wenzel, I., and Friedlin, M., (1987). *Random Perturbations of Dynamical Systems*, Springer-Verlag, New York.
74. Marshall, J., and Schott, F., (1998). "Open-ocean convection: observations, theory and models", MIT. *Center for Global Change Science Report 52.*
75. Legg, S. and Marshall, J., (1993). "A Heton model of the spreading phase of open-ocean deep convection", *J. Phys. Oceanography*, 23, 1040-1056.
76. Legg, S., Jones, H., and Visbeck, M., (1996), "Heton perspective of baroclinic eddy transfer in localized open ocean convection", *J. Phys. Oceanography*, 26. 2251-2266.
77. Legg, S., and Marshall, J., (1998),, "The influence of the ambient flow on the spreading of convective water masses", *J. Marine Res.*, 56, 107-139.
78. DiBattista, M., and Majda, A., (1998). "An equilibrium statistical theory for large scale features of open ocean convection", submitted to *J. Phys. Oceanography*.

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