

ON THE LACK OF COMPACTNESS IN THE 2D CRITICAL SOBOLEV EMBEDDING

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ABSTRACT. This paper is devoted to the description of the lack of compactness of $H_{rad}^1(\mathbb{R}^2)$ in the Orlicz space. Our result is expressed in terms of the concentration-type examples derived by P. -L. Lions in [30]. The approach that we adopt to establish this characterization is completely different from the methods used in the study of the lack of compactness of Sobolev embedding in Lebesgue spaces and take into account the variational aspect of Orlicz spaces. We also investigate the feature of the solutions of non linear wave equation with exponential growth, where the Orlicz norm plays a decisive role.

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1. INTRODUCTION

1.1. Lack of compactness in the Sobolev embedding in Lebesgue spaces.
 Due to the scaling invariance of the critical Sobolev embedding

$$\dot{H}^s(\mathbb{R}^d) \longrightarrow L^p(\mathbb{R}^d),$$

in the case where $d \geq 3$ with $0 \leq s < d/2$ and $p = 2d/(d - 2s)$, no compactness properties may be expected. Indeed if $u \in \dot{H}^s \setminus \{0\}$, then for any sequence (y_n) of points of \mathbb{R}^d tending to the infinity and for any sequence (h_n) of positive real numbers tending to 0 or to infinity, the sequences $(\tau_{y_n} u)$ and $(\delta_{h_n} u)$, where we denote $\delta_{h_n} u(\cdot) = \frac{1}{h_n^{\frac{d}{2}}} u(\frac{\cdot}{h_n})$, converge weakly to 0 in \dot{H}^s but are not relatively compact in L^p since $\|\tau_{y_n} u\|_{L^p} = \|u\|_{L^p}$ and $\|\delta_{h_n} u\|_{L^p} = \|u\|_{L^p}$.

After the pioneering works of P. -L. Lions [30] and [31], several works have been devoted to the study of the lack of compactness in critical Sobolev embeddings, for the sake of geometric problems and the understanding of features of solutions of nonlinear partial differential equations. This question was investigated through several angles: for instance, in [14] the lack of compactness is describe in terms of defect measures, in [15] by means of profiles and in [23] by the use of nonlinear wavelet approximation theory. Nevertheless, it has been shown in all these results that translational and scaling invariance are the sole responsible for the defect of compactness of the embedding of \dot{H}^s into L^p and more generally in Sobolev spaces in the L^q frame.

As it is pointed above, the study of the lack of compactness in critical Sobolev embedding supply us numerous information on solutions of nonlinear partial differential equations whether in elliptic frame or evolution frame. For example, one can mention the description of bounded energy sequences of solutions to the defocusing semi-linear wave equation

$$\square u + u^5 = 0$$

in $\mathbb{R} \times \mathbb{R}^3$, up to remainder terms small in energy norm in [3] or the sharp estimate of the span time life of the focusing critical semi-linear wave equation by means of the size of energy of the Cauchy data in the remarkable work of [25].

Roughly speaking, the lack of compactness in the critical Sobolev embedding

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$$

in the case where $d \geq 3$ with $0 \leq s < d/2$ and $p = 2d/(d - 2s)$, is characterized in the following terms: a sequence $(u_n)_{n \in \mathbb{N}}$ bounded in $\dot{H}^s(\mathbb{R}^d)$ can be decomposed up to a subsequence extraction on a finite sum of orthogonal profiles such that the remainder converges to zero in $L^p(\mathbb{R}^d)$ as the number of the sum and n tend to ∞ . This description still holds in the more general case of Sobolev spaces in the L^q frame (see [23]).

1.2. Critical 2D Sobolev embedding. It is well known that $H^1(\mathbb{R}^2)$ is continuously embedded in all Lebesgue spaces $L^p(\mathbb{R}^2)$ for $2 \leq p < \infty$, but not in $L^\infty(\mathbb{R}^2)$. A short proof of this fact is given in Appendix A for the convenience of the reader. On the other hand, it is also known (see for instance [25]) that $H^1(\mathbb{R}^2)$ embeds in $BMO(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, where $BMO(\mathbb{R}^d)$ denotes the space of bounded mean oscillations which is the space of locally integrable functions f such that

$$\|f\|_{BMO} \stackrel{\text{def}}{=} \sup_B \frac{1}{|B|} \int_B |f - f_B| dx < \infty \quad \text{with} \quad f_B \stackrel{\text{def}}{=} \frac{1}{|B|} \int_B f dx.$$

The above supremum being taken over the set of Euclidean balls B , $|\cdot|$ denoting the Lebesgue measure.

In this paper, we rather investigate the lack of compactness in Orlicz space \mathcal{L} (see Definition 1.1 below) which arises naturally in the study of non linear wave equation with exponential growth. As, it will be shown in Appendix B, the spaces \mathcal{L} and BMO are not comparable.

Let us now introduce the so-called Orlicz spaces on \mathbb{R}^d and some related basic facts. (For the sake of completeness, we postpone to an appendix some additional properties on Orlicz spaces).

Definition 1.1.

Let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a convex increasing function such that

$$\phi(0) = 0 = \lim_{s \rightarrow 0^+} \phi(s), \quad \lim_{s \rightarrow \infty} \phi(s) = \infty.$$

We say that a measurable function $u : \mathbb{R}^d \rightarrow \mathbb{C}$ belongs to L^ϕ if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty.$$

We denote then

$$(1) \quad \|u\|_{L^\phi} = \inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

It is easy to check that L^ϕ is a \mathbb{C} -vectorial space and $\|\cdot\|_{L^\phi}$ is a norm. Moreover, we have the following properties.

- For $\phi(s) = s^p$, $1 \leq p < \infty$, L^ϕ is nothing else than the Lebesgue space L^p .
- For $\phi_\alpha(s) = e^{\alpha s^2} - 1$, with $\alpha > 0$, we claim that $L^{\phi_\alpha} = L^{\phi_1}$. It is actually a direct consequence of Definition 1.1.
- We may replace in (1) the number 1 by any positive constant. This change the norm $\|\cdot\|_{L^\phi}$ to an equivalent norm.
- For $u \in L^\phi$ with $A := \|u\|_{L^\phi} > 0$, we have the following property

$$(2) \quad \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\} = [A, \infty[.$$

In what follows we shall fix $d = 2$, $\phi(s) = e^{s^2} - 1$ and denote the Orlicz space L^ϕ by \mathcal{L} endowed with the norm $\|\cdot\|_{\mathcal{L}}$ where the number 1 is replaced by the constant

κ that will be fixed in Identity (6) below. As it is already mentioned, this change does not have any impact on the definition of Orlicz space. It is easy to see that $\mathcal{L} \hookrightarrow L^p$ for every $2 \leq p < \infty$.

The 2D critical Sobolev embedding in Orlicz space \mathcal{L} states as follows:

Proposition 1.2.

$$(3) \quad \|u\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1}.$$

Remarks 1.3.

a) Inequality (3) is insensitive to space translation but not invariant under scaling nor oscillations.

b) The embedding of $H^1(\mathbb{R}^2)$ in \mathcal{L} is sharp within the context of Orlicz spaces. In other words, the target space \mathcal{L} cannot be replaced by an essentially smaller Orlicz space. However, this target space can be improved if we allow different function spaces than Orlicz spaces. More precisely

$$(4) \quad H^1(\mathbb{R}^2) \hookrightarrow BW(\mathbb{R}^2),$$

where the Brézis-Wainger space $BW(\mathbb{R}^2)$ is defined via

$$\|u\|_{BW} := \left(\int_0^1 \left(\frac{u^*(t)}{\log(e/t)} \right)^2 \frac{dt}{t} \right)^{1/2} + \left(\int_1^\infty u^*(t)^2 dt \right)^{1/2},$$

where u^* denotes the rearrangement function of u given by

$$u^*(t) = \inf \left\{ \lambda > 0; \quad |\{x; |u(x)| > \lambda\}| \leq t \right\}.$$

The embedding (4) is sharper than (3) as $BW(\mathbb{R}^2) \subsetneq \mathcal{L}$. It is also optimal with respect to all rearrangement invariant Banach function spaces. For more details on this subject, we refer the reader to [6, 10, 12, 16, 17, 33].

c) In higher dimensions ($d = 3$ for example), the equivalent of Embedding (4) is

$$H^1(\mathbb{R}^3) \hookrightarrow L^{6,2}(\mathbb{R}^3),$$

where $L^{6,2}$ is the classical Lorentz space. Notice that $L^{6,2}$ is a rearrangement invariant Banach space but not an Orlicz space.

To end this short introduction to Orlicz spaces, let us point out that the embedding (3) derives immediately from the following Trudinger-Moser type inequalities:

Proposition 1.4. Let $\alpha \in [0, 4\pi[$. A constant c_α exists such that

$$(5) \quad \int_{\mathbb{R}^2} \left(e^{\alpha|u|^2} - 1 \right) dx \leq c_\alpha \|u\|_{L^2(\mathbb{R}^2)}^2$$

for all u in $H^1(\mathbb{R}^2)$ such that $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. Moreover, if $\alpha \geq 4\pi$, then (5) is false.

A first proof of these inequalities using rearrangement can be found in [1]. In other respects, it is well known (see for instance [40]) that the value $\alpha = 4\pi$ becomes admissible in (5) if we require $\|u\|_{H^1(\mathbb{R}^2)} \leq 1$ rather than $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. In other words, we have

Proposition 1.5.

$$(6) \quad \sup_{\|u\|_{H^1} \leq 1} \int_{\mathbb{R}^2} \left(e^{4\pi|u|^2} - 1 \right) dx := \kappa < \infty,$$

and this is false for $\alpha > 4\pi$.

Now, it is obvious that Estimate (6) allows to prove Proposition 1.2. Indeed, without loss of generality, we may assume that $\|u\|_{H^1} = 1$ which leads under Proposition 1.5 to the inequality $\|u\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}}$, which is the desired result.

Remark 1.6. *Let us mention that a sharp form of Trudinger-Moser inequality in bounded domain was obtained in [2].*

1.3. Lack of compactness in 2D critical Sobolev embedding in Orlicz space.

The embedding $H^1 \hookrightarrow \mathcal{L}$ is non compact at least for two reasons. The first reason is the lack of compactness at infinity. A typical example is $u_k(x) = \varphi(x + x_k)$ where $0 \neq \varphi \in \mathcal{D}$ and $|x_k| \rightarrow \infty$. The second reason is of concentration-type derived by P.-L. Lions [30, 31] and illustrated by the following fundamental example f_α defined by:

$$f_\alpha(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ -\frac{\log|x|}{\sqrt{2\alpha\pi}} & \text{if } e^{-\alpha} \leq |x| \leq 1, \\ \sqrt{\frac{\alpha}{2\pi}} & \text{if } |x| \leq e^{-\alpha}, \end{cases}$$

where $\alpha > 0$.

Straightforward computations show that $\|f_\alpha\|_{L^2(\mathbb{R}^2)}^2 = \frac{1}{4\alpha}(1 - e^{-2\alpha}) - \frac{1}{2}e^{-2\alpha}$ and $\|\nabla f_\alpha\|_{L^2(\mathbb{R}^2)} = 1$. Moreover, it can be seen easily that $f_\alpha \rightarrow 0$ in $H^1(\mathbb{R}^2)$ as $\alpha \rightarrow \infty$ or $\alpha \rightarrow 0$. However, the lack of compactness of this sequence in the Orlicz space \mathcal{L} occurs only when α goes to infinity. More precisely, we have

Proposition 1.7. *f_α denoting the sequence defined above, we have the following convergence results:*

- a) $\|f_\alpha\|_{\mathcal{L}} \rightarrow \frac{1}{\sqrt{4\pi}}$ as $\alpha \rightarrow \infty$.
- b) $\|f_\alpha\|_{\mathcal{L}} \rightarrow 0$ as $\alpha \rightarrow 0$.

Proof of Proposition 1.7. Let us first go to the proof of the first assertion. If

$$\int \left(e^{\frac{|f_\alpha(x)|^2}{\lambda^2}} - 1 \right) dx \leq \kappa,$$

then

$$2\pi \int_0^{e^{-\alpha}} \left(e^{\frac{\alpha}{2\pi\lambda^2}} - 1 \right) r dr \leq \kappa,$$

which implies that

$$\lambda^2 \geq \frac{\alpha}{2\pi \log \left(1 + \frac{\kappa e^{2\alpha}}{\pi} \right)}.$$

It follows that

$$\liminf_{\alpha \rightarrow \infty} \|f_\alpha\|_{\mathcal{L}} \geq \frac{1}{\sqrt{4\pi}}.$$

To conclude, it suffices to show that

$$\limsup_{\alpha \rightarrow \infty} \|f_\alpha\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}}.$$

Let us fix $\varepsilon > 0$. Taking advantage of Trudinger-Moser inequality and the fact that $\|f_\alpha\|_{L^2} \rightarrow 0$, we infer

$$\begin{aligned} \int \left(e^{(4\pi-\varepsilon)|f_\alpha(x)|^2} - 1 \right) dx &\leq C_\varepsilon \|f_\alpha\|_{L^2}^2, \\ &\leq \kappa, \quad \text{for } \alpha \geq \alpha_\varepsilon. \end{aligned}$$

Hence, for any $\varepsilon > 0$,

$$\limsup_{\alpha \rightarrow \infty} \|f_\alpha\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi - \varepsilon}},$$

which ends the proof of the first assertion. To prove the second one, let us write

$$\begin{aligned} \int \left(e^{\frac{|f_\alpha(x)|^2}{\alpha^{1/2}}} - 1 \right) dx &= 2\pi \int_0^{e^{-\alpha}} \left(e^{\frac{\sqrt{r}}{2\pi}} - 1 \right) r dr + 2\pi \int_{e^{-\alpha}}^1 \left(e^{\frac{\log^2 r}{2\pi\alpha^{3/2}}} - 1 \right) r dr \\ &\leq \pi \left(e^{\frac{\sqrt{e^{-\alpha}}}{2\pi}} - 1 \right) e^{-2\alpha} + 2\pi (1 - e^{-\alpha}) e^{\frac{\alpha^{1/2}}{2\pi}}. \end{aligned}$$

This implies that, for α small enough, $\|f_\alpha\|_{\mathcal{L}} \leq \alpha^{1/4}$, which leads to the result. \square

The difference between the behavior of these families in Orlicz space when $\alpha \rightarrow 0$ or $\alpha \rightarrow \infty$ comes from the fact that the concentration effect is only displayed by this family when $\alpha \rightarrow \infty$. Indeed, in the case where $\alpha \rightarrow \infty$ we have the following result which does not occur when $\alpha \rightarrow 0$.

Proposition 1.8. *f_α being the family of functions defined above, we have*

$$|\nabla f_\alpha|^2 \rightarrow \delta(x=0) \quad \text{and} \quad e^{4\pi|f_\alpha|^2} - 1 \rightarrow 2\pi\delta(x=0) \quad \text{as } \alpha \rightarrow \infty \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Proof of Proposition 1.8. Straightforward computations give for any smooth compactly supported function φ

$$\begin{aligned} \int |\nabla f_\alpha(x)|^2 \varphi(x) dx &= \frac{1}{2\pi\alpha} \int_{e^{-\alpha}}^1 \int_0^{2\pi} \frac{\varphi(r \cos \theta, r \sin \theta)}{r} dr d\theta \\ &= \varphi(0) + \frac{1}{2\pi\alpha} \int_{e^{-\alpha}}^1 \int_0^{2\pi} \frac{\varphi(r \cos \theta, r \sin \theta) - \varphi(0)}{r} dr d\theta \end{aligned}$$

Since $\left| \frac{\varphi(r \cos \theta, r \sin \theta) - \varphi(0)}{r} \right| \leq \|\nabla \varphi\|_{L^\infty}$, we deduce that $\int |\nabla f_\alpha(x)|^2 \varphi(x) dx \rightarrow \varphi(0)$ as $\alpha \rightarrow \infty$, which ensures the result. Similarly, we have

$$\begin{aligned}
\int \left(e^{4\pi|f_\alpha(x)|^2} - 1 \right) \varphi(x) dx &= \int_0^{e^{-\alpha}} \int_0^{2\pi} (e^{2\alpha} - 1) \varphi(r \cos \theta, r \sin \theta) r dr d\theta \\
&+ \int_{e^{-\alpha}}^1 \int_0^{2\pi} \left(e^{\frac{2}{\alpha} \log^2 r} - 1 \right) \varphi(r \cos \theta, r \sin \theta) r dr d\theta \\
&= \pi \varphi(0) (1 - e^{-2\alpha}) + 2\pi \varphi(0) \int_{e^{-\alpha}}^1 \left(e^{\frac{2}{\alpha} \log^2 r} - 1 \right) r dr \\
&+ \int_0^{e^{-\alpha}} \int_0^{2\pi} (e^{2\alpha} - 1) (\varphi(r \cos \theta, r \sin \theta) - \varphi(0)) r dr d\theta \\
&+ \int_{e^{-\alpha}}^1 \int_0^{2\pi} \left(e^{\frac{2}{\alpha} \log^2 r} - 1 \right) (\varphi(r \cos \theta, r \sin \theta) - \varphi(0)) r dr d\theta.
\end{aligned}$$

We conclude by using the following lemma. □

Lemma 1.9. *When α goes to infinity*

$$(7) \quad I_\alpha := \int_{e^{-\alpha}}^1 r e^{\frac{2}{\alpha} \log^2 r} dr \rightarrow 1$$

and

$$(8) \quad J_\alpha := \int_{e^{-\alpha}}^1 r^2 e^{\frac{2}{\alpha} \log^2 r} dr \rightarrow \frac{1}{3}.$$

Proof of Lemma 1.9. The change of variable $y := \sqrt{\frac{2}{\alpha}} (-\log r - \frac{\alpha}{2})$ yields

$$I_\alpha = 2\sqrt{\frac{\alpha}{2}} e^{-\frac{\alpha}{2}} \int_0^{\sqrt{\frac{\alpha}{2}}} e^{y^2} dy.$$

Taking advantage of the following obvious equivalence at infinity which can be derived by integration by parts

$$(9) \quad \int_0^A e^{y^2} dy \sim \frac{e^{A^2}}{2A},$$

we deduce (7).

Similarly for the second term, the change of variable $y := \sqrt{\frac{2}{\alpha}} (-\log r - \frac{3}{4}\alpha)$ implies that

$$\begin{aligned}
J_\alpha &= \sqrt{\frac{\alpha}{2}} e^{-\frac{9}{8}\alpha} \int_{-\frac{3}{2}\sqrt{\frac{\alpha}{2}}}^{\frac{1}{2}\sqrt{\frac{\alpha}{2}}} e^{y^2} dy \\
&= \sqrt{\frac{\alpha}{2}} e^{-\frac{9}{8}\alpha} \int_0^{\frac{1}{2}\sqrt{\frac{\alpha}{2}}} e^{y^2} dy + \sqrt{\frac{\alpha}{2}} e^{-\frac{9}{8}\alpha} \int_0^{\frac{3}{2}\sqrt{\frac{\alpha}{2}}} e^{y^2} dy.
\end{aligned}$$

According to (9), we get (8). □

Remark 1.10. *When α goes to zero, we get a spreading rather than a concentration. Notice also that for small values of the function, our Orlicz space behaves like L^2 (see Proposition 4.9) and simple computations show that $\|f_\alpha\|_{L^2(\mathbb{R}^2)}$ goes to zero when α goes to zero.*

In fact, the conclusion of Proposition 1.8 is available for more general radial sequences. More precisely, we have the following result due to P. -L. Lions (in a slightly different form):

Proposition 1.11. *Let (u_n) be a sequence in $H_{rad}^1(\mathbb{R}^2)$ such that*

$$u_n \rightharpoonup 0 \quad \text{in } H^1, \quad \liminf_{n \rightarrow \infty} \|u_n\|_{\mathcal{L}} > 0 \quad \text{and} \quad \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n(x)|^2 dx = 0.$$

Then, there exists a constant $c > 0$ such that

$$(10) \quad |\nabla u_n(x)|^2 dx \rightharpoonup \mu \geq c \delta(x=0) \quad (n \rightarrow \infty)$$

weakly in the sense of measures.

Remark 1.12. *The hypothesis of compactness at infinity*

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n(x)|^2 dx = 0$$

is necessary to get (10). For instance, $u_n(x) = \frac{1}{n} e^{-|\frac{x}{n}|^2}$ satisfies $\|u_n\|_{L^2} = C > 0$, $\|\nabla u_n\|_{L^2} \rightarrow 0$ and $\liminf_{n \rightarrow \infty} \|u_n\|_{\mathcal{L}} > 0$.

1.4. Fundamental Remark. In order to describe the lack of compactness of the Sobolev embedding of $H_{rad}^1(\mathbb{R}^2)$ in Orlicz space, we will make the change of variable $s := -\log r$, with $r = |x|$. We associate then to any radial function u on \mathbb{R}^2 a one space variable function v defined by $v(s) = u(e^{-s})$. It follows that

$$(11) \quad \|u\|_{L^2}^2 = 2\pi \int_{\mathbb{R}} |v(s)|^2 e^{-2s} ds,$$

$$(12) \quad \|\nabla u\|_{L^2}^2 = 2\pi \int_{\mathbb{R}} |v'(s)|^2 ds, \quad \text{and}$$

$$(13) \quad \int_{\mathbb{R}^2} \left(e^{|\frac{u(x)}{\lambda}|^2} - 1 \right) dx = 2\pi \int_{\mathbb{R}} \left(e^{|\frac{v(s)}{\lambda}|^2} - 1 \right) e^{-2s} ds.$$

The starting point in our analysis is the following observation related to the Lions's example

$$\tilde{f}_\alpha(s) := f_\alpha(e^{-s}) = \sqrt{\frac{\alpha}{2\pi}} \mathbf{L}\left(\frac{s}{\alpha}\right),$$

where

$$\mathbf{L}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

The sequence $\alpha \rightarrow \infty$ is called the scale and the function \mathbf{L} the profile. In fact, the Lions's example generates more elaborate situations which help us to understand the defect of compactness of Sobolev embedding in Orlicz space. For example, it can be seen that for the sequence $g_k := f_k + f_{2k}$ we have $g_k(s) = \sqrt{\frac{k}{2\pi}} \psi\left(\frac{s}{k}\right)$, where

$$\psi(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t + \frac{t}{\sqrt{2}} & \text{if } 0 \leq t \leq 1, \\ 1 + \frac{t}{\sqrt{2}} & \text{if } 1 \leq t \leq 2, \\ 1 + \sqrt{2} & \text{if } t \geq 2. \end{cases}$$

This is due to the fact that the scales $(k)_{k \in \mathbb{N}}$ and $(2k)_{k \in \mathbb{N}}$ are not orthogonal (see Definition 1.13 below) and thus they give a unique profile. However, for the sequence $h_k := f_k + f_{k^2}$, the situation is completely different and a decomposition under the form

$$h_k(x) \asymp \sqrt{\frac{\alpha_k}{2\pi}} \psi\left(\frac{-\log|x|}{\alpha_k}\right)$$

is not possible, where the symbol \asymp means that the difference is compact in the Orlicz space \mathcal{L} . The reason behind is that the scales $(k)_{k \in \mathbb{N}}$ and $(k^2)_{k \in \mathbb{N}}$ are orthogonal. It is worth noticing that in the above examples the support is a fixed compact, and thus at first glance the construction cannot be adapted in the general case. But as shown by the following example, no assumption on the support is needed to "display" lack of compactness in the Orlicz space. Indeed, let R_α in $(0, \infty)$ such that

$$(14) \quad \frac{R_\alpha}{\sqrt{\alpha}} \rightarrow 0, \quad \alpha \rightarrow \infty,$$

and

$$(15) \quad a := \liminf_{\alpha \rightarrow \infty} \left(\frac{\log R_\alpha}{\alpha} \right) > -\infty.$$

We can take for instance $R_\alpha = \alpha^\theta$ with $\theta < 1/2$ and then $a = 0$, or $R_\alpha = e^{-\gamma\alpha}$ with $\gamma \geq 0$ and then $a = -\gamma$. Remark that Assumption (14) implies that a is always negative. Now, let us define the sequence $g_\alpha(x) := f_\alpha\left(\frac{x}{R_\alpha}\right)$. It is obvious that the family g_α is not uniformly supported in a fixed compact subset of \mathbb{R}^2 , in the case when $R_\alpha = \alpha^\theta$ with $0 < \theta < 1/2$. Now, arguing exactly as for Lions's example, we can easily show that

$$\|g_\alpha\|_{L^2} \sim \frac{R_\alpha}{2\sqrt{\alpha}}, \quad \|\nabla g_\alpha\|_{L^2} = 1 \quad \text{and} \quad \|g_\alpha\|_{\mathcal{L}}^2 \geq \frac{\alpha}{2\pi \log \left(1 + \frac{\kappa}{\pi} \left(\frac{e^\alpha}{R_\alpha} \right)^2 \right)}.$$

Hence, $g_\alpha \rightarrow 0$ in H^1 and $\liminf_{\alpha \rightarrow \infty} \|g_\alpha\|_{\mathcal{L}} > 0$.

Up to a subsequence extraction, straightforward computation yields the strong convergence to zero in H^1 for the difference $f_\alpha - g_\alpha$, in the case when $a = 0$, which implies that $g_\alpha(x) \asymp \sqrt{\frac{\alpha}{2\pi}} \mathbf{L}\left(\frac{-\log|x|}{\alpha}\right)$. However, in the case when $a < 0$, the

sequence (g_α) converges strongly to $f_\alpha(\frac{\cdot}{e^{\alpha a}})$ in H^1 and then the profile is slightly different in the sense that $g_\alpha(x) \asymp \sqrt{\frac{\alpha}{2\pi}} \mathbf{L}_a\left(\frac{-\log|x|}{\alpha}\right)$ where $\mathbf{L}_a(s) = \mathbf{L}(s+a)$.

To be more complete and in order to state our main result in a clear way, let us introduce some definitions as in [15] for instance.

Definition 1.13. *A scale is a sequence $\underline{\alpha} := (\alpha_n)$ of positive real numbers going to infinity. We shall say that two scales $\underline{\alpha}$ and $\underline{\beta}$ are orthogonal (in short $\underline{\alpha} \perp \underline{\beta}$) if*

$$\left| \log(\beta_n/\alpha_n) \right| \rightarrow \infty.$$

According to (11) and (12), we introduce the profiles as follows.

Definition 1.14. *The set of profiles is*

$$\mathcal{P} := \left\{ \psi \in L^2(\mathbb{R}, e^{-2s} ds); \quad \psi' \in L^2(\mathbb{R}), \psi|_{]-\infty, 0]} = 0 \right\}.$$

Some remarks are in order:

- a) The limitation for scales tending to infinity is justified by the behavior of $\|f_\alpha\|_{\mathcal{L}}$ stated in Proposition 1.7.
- b) The set \mathcal{P} is invariant under negative translations. More precisely, if $\psi \in \mathcal{P}$ and $a \leq 0$ then $\psi_a(s) := \psi(s+a)$ belongs to \mathcal{P} .
- c) It will be useful to observe that a profile (in the sense of Definition 1.14) is a continuous function since it belongs to $H_{loc}^1(\mathbb{R})$.
- d) For a scale $\underline{\alpha}$ and a profile ψ , define

$$g_{\underline{\alpha}, \psi}(x) := \sqrt{\frac{\alpha_n}{2\pi}} \psi\left(\frac{-\log|x|}{\alpha_n}\right).$$

It is clear that, for any $\lambda > 0$,

$$g_{\underline{\alpha}, \psi} = g_{\lambda \underline{\alpha}, \psi_\lambda},$$

where $\psi_\lambda(t) = \frac{1}{\sqrt{\lambda}} \psi(\lambda t)$.

The next proposition illustrates the above definitions of scales and profiles.

Proposition 1.15. *Let $\psi \in \mathcal{P}$ a profile, (α_n) any scale and set*

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}} \psi\left(\frac{-\log|x|}{\alpha_n}\right).$$

Then

$$(16) \quad \frac{1}{\sqrt{4\pi}} \sup_{s>0} \frac{|\psi(s)|}{\sqrt{s}} \leq \liminf_{n \rightarrow \infty} \|g_n\|_{\mathcal{L}} \leq \limsup_{n \rightarrow \infty} \|g_n\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}} \|\psi'\|_{L^2}.$$

Proof. A simple computation shows that

$$\|g_n\|_{L^2}^2 = \alpha_n^2 \int_0^\infty |\psi(t)|^2 e^{-2\alpha_n t} dt \quad \text{and} \quad \|\nabla g_n\|_{L^2} = \|\psi'\|_{L^2(\mathbb{R})}.$$

In view of (3), we have

$$\limsup_{n \rightarrow \infty} \|g_n\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}} \|\psi'\|_{L^2} + \frac{1}{\sqrt{4\pi}} \limsup_{n \rightarrow \infty} \|g_n\|_{L^2}.$$

Therefore, to get the right hand side of (16), it suffices to prove that the sequence (g_n) converges strongly to 0 in L^2 . To do so, let us first observe that

$$\frac{\psi(t)}{\sqrt{t}} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Indeed

$$|\psi(t)| = \left| \int_0^t \psi'(\tau) d\tau \right| \leq \sqrt{t} \left(\int_0^t |\psi'(\tau)|^2 d\tau \right)^{1/2},$$

which ensures the result since $\psi' \in L^2(\mathbb{R})$. Now, $\varepsilon > 0$ being fixed, we deduce the existence of $t_0 > 0$ such that

$$|\psi(t)|^2 \leq \varepsilon t \quad \text{for } 0 < t < t_0.$$

Therefore

$$\alpha_n^2 \int_0^{t_0} |\psi(t)|^2 e^{-2\alpha_n t} dt \leq \varepsilon \int_0^\infty s e^{-2s} ds = \frac{\varepsilon}{4}.$$

In other respects, by Lebesgue theorem

$$\alpha_n^2 \int_{t_0}^\infty |\psi(t)|^2 e^{-2\alpha_n t} dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This leads to the strong convergence of (g_n) to 0 in L^2 .

Let us now go to the proof of the left hand side inequality of (16). Setting $L = \liminf_{n \rightarrow \infty} \|g_n\|_{\mathcal{L}}$, we have according to (2) for fixed $\varepsilon > 0$ and n large enough (up to a subsequence extraction)

$$\int_{\mathbb{R}^2} \left(e^{\left| \frac{g_n(x)}{L+\varepsilon} \right|^2} - 1 \right) dx \leq \kappa.$$

A straightforward computation yields to

$$\alpha_n \int_0^\infty e^{2\alpha_n t} \left(\frac{1}{4\pi(L+\varepsilon)^2} \left(\frac{\psi(t)}{\sqrt{t}} \right)^2 - 1 \right) dt \leq C,$$

for some absolute constant C and for n large enough. Since $\psi(t)$ is continuous, we deduce that necessarily, for any $t > 0$,

$$\frac{1}{\sqrt{4\pi}} \frac{|\psi(t)|}{\sqrt{t}} \leq L + \varepsilon,$$

which ensures the result. □

1.5. Statement of the results. Our first main goal is to establish that the characterization of the lack of compactness of the embedding

$$H_{rad}^1 \hookrightarrow \mathcal{L},$$

can be reduced to the Lion's example. More precisely, we shall prove that the lack of compactness of this embedding can be described in terms of an asymptotic decomposition as follows:

Theorem 1.16. *Let (u_n) be a bounded sequence in $H_{rad}^1(\mathbb{R}^2)$ such that*

$$(17) \quad u_n \rightharpoonup 0,$$

$$(18) \quad \limsup_{n \rightarrow \infty} \|u_n\|_{\mathcal{L}} = A_0 > 0, \quad \text{and}$$

$$(19) \quad \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^2 dx = 0.$$

Then, there exists a sequence $(\alpha^{(j)})$ of pairwise orthogonal scales and a sequence of profiles $(\psi^{(j)})$ in \mathcal{P} such that, up to a subsequence extraction, we have for all $\ell \geq 1$,

$$(20) \quad u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}} \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + r_n^{(\ell)}(x), \quad \limsup_{n \rightarrow \infty} \|r_n^{(\ell)}\|_{\mathcal{L}} \xrightarrow{\ell \rightarrow \infty} 0.$$

Moreover, we have the following stability estimates

$$(21) \quad \|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)'}\|_{L^2}^2 + \|\nabla r_n^{(\ell)}\|_{L^2}^2 + o(1), \quad n \rightarrow \infty.$$

Remarks 1.17.

a) *As in higher dimensions, the decomposition (20) is not unique (see [15]).*

b) *The assumption (19) means that there is no lack of compactness at infinity. It is in particular satisfied when the sequence $(u_n)_{n \in \mathbb{N}}$ is supported in a fixed compact of \mathbb{R}^2 and also by g_α .*

c) *Also, this assumption implies the condition $\psi|_{] -\infty, 0]} = 0$ included in the definition of the set of profiles. Indeed, first let us observe that under Condition (19), necessarily each element $g_n^{(j)}(x) := \sqrt{\frac{\alpha_n^{(j)}}{2\pi}} \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right)$ of Decomposition (20) does not display lack of compactness at infinity. The problem is then reduced to prove that if a sequence $g_n = \sqrt{\frac{\alpha_n}{2\pi}} \psi\left(\frac{-\log|x|}{\alpha_n}\right)$, where (α_n) is any scale and $\psi \in L^2(\mathbb{R}, e^{-2s} ds)$ with $\psi' \in L^2(\mathbb{R})$, satisfies Hypothesis (19) then consequently $\psi|_{] -\infty, 0]} = 0$. Let us then consider a sequence g_n satisfying the above assumptions. This yields*

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\alpha_n^2 \int_{-\infty}^{-\frac{\log R}{\alpha_n}} |\psi(t)|^2 e^{-2\alpha_n t} dt \right) = 0.$$

Now, if $\psi(t_0) \neq 0$ for some $t_0 < 0$ then by continuity, we get $|\psi(t)| \gtrsim 1$ for $t_0 - \eta \leq t \leq t_0 + \eta < 0$. Hence, for n large enough,

$$\alpha_n^2 \int_{-\infty}^{-\frac{\log R}{\alpha_n}} |\psi(t)|^2 e^{-2\alpha_n t} dt \gtrsim \frac{\alpha_n}{2} \left(e^{-2\alpha_n(t_0 - \eta)} - e^{-2\alpha_n(t_0 + \eta)} \right),$$

which leads easily to the desired result.

d) *Compared with the decomposition in [15], it can be seen that there's no core in (20). This is justified by the radial setting.*

e) *The description of the lack of compactness of the embedding of $H^1(\mathbb{R}^2)$ into Orlicz space in the general frame is much harder than the radial setting. This will be dealt*

with in a forthcoming paper.

f) Let us mention that M. Struwe in [46] studied the loss of compactness for the functional

$$E(u) = \frac{1}{|\Omega|} \int_{\Omega} e^{4\pi|u|^2} dx,$$

where Ω is a bounded domain in \mathbb{R}^2 .

It should be emphasized that, contrary to the case of Sobolev embedding in Lebesgue spaces, where the asymptotic decomposition derived by P. Gérard in [15] leads to

$$\|u_n\|_{L^p}^p \rightarrow \sum_{j \geq 1} \|\psi^{(j)}\|_{L^p}^p,$$

Theorem 1.16 induces to

$$(22) \quad \|u_n\|_{\mathcal{L}} \rightarrow \sup_{j \geq 1} \left(\lim_{n \rightarrow \infty} \|g_n^{(j)}\|_{\mathcal{L}} \right),$$

thanks to the following proposition.

Proposition 1.18. *Let $(\alpha^{(j)})_{1 \leq j \leq \ell}$ be a family of pairwise orthogonal scales and $(\psi^{(j)})_{1 \leq j \leq \ell}$ be a family of profiles, and set*

$$g_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}} \psi^{(j)} \left(\frac{-\log|x|}{\alpha_n^{(j)}} \right) := \sum_{j=1}^{\ell} g_n^{(j)}(x).$$

Then

$$(23) \quad \|g_n\|_{\mathcal{L}} \rightarrow \sup_{1 \leq j \leq \ell} \left(\lim_{n \rightarrow \infty} \|g_n^{(j)}\|_{\mathcal{L}} \right).$$

A consequence of this proposition is that the first profile in the decomposition (20) can be chosen such that up to extraction

$$\limsup_{n \rightarrow \infty} \|u_n\|_{\mathcal{L}} = A_0 = \lim_{n \rightarrow \infty} \left\| \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \psi^{(1)} \left(\frac{-\log|x|}{\alpha_n^{(1)}} \right) \right\|_{\mathcal{L}}.$$

1.6. Structure of the paper. Our paper is organized as follows: we first describe in Section 2 the algorithmic construction of the decomposition of a bounded sequence $(u_n)_{n \in \mathbb{N}}$ in $H_{rad}^1(\mathbb{R}^2)$, up to a subsequence extraction, in terms of asymptotically orthogonal profiles in the spirit of the Lions' examples $\sqrt{\frac{\alpha}{2\pi}} \psi\left(\frac{-\log|x|}{\alpha}\right)$, and then prove Proposition 1.18. Section 3 is devoted to the study of nonlinear wave equations with exponential growth, both in the sub-critical and critical cases. The purpose is then to investigate the influence of the nonlinear term on the main features of solutions of nonlinear wave equations by comparing their evolution with the evolution of the solutions of the Klein-Gordon equation. Finally, we deal in appendix with several complements for the sake of completeness.

Finally, we mention that, C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$

for some absolute constant C and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. For simplicity, we shall also still denote by (u_n) any subsequence of (u_n) .

2. EXTRACTION OF SCALES AND PROFILES

This section is devoted to the proofs of Theorem 1.16 and Proposition 1.18. Our approach to extract scales and profiles relies on a diagonal subsequence extraction and uses in a crucial way the radial setting and particularly the fact that we deal with bounded functions far away from the origin. The heart of the matter is reduced to the proof of the following lemma.

Lemma 2.1. *Let (u_n) be a sequence in $H_{rad}^1(\mathbb{R}^2)$ satisfying the assumptions of the Theorem 1.16. Then there exists a scale (α_n) and a profile ψ such that*

$$(24) \quad \|\psi'\|_{L^2} \geq C A_0,$$

where C is a universal constant.

Roughly speaking, the proof is done in three steps. In the first step, according to Lemma 2.1, we extract the first scale and the first profile satisfying the condition (24). This reduces the problem to the study of the remainder term. If the limit of its Orlicz norm is null we stop the process. If not, we prove that this remainder term satisfies the same properties as the sequence start which allow us to apply the lines of reasoning of the first step and extract a second scale and a second profile which verifies the above key property (24). By contradiction arguments, we get the property of orthogonality between the two first scales. Finally, we prove that this process converges.

2.1. Extraction of the first scale and the first profile. Let us consider a bounded sequence (u_n) in $H_{rad}^1(\mathbb{R}^2)$ satisfying hypothesis (17)-(18)-(19) and let us set $v_n(s) = u_n(e^{-s})$. The following lemma summarize some properties of the sequence (u_n) that will be useful to implement the proof strategy.

Lemma 2.2. *Under the above assumptions and up to a subsequence extraction, the sequence (u_n) converges strongly to 0 in L^2 and we have, for any $M \in \mathbb{R}$,*

$$(25) \quad \|v_n\|_{L^\infty(-\infty, M]} \rightarrow 0, \quad n \rightarrow \infty.$$

Proof. Let us first observe that for any $R > 0$, we have

$$\|u_n\|_{L^2} = \|u_n\|_{L^2(|x| \leq R)} + \|u_n\|_{L^2(|x| > R)}.$$

Now, by virtue of Rellich's theorem, the Sobolev space $H^1(|x| \leq R)$ is compactly embedded in $L^2(|x| \leq R)$. Therefore,

$$\limsup_{n \rightarrow \infty} \|u_n\|_{L^2} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{L^2(|x| \geq R)}.$$

Taking advantage of the compactness at infinity of the sequence, we deduce the strong convergence of the sequence (u_n) to zero in L^2 .

On the other hand, Property (25) derives immediately from the boundedness of (u_n)

in H^1 , the strong convergence to zero of (u_n) in L^2 and the following well known radial estimate recalled in Lemma 4.2

$$|u(r)| \leq \frac{C}{\sqrt{r}} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}.$$

□

The first step is devoted to the determination of the first scale and the first profile.

Proposition 2.3. *For any $\delta > 0$, we have*

$$(26) \quad \sup_{s \geq 0} \left(\left| \frac{v_n(s)}{A_0 - \delta} \right|^2 - s \right) \rightarrow \infty, \quad n \rightarrow \infty.$$

Proof. We proceed by contradiction. If not, there exists $\delta > 0$ such that, up to a subsequence extraction

$$(27) \quad \sup_{s \geq 0, n \in \mathbb{N}} \left(\left| \frac{v_n(s)}{A_0 - \delta} \right|^2 - s \right) \leq C < \infty.$$

In one hand, thanks to (25) and (27), we get by virtue of Lebesgue theorem

$$\int_{|x| < 1} \left(e^{\left| \frac{u_n(x)}{A_0 - \delta} \right|^2} - 1 \right) dx = 2\pi \int_0^\infty \left(e^{\left| \frac{v_n(s)}{A_0 - \delta} \right|^2} - 1 \right) e^{-2s} ds \rightarrow 0, \quad n \rightarrow \infty.$$

On the other hand, using Lemma 2.2 and the simple fact that for any positive M , there exists a finite constant C_M such that

$$\sup_{|t| \leq M} \left(\frac{e^{t^2} - 1}{t^2} \right) < C_M,$$

we deduce that

$$\int_{|x| \geq 1} \left(e^{\left| \frac{u_n(x)}{A_0 - \delta} \right|^2} - 1 \right) dx \leq C \|u_n\|_{L^2}^2 \rightarrow 0.$$

This leads finally to

$$\int_{\mathbb{R}^2} \left(e^{\left| \frac{u_n(x)}{A_0 - \delta} \right|^2} - 1 \right) dx \rightarrow 0, \quad n \rightarrow \infty.$$

Hence

$$\limsup_{n \rightarrow \infty} \|u_n\|_{\mathcal{L}} \leq A_0 - \delta,$$

which is in contradiction with Hypothesis (18). □

Corollary 2.4. *Let us fix $\delta = A_0/2$, then up to a subsequence extraction, there exists a sequence $(\alpha_n^{(1)})$ in \mathbb{R}_+ tending to infinity such that*

$$(28) \quad 4 \left| \frac{v_n(\alpha_n^{(1)})}{A_0} \right|^2 - \alpha_n^{(1)} \rightarrow \infty$$

Proof. Let us set

$$W_n(s) = 4 \left| \frac{v_n(s)}{A_0} \right|^2 - s, \quad a_n = \sup_s W_n(s).$$

Then, there exists $\alpha_n^{(1)} > 0$ such that

$$W_n(\alpha_n^{(1)}) \geq a_n - \frac{1}{n}.$$

In other respects under (26), $a_n \rightarrow \infty$ and then $W_n(\alpha_n^{(1)}) \rightarrow \infty$. It remains to prove that $\alpha_n^{(1)} \rightarrow \infty$. If not, up to a subsequence extraction, the sequence $(\alpha_n^{(1)})$ is bounded and so is $(W_n(\alpha_n^{(1)}))$ by (25). This completes the proof. \square

An immediate consequence of the previous corollary is the following result

Corollary 2.5. *Under the above hypothesis, we have*

$$\frac{A_0}{2} \sqrt{\alpha_n^{(1)}} \leq |v_n(\alpha_n^{(1)})| \leq C \sqrt{\alpha_n^{(1)}} + o(1),$$

with $C = (\limsup_{n \rightarrow \infty} \|\nabla u_n\|_{L^2}) / \sqrt{2\pi}$ and where, as in all that follows, $o(1)$ denotes a sequence which tends to 0 as n goes to infinity.

Proof. The left hand side inequality follows immediately from Corollary 2.4. In other respects, noticing that by virtue of (25), the sequence $v_n(0) \rightarrow 0$, one can write for any positive real s

$$|v_n(s)| = \left| v_n(0) + \int_0^s v_n'(\tau) d\tau \right| \leq |v_n(0)| + s^{1/2} \|v_n'\|_{L^2} \leq |v_n(0)| + s^{1/2} \frac{\|\nabla u_n\|_{L^2}}{\sqrt{2\pi}},$$

from which the right hand side of the desired inequality follows. \square

Now we are able to extract the first profile. To do so, let us set

$$\psi_n(y) = \sqrt{\frac{2\pi}{\alpha_n^{(1)}}} v_n(\alpha_n^{(1)} y).$$

The following lemma summarize the principle properties of (ψ_n) .

Lemma 2.6. *Under notations of Corollary 2.5, there exists a constant C such that*

$$\frac{A_0}{2} \sqrt{2\pi} \leq |\psi_n(1)| \leq C + o(1).$$

Moreover, there exists a profile $\psi^{(1)} \in \mathcal{P}$ such that, up to a subsequence extraction

$$\psi_n' \rightharpoonup (\psi^{(1)})' \quad \text{in } L^2(\mathbb{R}) \quad \text{and} \quad \|(\psi^{(1)})'\|_{L^2} \geq \frac{\sqrt{2\pi}}{2} A_0.$$

Proof. The first assertion is contained in Corollary 2.5. To prove the second one, let us first remark that since $\|\psi_n'\|_{L^2} = \|\nabla u_n\|_{L^2}$ then the sequence (ψ_n') is bounded in L^2 . Thus, up to a subsequence extraction, (ψ_n') converges weakly in L^2 to some function $g \in L^2$. In addition, $(\psi_n(0))$ converges in \mathbb{R} to 0 and (still up to a subsequence

extraction) $(\psi_n(1))$ converges in \mathbb{R} to some constant a satisfying $|a| \geq \frac{\sqrt{2\pi}}{2} A_0$. Let us then introduce the function

$$\psi^{(1)}(s) := \int_0^s g(\tau) d\tau.$$

Our task now is to show that $\psi^{(1)}$ belongs to the set \mathcal{P} . Clearly $\psi^{(1)} \in \mathcal{C}(\mathbb{R})$ and $(\psi^{(1)})' = g \in L^2(\mathbb{R})$. Moreover, since

$$|\psi^{(1)}(s)| = \left| \int_0^s g(\tau) d\tau \right| \leq s^{1/2} \|g\|_{L^2(\mathbb{R})},$$

we get $\psi^{(1)} \in L^2(\mathbb{R}^+, e^{-2s} ds)$. It remains to prove that $\psi^{(1)}(s) = 0$ for all $s \leq 0$. Using the boundedness of the sequence (u_n) in $L^2(\mathbb{R}^2)$ and the fact that

$$\|u_n\|_{L^2}^2 = (\alpha_n^{(1)})^2 \int_{\mathbb{R}} |\psi_n(s)|^2 e^{-2\alpha_n^{(1)} s} ds,$$

we deduce that

$$\int_{-\infty}^0 |\psi_n(s)|^2 ds \leq \frac{C}{(\alpha_n^{(1)})^2}.$$

Hence, (ψ_n) converges strongly to zero in $L^2(-\infty, 0]$, and then almost everywhere (still up to a subsequence extraction). In other respects, since (ψ'_n) converges weakly to g in $L^2(\mathbb{R})$ and $\psi_n \in H_{loc}^1(\mathbb{R})$, we infer that

$$\psi_n(s) - \psi_n(0) = \int_0^s \psi'_n(\tau) d\tau \rightarrow \int_0^s g(\tau) d\tau = \psi^{(1)}(s),$$

from which it follows that

$$\psi_n(s) \rightarrow \psi^{(1)}(s), \quad \text{for all } s \in \mathbb{R}.$$

As ψ_n goes to zero for all $s \leq 0$, we deduce that $\psi^{(1)}(s) = 0$ for all $s \leq 0$. Finally, we have proved that $\psi^{(1)} \in \mathcal{P}$ and $|\psi^{(1)}(1)| = |a| \geq \frac{\sqrt{2\pi}}{2} A_0$. The fact that

$$|\psi^{(1)}(1)| = \left| \int_0^1 (\psi^{(1)})'(\tau) d\tau \right| \leq \|(\psi^{(1)})'\|_{L^2},$$

yields $\|(\psi^{(1)})'\|_{L^2} \geq \frac{\sqrt{2\pi}}{2} A_0$. □

Set

$$(29) \quad r_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \left(\psi_n \left(\frac{-\log|x|}{\alpha_n^{(1)}} \right) - \psi^{(1)} \left(\frac{-\log|x|}{\alpha_n^{(1)}} \right) \right).$$

It can be easily seen that

$$\|\nabla r_n^{(1)}\|_{L^2(\mathbb{R}^2)}^2 = \|\psi'_n - (\psi^{(1)})'\|_{L^2(\mathbb{R})}^2.$$

Taking advantage of the fact that (ψ'_n) converges weakly in $L^2(\mathbb{R})$ to $(\psi^{(1)})'$, we get the following result

Lemma 2.7. *Let (u_n) be a sequence in $H_{rad}^1(\mathbb{R}^2)$ satisfying the assumptions of the Theorem 1.16. Then there exists a scale $(\alpha_n^{(1)})$ and a profile $\psi^{(1)}$ such that*

$$\|(\psi^{(1)})'\|_{L^2} \geq \frac{\sqrt{2\pi}}{2} A_0,$$

and

$$(30) \quad \limsup_{n \rightarrow \infty} \|\nabla r_n^{(1)}\|_{L^2}^2 \leq \limsup_{n \rightarrow \infty} \|\nabla u_n\|_{L^2}^2 - \|(\psi^{(1)})'\|_{L^2}^2.$$

where $r_n^{(1)}$ is given by (29).

2.2. Conclusion. Our concern now is to iterate the previous process and to prove that the algorithmic construction converges. Observing that, for $R \geq 1$, and thanks to the fact that $\psi|_{]-\infty, 0]} = 0$,

$$\begin{aligned} \|r_n^{(1)}\|_{L^2(|x| \geq R)}^2 &= (\alpha_n^{(1)})^2 \int_{-\infty}^{-\frac{\log R}{\alpha_n^{(1)}}} |\psi_n(t) - \psi^{(1)}(t)|^2 e^{-2\alpha_n^{(1)}t} dt, \\ &= (\alpha_n^{(1)})^2 \int_{-\infty}^{-\frac{\log R}{\alpha_n^{(1)}}} |\psi_n(t)|^2 e^{-2\alpha_n^{(1)}t} dt, \\ &= \|u_n\|_{L^2(|x| \geq R)}^2, \end{aligned}$$

we deduce that $(r_n^{(1)})$ satisfies the hypothesis of compactness at infinity (19). This leads, according to (30), that $(r_n^{(1)})$ is bounded in H_{rad}^1 and satisfies (17).

Let us define $A_1 = \limsup_{n \rightarrow \infty} \|r_n^{(1)}\|_{\mathcal{L}}$. If $A_1 = 0$, we stop the process. If not, we apply the above argument to $r_n^{(1)}$ and then there exists a scale $(\alpha_n^{(2)})$ satisfying the statement of Corollary 2.4 with A_1 instead of A_0 . In particular, there exists a constant C such that

$$(31) \quad \frac{A_1}{2} \sqrt{\alpha_n^{(2)}} \leq |\tilde{r}_n^{(1)}(\alpha_n^{(2)})| \leq C \sqrt{\alpha_n^{(2)}} + o(1),$$

where $\tilde{r}_n^{(1)}(s) = r_n^{(1)}(e^{-s})$. Moreover, we claim that $\alpha_n^{(2)} \perp \alpha_n^{(1)}$, or equivalently that $\log \left| \frac{\alpha_n^{(2)}}{\alpha_n^{(1)}} \right| \rightarrow \infty$. Otherwise, there exists a constant C such that

$$\frac{1}{C} \leq \left| \frac{\alpha_n^{(2)}}{\alpha_n^{(1)}} \right| \leq C.$$

Now, according to (29), we have

$$\tilde{r}_n^{(1)}(\alpha_n^{(2)}) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \left(\psi_n \left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}} \right) - \psi^{(1)} \left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}} \right) \right).$$

This yields a contradiction in view of (31) and the following convergence result (up to a subsequence extraction)

$$\psi_n \left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}} \right) - \psi^{(1)} \left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}} \right) \rightarrow 0.$$

Moreover, there exists a profile $\psi^{(2)}$ in \mathcal{P} such that

$$r_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(2)}}{2\pi}} \psi^{(2)} \left(\frac{-\log|x|}{\alpha_n^{(2)}} \right) + r_n^{(2)}(x),$$

with $\|(\psi^{(2)})'\|_{L^2} \geq \frac{\sqrt{2\pi}}{2} A_1$ and

$$\limsup_{n \rightarrow \infty} \|\nabla r_n^{(2)}\|_{L^2}^2 \leq \limsup_{n \rightarrow \infty} \|\nabla r_n^{(1)}\|_{L^2}^2 - \|(\psi^{(2)})'\|_{L^2}^2.$$

This leads to the following crucial estimate

$$\limsup_{n \rightarrow \infty} \|r_n^{(2)}\|_{H^1}^2 \leq C - \frac{\sqrt{2\pi}}{2} A_0^2 - \frac{\sqrt{2\pi}}{2} A_1^2,$$

with $C = \limsup_{n \rightarrow \infty} \|\nabla u_n\|_{L^2}^2$.

At iteration ℓ , we get

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}} \psi^{(j)} \left(\frac{-\log|x|}{\alpha_n^{(j)}} \right) + r_n^{(\ell)}(x),$$

with

$$\limsup_{n \rightarrow \infty} \|r_n^{(\ell)}\|_{H^1}^2 \lesssim 1 - A_0^2 - A_1^2 - \dots - A_{\ell-1}^2.$$

Therefore $A_\ell \rightarrow 0$ as $\ell \rightarrow \infty$ and the proof of the decomposition (20) is achieved.

The stability estimate (21) is a simple consequence of the fact that for every $f, g \in L^2(\mathbb{R})$ and every orthogonal scales $\underline{\alpha}$ and $\underline{\beta}$, it holds

$$\int_{\mathbb{R}} \frac{1}{\sqrt{\alpha_n \beta_n}} f \left(\frac{s}{\alpha_n} \right) g \left(\frac{s}{\beta_n} \right) ds \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This ends the proof of the theorem.

Let us now go to the proof of Proposition 1.18.

Proof of Proposition 1.18. We restrict ourselves to the example $h_\alpha := a f_\alpha + b f_{\alpha^2}$ where a, b are two real numbers. The general case is similar except for more technical complications. Set $M := \sup(|a|, |b|)$. We want to show that

$$\|h_\alpha\|_{\mathcal{L}} \rightarrow \frac{M}{\sqrt{4\pi}} \quad \text{as } \alpha \rightarrow \infty.$$

We start by proving that

$$(32) \quad \liminf_{\alpha \rightarrow \infty} \|h_\alpha\|_{\mathcal{L}} \geq \frac{M}{\sqrt{4\pi}}.$$

Let $\lambda > 0$ such that

$$\int_{\mathbb{R}^2} \left(e^{\frac{|h_\alpha(x)|^2}{\lambda^2}} - 1 \right) dx \leq \kappa.$$

This implies

$$(33) \quad \int_0^{e^{-\alpha^2}} \left(e^{\frac{|h_\alpha(r)|^2}{\lambda^2}} - 1 \right) r \, dr \leq \frac{\kappa}{2\pi},$$

and

$$(34) \quad \int_{e^{-\alpha^2}}^{e^{-\alpha}} \left(e^{\frac{|h_\alpha(r)|^2}{\lambda^2}} - 1 \right) r \, dr \leq \frac{\kappa}{2\pi}.$$

Since

$$h_\alpha(r) = \begin{cases} a\sqrt{\frac{\alpha}{2\pi}} + b\frac{\alpha}{\sqrt{2\pi}} & \text{if } r \leq e^{-\alpha^2}, \\ a\sqrt{\frac{\alpha}{2\pi}} - \frac{b}{\alpha\sqrt{2\pi}} \log r & \text{if } e^{-\alpha^2} \leq r \leq e^{-\alpha}, \end{cases}$$

we get from (33) and (34)

$$\lambda^2 \geq \frac{\alpha(a + b\sqrt{\alpha})^2}{2\pi \log(1 + Ce^{2\alpha^2})} = \frac{b^2}{4\pi} + o(1),$$

and

$$\lambda^2 \geq \frac{a^2\alpha + 2ab\sqrt{\alpha} + b^2}{2\pi \log(1 + Ce^{2\alpha})} = \frac{a^2}{4\pi} + o(1).$$

This leads to (32) as desired.

In the general case, we have to replace (33) and (34) by ℓ estimates of that type. Indeed, assuming that $\frac{\alpha_n^{(j)}}{\alpha_n^{(j+1)}} \rightarrow 0$ when n goes to infinity for $j = 1, 2, \dots, \ell - 1$, we replace (33) and (34) by the fact that

$$(35) \quad \int_{e^{-\alpha_n^{(j+1)}}}^{e^{-\alpha_n^{(j)}}} \left(e^{\frac{|h_\alpha(r)|^2}{\lambda^2}} - 1 \right) r \, dr \leq \frac{\kappa}{2\pi}, \quad j = 1, \dots, \ell - 1$$

and

$$(36) \quad \int_0^{e^{-\alpha_n^{(\ell)}}} \left(e^{\frac{|h_\alpha(r)|^2}{\lambda^2}} - 1 \right) r \, dr \leq \frac{\kappa}{2\pi}.$$

Our concern now is to prove the second (and more difficult) part, that is

$$(37) \quad \limsup_{\alpha \rightarrow \infty} \|h_\alpha\|_{\mathcal{L}} \leq \frac{M}{\sqrt{4\pi}}.$$

To do so, it is sufficient to show that for any $\eta > 0$ small enough and α large enough

$$(38) \quad \int_{\mathbb{R}^2} \left(e^{\frac{4\pi-\eta}{M^2}|h_\alpha(x)|^2} - 1 \right) dx \leq \kappa.$$

Actually, we will prove that the left hand side of (38) goes to zero when α goes to infinity. We shall make use of the following lemma.

Lemma 2.8. *Let p, q be two real numbers such that $0 < p, q < 2$. Set*

$$(39) \quad \mathbf{I}_\alpha = e^{p\alpha} \int_{e^{-\alpha^2}}^{e^{-\alpha}} e^{q \frac{\log^2 r}{\alpha^2}} r \, dr .$$

Then $I_\alpha \rightarrow 0$ as $\alpha \rightarrow \infty$.

Proof of Lemma 2.8. The change of variable $y = \frac{\sqrt{q}}{\alpha} \left(-\log r - \frac{\alpha^2}{q} \right)$ yields

$$\mathbf{I}_\alpha = \alpha e^{p\alpha} \frac{e^{-\frac{\alpha^2}{q}}}{\sqrt{q}} \int_{\sqrt{q} - \frac{\alpha}{\sqrt{q}}}^{\alpha \frac{q-1}{\sqrt{q}}} e^{y^2} \, dy .$$

Since $\int_0^A e^{y^2} \, dy \leq \frac{e^{A^2}}{A}$ for every nonnegative real A , we get (for $q > 1$ for example)

$$\mathbf{I}_\alpha \lesssim e^{p\alpha} e^{(q-2)\alpha^2} + e^{(p-2)\alpha} ,$$

and the conclusion follows. We argue similarly if $q \leq 1$. □

We return now to the proof of (37). To this end, write

$$(40) \quad \frac{(4\pi - \eta)}{M^2} |h_\alpha|^2 = \frac{4\pi - \eta}{M^2} a^2 f_\alpha^2 + \frac{4\pi - \eta}{M^2} b^2 f_{\alpha^2}^2 + 2 \frac{4\pi - \eta}{M^2} ab f_\alpha f_{\alpha^2}$$

$$(41) \quad := A_\alpha + B_\alpha + C_\alpha .$$

The simple observation

$$\begin{aligned} e^{x+y+z} - 1 &= (e^x - 1)(e^y - 1)(e^z - 1) + (e^x - 1)(e^y - 1) \\ &+ (e^x - 1)(e^z - 1) + (e^y - 1)(e^z - 1) \\ &+ (e^x - 1) + (e^y - 1) + (e^z - 1) , \end{aligned}$$

yields

$$(42) \quad \begin{aligned} \int_{\mathbb{R}^2} \left(e^{\frac{4\pi - \eta}{M^2} |h_\alpha(x)|^2} - 1 \right) dx &= \int (e^{A_\alpha} - 1)(e^{B_\alpha} - 1)(e^{C_\alpha} - 1) + \int (e^{A_\alpha} - 1)(e^{B_\alpha} - 1) \\ &+ \int (e^{A_\alpha} - 1)(e^{C_\alpha} - 1) + \int (e^{B_\alpha} - 1)(e^{C_\alpha} - 1) \\ &+ \int (e^{A_\alpha} - 1) + \int (e^{B_\alpha} - 1) + \int (e^{C_\alpha} - 1) . \end{aligned}$$

By Trudinger-Moser estimate (5), we have for $\varepsilon \geq 0$ small enough,

$$(43) \quad \left\| e^{A_\alpha} - 1 \right\|_{L^{1+\varepsilon}} + \left\| e^{B_\alpha} - 1 \right\|_{L^{1+\varepsilon}} \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty .$$

To check that the last term in (42) tends to zero, we use Lebesgue Theorem in the region $e^{-\alpha} \leq r \leq 1$. Observe that one can replace C_α with γC_α for any $\gamma > 0$.

The two terms containing both A_α and C_α or B_α and C_α can be handled in a similar way. Indeed, by Hölder inequality and (43), we infer (for $\varepsilon > 0$ small enough)

$$\begin{aligned} \int (e^{A_\alpha} - 1) (e^{C_\alpha} - 1) + \int (e^{B_\alpha} - 1) (e^{C_\alpha} - 1) &\leq \left\| e^{A_\alpha} - 1 \right\|_{L^{1+\varepsilon}} \left\| e^{C_\alpha} - 1 \right\|_{L^{1+\frac{1}{\varepsilon}}} \\ &+ \left\| e^{B_\alpha} - 1 \right\|_{L^{1+\varepsilon}} \left\| e^{C_\alpha} - 1 \right\|_{L^{1+\frac{1}{\varepsilon}}} \rightarrow 0. \end{aligned}$$

Now, we claim that

$$(44) \quad \int (e^{A_\alpha} - 1) (e^{B_\alpha} - 1) \rightarrow 0, \quad \alpha \rightarrow \infty.$$

The main difficulty in the proof of (44) comes from the term

$$\int_{e^{-\alpha^2}}^{e^{-\alpha}} (e^{A_\alpha(r)} - 1) (e^{B_\alpha(r)} - 1) r dr \lesssim \mathbf{I}_\alpha := e^{\frac{4\pi-\eta}{M^2} a^2 \frac{\alpha}{2\pi}} \int_{e^{-\alpha^2}}^{e^{-\alpha}} e^{\frac{4\pi-\eta}{M^2} b^2 \frac{\log^2 r}{2\pi\alpha^2}} r dr.$$

Setting $p := \frac{4\pi-\eta}{2\pi} \frac{a^2}{M^2}$ and $q := \frac{4\pi-\eta}{2\pi} \frac{b^2}{M^2}$, we conclude thanks to Lemma 2.8 since $0 < p, q < 2$. It is easy to see that (44) still holds if A_α and B_α are replaced by $(1 + \varepsilon)A_\alpha$ and $(1 + \varepsilon)B_\alpha$ respectively, where $\varepsilon \geq 0$ is small.

Finally, for the first term in (42), we use Hölder inequality and (44).

Consequently, we obtain

$$\limsup_{\alpha \rightarrow \infty} \|h_\alpha\|_{\mathcal{L}} \leq \frac{M}{\sqrt{4\pi}}.$$

In the general case we replace (41), by $\ell + \frac{\ell(\ell-1)}{2}$ terms and the rest of the proof is very similar. This completes the proof of Proposition 1.18. \square

3. QUALITATIVE STUDY OF NONLINEAR WAVE EQUATION

This section is devoted to the qualitative study of the solutions of the two-dimensional nonlinear Klein-Gordon equation

$$(45) \quad \square u + u + f(u) = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x^2 \rightarrow \mathbb{R},$$

where

$$f(u) = u \left(e^{4\pi u^2} - 1 \right).$$

Exponential type nonlinearities have been considered in several physical models (see e.g. [27] on a model of self-trapped beams in plasma). For decreasing exponential nonlinearities, T. Cazenave in [8] proved global well-posedness together with scattering in the case of NLS.

It is known (see [35, 37]) that the Cauchy problem associated to Equation (45) with Cauchy data small enough in $H^1 \times L^2$ is globally well-posed. Moreover, subcritical, critical and supercritical regimes in the energy space are identified (see [21]). Global well-posedness is established in both subcritical and critical regimes while well-posedness fails to hold in the supercritical one (we refer to [21] for more details). Very recently, M. Struwe [47] has constructed global smooth solutions for the 2D energy critical wave equation with radially symmetric data. Although the techniques are different, this result might be seen as an analogue of Tao's result [48] for the 3D energy supercritical wave equation. Let us emphasize that the solutions of the

two-dimensional nonlinear Klein-Gordon equation formally satisfy the conservation of energy

$$(46) \quad \begin{aligned} E(u, t) &= \|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4\pi} \|e^{4\pi u(t)^2} - 1\|_{L^1} \\ &= E(u, 0) := E_0. \end{aligned}$$

The notion of criticality here depends on the size of the initial energy¹ E_0 with respect to 1. This relies on the so-called Trudinger-Moser type inequalities stated in Proposition 1.4 (see [20] and references therein for more details). Let us now precise the notions of these regimes:

Definition 3.1. *The Cauchy problem associated to Equation (45) with initial data $(u_0, u_1) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ is said to be subcritical if*

$$E_0 < 1.$$

It is said critical if $E_0 = 1$ and supercritical if $E_0 > 1$.

It is then natural to investigate the feature of solutions of the two-dimensional nonlinear Klein-Gordon equation taking into account the different regimes, as in earlier works of P. Gérard [14] and H. Bahouri-P. Gérard [3]. The approach that we adopt here is the one introduced by P. Gérard in [14] which consists to compare the evolution of oscillations and concentration effects displayed by sequences of solutions of the nonlinear Klein-Gordon equation (45) and solutions of the linear Klein-Gordon equation.

$$(47) \quad \square v + v = 0$$

More precisely, let (φ_n, ψ_n) be a sequence of data in $H^1 \times L^2$ supported in some fixed ball and satisfying

$$(48) \quad \varphi_n \rightharpoonup 0 \quad \text{in } H^1, \quad \psi_n \rightharpoonup 0 \quad \text{in } L^2,$$

such that

$$(49) \quad E^n \leq 1, \quad n \in \mathbb{N}$$

where E^n stands for the energy of (φ_n, ψ_n) given by

$$E^n = \|\psi_n\|_{L^2}^2 + \|\nabla \varphi_n\|_{L^2}^2 + \frac{1}{4\pi} \|e^{4\pi \varphi_n^2} - 1\|_{L^1}$$

and let us consider (u_n) and (v_n) the sequences of finite energy solutions of (45) and (47) such that

$$(u_n, \partial_t u_n)(0) = (v_n, \partial_t v_n)(0) = (\varphi_n, \psi_n).$$

Arguing as in [14], we introduce the following definition

Definition 3.2. *Let T be a positive time. We shall say that the sequence (u_n) is linearizable on $[0, T]$, if*

$$\sup_{t \in [0, T]} E_c(u_n - v_n, t) \longrightarrow 0 \quad \text{as } n \rightarrow \infty$$

¹This is in contrast with higher dimensions where the criticality depends on the nonlinearity.

where $E_c(w, t)$ denotes the kinetic energy defined by:

$$E_c(w, t) = \int_{\mathbb{R}^2} [|\partial_t w|^2 + |\nabla_x w|^2 + |w|^2] (t, x) dx.$$

The following results illustrate the critical feature of the condition $E_0 = 1$.

Theorem 3.3. *Under the above notations, let us assume that $\limsup_{n \rightarrow \infty} E^n < 1$. Then, for every positive time T , the sequence (u_n) is linearizable on $[0, T]$.*

Remark 3.4. *Let us recall that in the case of dimension $d \geq 3$, the same kind of result holds. More precisely, P. Gérard proved in [14] that in the subcritical case, the nonlinearity does not induce any effect on the behavior of the solutions.*

In the critical case i.e. $\limsup_{n \rightarrow \infty} E^n = 1$, it turns out that a nonlinear effect can be produced and we have the following result:

Theorem 3.5. *Assume that $\limsup_{n \rightarrow \infty} E^n = 1$ and let $T > 0$. Then the sequence (u_n) is linearizable on $[0, T]$ provided that the sequence (v_n) satisfies*

$$(50) \quad \limsup_{n \rightarrow \infty} \|v_n\|_{L^\infty([0, T]; \mathcal{L})} < \frac{1}{\sqrt{4\pi}}.$$

Remark 3.6. *In Theorem 3.5, we give a sufficient condition on the sequence (v_n) which ensures the linearizability of the sequence (u_n) . Similarly to higher dimensions, this condition concerns the solutions of linear Klein-Gordon equation. However, unlike to higher dimensions, we are not able to prove the converse, that is if the sequence (u_n) is linearizable on $[0, T]$ then $\limsup_{n \rightarrow \infty} \|v_n\|_{L^\infty([0, T]; \mathcal{L})} < \frac{1}{\sqrt{4\pi}}$. The main difficulty in our approach is that we do not know whether*

$$\limsup_{n \rightarrow \infty} \|v_n\|_{L^\infty([0, T]; \mathcal{L})} = \frac{1}{\sqrt{4\pi}} \quad \text{and} \quad \|f(v_n)\|_{L^1([0, T]; L^2(\mathbb{R}^2))} \rightarrow 0.$$

Before going into the proofs of Theorems 3.3 and 3.5, let us recall some well known and useful tools. The main basic tool that we shall deal with is Strichartz estimate.

3.1. Technical tools.

3.1.1. *Strichartz estimate.* Let us first begin by introducing the definition of admissible pairs.

Definition 3.7. *Let $\rho \in \mathbb{R}$. We say that $(q, r) \in [4, \infty] \times [2, \infty]$ is a ρ -admissible pair if*

$$(51) \quad \frac{1}{q} + \frac{2}{r} = \rho.$$

When $\rho = 1$, we shall say admissible instead of 1-admissible.

For example, $(4, \infty)$ is a 1/4-admissible pair, and for every $0 < \varepsilon \leq 1/3$, the couple $(1 + 1/\varepsilon, 2(1 + \varepsilon))$ is an admissible pair. The following Strichartz inequalities that can be for instance found in [36] will be of constant use in what follows.

Proposition 3.8 (Strichartz estimate). *Let $\rho \in \mathbb{R}$, (q, r) a ρ -admissible pair and $T > 0$. Then*

$$(52) \quad \|v\|_{L^q([0,T]; B_{r,2}^\rho(\mathbb{R}^2))} \lesssim [\|\partial_t v(0, \cdot)\|_{L^2(\mathbb{R}^2)} + \|v(0, \cdot)\|_{H^1(\mathbb{R}^2)} + \|\square v + v\|_{L^1([0,T]; L^2(\mathbb{R}^2))}],$$

where $B_{r,2}^\rho(\mathbb{R}^2)$ stands for the usual inhomogeneous Besov space (see for example [9] or [41] for a detailed exposition on Besov spaces).

Now, for any time slab $I \subset \mathbb{R}$, we shall denote

$$\|v\|_{\text{ST}(I)} := \sup_{(q,r) \text{ admissible}} \|v\|_{L^q(I; B_{r,2}^1(\mathbb{R}^2))}.$$

By interpolation argument, this Strichartz norm is equivalent to

$$\|v\|_{L^\infty(I; H^1(\mathbb{R}^2))} + \|v\|_{L^4(I; B_{8/3,2}^1(\mathbb{R}^2))}.$$

As $B_{r,2}^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for all $r \leq p < \infty$ (and $r \leq p \leq \infty$ if $r > 2$), it follows that

$$(53) \quad \|v\|_{L^q(I; L^p)} \lesssim \|v\|_{\text{ST}(I)}, \quad \frac{1}{q} + \frac{2}{p} \leq 1.$$

Proposition 3.8 will often be combined with the following elementary bootstrap lemma.

Lemma 3.9. *Let $X(t)$ be a nonnegative continuous function on $[0, T]$ such that, for every $0 \leq t \leq T$,*

$$(54) \quad X(t) \leq a + bX(t)^\theta,$$

where $a, b > 0$ and $\theta > 1$ are constants such that

$$(55) \quad a < \left(1 - \frac{1}{\theta}\right) \frac{1}{(\theta b)^{1/(\theta-1)}}, \quad X(0) \leq \frac{1}{(\theta b)^{1/(\theta-1)}}.$$

Then, for every $0 \leq t \leq T$, we have

$$(56) \quad X(t) \leq \frac{\theta}{\theta-1} a.$$

Proof of Lemma 3.9. We sketch the proof for the convenience of the reader. The function $f : x \mapsto bx^\theta - x + a$ is decreasing on $[0, (\theta b)^{1/(1-\theta)}]$ and increasing on $[(\theta b)^{1/(1-\theta)}, \infty[$. The first assumption in (55) implies that $f((\theta b)^{1/(1-\theta)}) < 0$. As $f(X(t)) \geq 0$, $f(0) > 0$ and $X(0) \leq (\theta b)^{1/(1-\theta)}$, we deduce by continuity (56). \square

3.1.2. Logarithmic Inequalities. It is well known that the space $H^1(\mathbb{R}^2)$ is not included in $L^\infty(\mathbb{R}^2)$. However, resorting to an interpolation argument, we can estimate the L^∞ norm of functions in $H^1(\mathbb{R}^2)$, using a stronger norm but with a weaker growth (namely logarithmic). More precisely, we have the following logarithmic estimate which also holds in any bounded domain.

Lemma 3.10 (Logarithmic inequality [20], Theorem 1.3). *Let $0 < \alpha < 1$. For any real number $\lambda > \frac{1}{2\pi\alpha}$, a constant C_λ exists such that for any function φ belonging to $H_0^1(|x| < 1) \cap \dot{C}^\alpha(|x| < 1)$, we have*

$$(57) \quad \|\varphi\|_{L^\infty}^2 \leq \lambda \|\nabla\varphi\|_{L^2}^2 \log \left(C_\lambda + \frac{\|\varphi\|_{\dot{C}^\alpha}}{\|\nabla\varphi\|_{L^2}} \right),$$

where \dot{C}^α denotes the homogeneous Hölder space of regularity index α .

We shall also need the following version of the above inequality which is available in the whole space.

Lemma 3.11 ([20], Theorem 1.3). *Let $0 < \alpha < 1$. For any $\lambda > \frac{1}{2\pi\alpha}$ and any $0 < \mu \leq 1$, a constant $C_\lambda > 0$ exists such that for any function $u \in H^1(\mathbb{R}^2) \cap C^\alpha(\mathbb{R}^2)$, we have*

$$(58) \quad \|u\|_{L^\infty}^2 \leq \lambda \|u\|_{H_\mu}^2 \log \left(C_\lambda + \frac{8^\alpha \mu^{-\alpha} \|u\|_{C^\alpha}}{\|u\|_{H_\mu}} \right),$$

where C^α denotes the inhomogeneous Hölder space of regularity index α and H_μ the Sobolev space endowed with the norm $\|u\|_{H_\mu}^2 := \|\nabla u\|_{L^2}^2 + \mu^2 \|u\|_{L^2}^2$.

3.1.3. *Convergence in measure.* Similarly to higher dimensions (see [14]), the concept of convergence in measure occurs in the process of the proof of Theorems 3.3 and 3.5. For the convenience of the reader, let us give an outline of this notion. In many cases, the convergence in Lebesgue space L^1 is reduced to the convergence in measure.

Definition 3.12. *Let Ω be a measurable subset of \mathbb{R}^m and (u_n) be a sequence of measurable functions on Ω . We say that the sequence (u_n) converges in measure to u if, for every $\varepsilon > 0$,*

$$\left| \{y \in \Omega; |u_n(y) - u(y)| \geq \varepsilon\} \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $|B|$ stands for the Lebesgue measure of a measurable set $B \subset \mathbb{R}^m$.

It is clear that the convergence in L^1 implies the convergence in measure. The contrary is also true if we require the boundedness in some Lebesgue space L^q with $q > 1$. More precisely, we have the following well known result.

Proposition 3.13. *Let Ω be a measurable subset of \mathbb{R}^m with finite measure and (u_n) be a bounded sequence in $L^q(\Omega)$ for some $q > 1$. Then, the sequence (u_n) converges to u in $L^1(\Omega)$ if, and only if, it converges to u in measure in Ω .*

Proof. The fact that the convergence in L^1 implies the convergence in measure follows immediately from the following Tchebychev's inequality

$$\varepsilon \left| \{y \in \Omega; |u_n(y) - u(y)| \geq \varepsilon\} \right| \leq \|u_n - u\|_{L^1}.$$

To prove the converse, let us show first that u belongs to $L^q(\Omega)$. Since the sequence (u_n) converges to u in measure, we get thanks to Egorov's lemma, up to subsequence extraction

$$u_n \rightarrow u \quad \text{a.e. in } \Omega.$$

The Fatou's lemma and the boundedness of (u_n) in L^q imply then

$$\int_{\Omega} |u(y)|^q dy \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |u_n(y)|^q dy \leq C.$$

According to Hölder inequality, we have for any fixed $\varepsilon > 0$

$$\begin{aligned} \|u_n - u\|_{L^1} &= \int_{\{|u_n - u| < \varepsilon\}} |u_n - u| + \int_{\{|u_n - u| \geq \varepsilon\}} |u_n - u| \\ &\leq \varepsilon |\Omega| + \|u_n - u\|_{L^q} |\{|u_n - u| \geq \varepsilon\}|^{1 - \frac{1}{q}}, \end{aligned}$$

which ensures the result. \square

3.2. Subcritical case. The aim of this section is to prove that the nonlinear term does not affect the behavior of the solutions in the subcritical case. By hypothesis in that case, there exists some nonnegative real ρ such that $\limsup_{n \rightarrow \infty} E^n = 1 - \rho$. The main point for the proof of Theorem 3.3 is based on the following lemma.

Lemma 3.14. *For every $T > 0$ and $E_0 < 1$, there exists a constant $C(T, E_0)$, such that every solution u of the nonlinear Klein-Gordon equation (45) of energy $E(u) \leq E_0$, satisfies*

$$(59) \quad \|u\|_{L^4([0, T]; C^{1/4})} \leq C(T, E_0).$$

Proof of Lemma 3.14. By virtue of Strichartz estimate (52), we have

$$\|u\|_{L^4([0, t]; C^{1/4})} \lesssim E_0^{1/2} + \|f(u)\|_{L^1([0, t]; L^2(\mathbb{R}^2))}.$$

To estimate $f(u)$ in $L^1([0, t]; L^2(\mathbb{R}^2))$, let us apply Hölder inequality

$$\|f(u)\|_{L^2} \leq \|u\|_{L^{2+2/\varepsilon}} \|e^{4\pi u^2} - 1\|_{L^{2(1+\varepsilon)}},$$

where $\varepsilon > 0$ is chosen small enough. This leads in view of Lemma 4.1 to

$$\|f(u)\|_{L^2} \lesssim \|u\|_{H^1} e^{2\pi \|u\|_{L^\infty}^2} \|e^{4\pi(1+\varepsilon)u^2} - 1\|_{L^1}^{\frac{1}{2(1+\varepsilon)}}.$$

The logarithmic inequality (58) yields for any fixed $\lambda > \frac{2}{\pi}$,

$$e^{2\pi \|u\|_{L^\infty}^2} \lesssim \left(C + \frac{\|u\|_{C^{1/4}}}{E_0^{1/2}} \right)^{2\pi\lambda E_0}$$

and Trudinger-Moser inequality implies that for $\varepsilon > 0$ small enough

$$\|e^{4\pi(1+\varepsilon)u^2} - 1\|_{L^1} \leq \kappa.$$

Plugging these estimates together, we obtain

$$\|u\|_{L^4([0, t]; C^{1/4})} \lesssim E_0^{1/2} \left(1 + \int_0^t \left(C + \frac{\|u\|_{C^{1/4}}}{E_0^{1/2}} \right)^\theta d\tau \right)$$

where $\theta := 2\pi\lambda E_0$. Since $E_0 < 1$, we can choose $\lambda > \frac{2}{\pi}$ such that $\theta < 4$. Using Hölder inequality in time, we deduce that

$$\begin{aligned} \|u\|_{L^4([0,t];C^{1/4})} &\lesssim E_0^{1/2} \left(1 + t^{1-\theta/4} \left(t^{1/4} + E_0^{-1/2} \|u\|_{L^4([0,t];C^{1/4})} \right)^\theta \right) \\ &\lesssim E_0^{1/2} + T + E_0^{\frac{1-\theta}{2}} t^{1-\theta/4} \|u\|_{L^4([0,t];C^{1/4})}^\theta. \end{aligned}$$

In the case where $\theta > 1$, we set

$$t_{max} := \left(\frac{CE_0^{\frac{1}{2}}}{E_0^{1/2} + T} \right)^{\frac{4(\theta-1)}{4-\theta}},$$

where C is some constant. Then we obtain the desired result on the interval $[0, t_{max}]$ by absorption argument (see Lemma 3.9). Finally, to get the general case we decompose $[0, T] = \cup_{i=0}^{i=n-1} [t_i, t_{i+1}]$ such that $t_{i+1} - t_i \leq t_{max}$. Applying the Strichartz estimate on $[t_i, t]$ with $t \leq t_{i+1}$ and using the conservation of the energy, we deduce

$$\|u\|_{L^4([t_i, t_{i+1}], C^{1/4})} \leq C(T, E_0),$$

which yields the desired inequality. In the case where $\theta \leq 1$ we use a convexity argument and proceed exactly as above.

Notice that similar argument was used in higher dimension (see [18]). \square

Remark 3.15. *Let us emphasize that in the critical case ($E_0 = 1$), the conclusion of Lemma 3.14 holds provided the additional assumption*

$$(60) \quad \|u\|_{L^\infty([0,T];\mathcal{L})} < \frac{1}{\sqrt{4\pi}}.$$

The key point consists to estimate differently the term $\|e^{4\pi u^2} - 1\|_{L^{2(1+\varepsilon)}}$. More precisely, taking advantage of (60) we write

$$\begin{aligned} \|e^{4\pi u^2} - 1\|_{L^{2(1+\varepsilon)}} &\leq e^{\frac{2\pi}{1+\varepsilon}\|u\|_{L^\infty}^2} \|e^{4\pi(1+2\varepsilon)u^2} - 1\|_{L^1}^{\frac{1}{2(1+\varepsilon)}} \\ &\leq \kappa^{\frac{1}{2(1+\varepsilon)}} e^{\frac{2\pi}{1+\varepsilon}\|u\|_{L^\infty}^2}, \end{aligned}$$

which leads to the result along the same lines as above.

Let us now go to the proof of Theorem 3.3. Denoting by $w_n = u_n - v_n$, we can easily verify that w_n is the solution of the nonlinear wave equation

$$\square w_n + w_n = -f(u_n)$$

with null Cauchy data.

Under energy estimate, we obtain

$$\|w_n\|_T \lesssim \|f(u_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))},$$

where $\|w_n\|_T^2 \stackrel{\text{def}}{=} \sup_{t \in [0,T]} E_c(w_n, t)$. Therefore, to prove that the sequence (u_n) is linearizable on $[0, T]$, it suffices to establish that

$$\|f(u_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thanks to finite propagation speed, for any time $t \in [0, T]$, the sequence $f(u_n(t, \cdot))$ is uniformly supported in a compact subset K of \mathbb{R}^2 . So, to prove that the sequence $(f(u_n))$ converges strongly to 0 in $L^1([0, T], L^2(\mathbb{R}^2))$, we shall follow the strategy of P. Gérard in [14] which is firstly to demonstrate that this sequence is bounded in $L^{1+\epsilon}([0, T], L^{2+\epsilon}(\mathbb{R}^2))$, for some nonnegative ϵ , and secondly to prove that it converges to 0 in measure in $[0, T] \times \mathbb{R}^2$.

Let us then begin by estimate $\|f(u_n)\|_{L^{1+\epsilon}([0, T], L^{2+\epsilon}(\mathbb{R}^2))}$, for ϵ small enough. Since, by definition we have $f(u_n) = -u_n (e^{4\pi u_n^2} - 1)$, straightforward computations imply that

$$\|f(u_n)\|_{L^{2+\epsilon}(\mathbb{R}^2)}^{2+\epsilon} \leq C e^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2} \int_{\mathbb{R}^2} |u_n|^{2+\epsilon} (e^{4\pi u_n^2} - 1) dx.$$

In other respects, using the obvious estimate

$$\sup_{x \geq 0} (x^m e^{-\gamma x^2}) = \left(\frac{m}{2\gamma}\right)^{\frac{m}{2}} e^{-\frac{m}{2}},$$

we get, for any positive real η

$$\int_{\mathbb{R}^2} |u_n|^{2+\epsilon} (e^{4\pi u_n^2} - 1) dx \leq C_\eta \int_{\mathbb{R}^2} (e^{(4\pi+\eta)u_n^2} - 1) dx.$$

In conclusion

$$\|f(u_n)\|_{L^{2+\epsilon}(\mathbb{R}^2)}^{2+\epsilon} \leq C_\eta e^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2} \int_{\mathbb{R}^2} (e^{(4\pi+\eta)(1-\rho)^2 \left(\frac{u_n}{1-\rho}\right)^2} - 1) dx.$$

Thanks to Trudinger-Moser estimate (5), we obtain for η small enough

$$\begin{aligned} \|f(u_n)\|_{L^{2+\epsilon}(\mathbb{R}^2)}^{2+\epsilon} &\leq C_\eta e^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2} \|u_n\|_{L^2}^2 \\ &\leq C_\eta e^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2} E^n \\ &\leq C_\eta e^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2}, \end{aligned}$$

by energy estimate, using the fact that $\limsup_{n \rightarrow \infty} E^n < 1 - \rho$.

Now, taking advantage of the logarithmic estimate (58), we get for any $\lambda > \frac{2}{\pi}$ and any $0 < \mu \leq 1$

$$e^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2} \lesssim \left(C_\lambda + \frac{\|u_n\|_{C^{\frac{1}{4}}}}{\sqrt{(1-\rho)(1+\mu^2)}} \right)^{4\lambda\pi(1+\epsilon)(1-\rho)(1+\mu^2)}.$$

We deduce that

$$\|f(u_n)\|_{L^{1+\epsilon}([0, T], L^{2+\epsilon}(\mathbb{R}^2))}^{1+\epsilon} \leq C_{\eta, \rho} \int_0^T \left(C_\lambda + \|u_n\|_{C^{\frac{1}{4}}} \right)^{\frac{4\lambda\pi(1+\epsilon)^2(1-\rho)(1+\mu^2)}{2+\epsilon}} dt.$$

Choosing λ close to $\frac{2}{\pi}$, ϵ and μ small enough such that $\theta \stackrel{\text{def}}{=} \frac{4\lambda\pi(1+\epsilon)^2(1-\rho)(1+\mu^2)}{2+\epsilon} < 4$, it comes by virtue of Hölder inequality

$$(61) \quad \|f(u_n)\|_{L^{1+\epsilon}([0, T], L^{2+\epsilon}(\mathbb{R}^2))}^{1+\epsilon} \leq C(\eta, \rho, T) (T^{\frac{1}{4}} + \|u_n\|_{L^4([0, T], C^{1/4})})^\theta.$$

Lemma 3.14 allows to end the proof of the first step, namely that in the subcritical case the sequence $(f(u_n))$ is bounded in $L^{1+\epsilon}([0, T], L^{2+\epsilon}(\mathbb{R}^2))$, for ϵ small enough.

Since $\epsilon > 0$, we are then reduced as it is mentioned above to prove that the sequence $(f(u_n))$ converges to 0 in measure in $[0, T] \times \mathbb{R}^2$. Thus, by definition we have to prove that for every $\epsilon > 0$,

$$\left| \{(t, x) \in [0, T] \times \mathbb{R}^2, \quad |f(u_n)| \geq \epsilon\} \right| \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The function f being continuous at the origin with $f(0) = 0$, it suffices then to show that the sequence (u_n) converges to 0 in measure.

Using the fact that (u_n) is supported in a fixed compact subset of $[0, T] \times \mathbb{R}^2$, we are led thanks to Rellich's theorem and Tchebychev's inequality to prove that the sequence (u_n) converges weakly to 0 in $H^1([0, T] \times \mathbb{R}^2)$. Indeed, assume that the sequence (u_n) converges weakly to 0 in $H^1([0, T] \times \mathbb{R}^2)$, then by Rellich's theorem (u_n) converges strongly to 0 in $L^2([0, T] \times \mathbb{R}^2)$. The Tchebychev's inequality

$$(62) \quad \epsilon^2 \left| \{(t, x) \in [0, T] \times \mathbb{R}^2, \quad |u_n(t, x)| \geq \epsilon\} \right| \leq \|u_n\|_{L^2}^2$$

implies the desired result.

Let u be a weak limit of a subsequence (u_n) . By virtue of Rellich's theorem and Tchebychev's inequality (62), the sequence (u_n) converges to u in measure. This leads to the convergence in measure of the sequence $f(u_n)$ to $f(u)$ under the continuity of the function f . Combining this information with the fact that $(f(u_n))$ is bounded in some L^q with $q > 1$ and is uniformly compactly supported, we infer by Proposition 3.12 that the convergence is also distributional and u is a solution of the nonlinear Klein-Gordon equation (45). Taking advantage of Lemma 3.14, the compactness of the support and Estimate (61), we deduce that $f(u) \in L^1([0, T], L^2(\mathbb{R}^2))$. This allows to apply energy method, and shows that the energy of u at time t equals the energy of the Cauchy data at $t = 0$, which is 0. Hence $u \equiv 0$ and the proof is complete.

3.3. Critical case. Our purpose here is to prove Theorem 3.5. Let $T > 0$ and assume that

$$(63) \quad L := \limsup_{n \rightarrow \infty} \|v_n\|_{L^\infty([0, T]; \mathcal{L})} < \frac{1}{\sqrt{4\pi}}.$$

As it is mentioned above, $w_n = u_n - v_n$, is the solution of the nonlinear wave equation

$$\square w_n + w_n = -f(u_n)$$

with null Cauchy data.

Under energy estimate, we have

$$\|w_n\|_T \leq C \|f(u_n)\|_{L^1([0, T], L^2(\mathbb{R}^2))},$$

where $\|w_n\|_T^2 \stackrel{\text{def}}{=} \sup_{t \in [0, T]} E_c(w_n, t)$. It suffices then to prove that

$$\|f(u_n)\|_{L^1([0, T], L^2(\mathbb{R}^2))} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

The idea here is to split $f(u_n)$ as follows applying Taylor's formula

$$f(u_n) = f(v_n + w_n) = f(v_n) + f'(v_n) w_n + \frac{1}{2} f''(v_n + \theta_n w_n) w_n^2,$$

for some $0 \leq \theta_n \leq 1$. The Strichartz inequality (52) yields (with $I = [0, T]$)

$$(64) \quad \begin{aligned} \|w_n\|_{\text{ST}(I)} &\lesssim \|f(v_n)\|_{L^1([0, T]; L^2(\mathbb{R}^2))} + \|f'(v_n) w_n\|_{L^1([0, T]; L^2(\mathbb{R}^2))} \\ &+ \|f''(v_n + \theta_n w_n) w_n^2\|_{L^1([0, T]; L^2(\mathbb{R}^2))} \\ &\lesssim I_n + J_n + K_n. \end{aligned}$$

The term I_n is the easiest term to treat. Indeed, by Assumption (63) we have

$$(65) \quad \|v_n\|_{L^\infty([0, T]; \mathcal{L})} \leq \frac{1}{\sqrt{4\pi(1 + \varepsilon)}},$$

for some $\varepsilon > 0$ and n large enough. This leads by similar arguments to the ones used in the proof of the subcritical case

$$\|f(v_n)\|_{L^{2+\eta}(\mathbb{R}^2)}^{2+\eta} \leq C e^{4\pi(1-\eta)\|v_n\|_{L^\infty}^2} \int_{\mathbb{R}^2} (e^{4\pi(1+3\eta)v_n^2} - 1) dx.$$

In view of (65) and the Logarithmic inequality, we obtain for $0 < \eta < \frac{\varepsilon}{4}$ and n large enough

$$\|f(v_n)\|_{L^{1+\eta}([0, T], L^{2+\eta}(\mathbb{R}^2))}^{1+\eta} \leq C(\eta, T) (T^{\frac{1}{4}} + \|v_n\|_{L^4([0, T], \mathcal{L}^{1/4})})^\theta,$$

with $\theta = \frac{4\pi\lambda(1-\eta^2)}{2+\eta}$ and $0 < \lambda - \frac{2}{\pi} \ll 1$. It follows by Strichartz estimate that $(f(v_n))$ is bounded in $L^{1+\eta}([0, T]; L^{2+\eta}(\mathbb{R}^2))$.

Since v_n solves the linear Klein-Gordon equation with Cauchy data weakly convergent to 0 in $H^1 \times L^2$, we deduce that (v_n) converges weakly to 0 in $H^1([0, T] \times \mathbb{R}^2)$. This implies that $f(v_n)$ converges to 0 in measure. This finally leads, using Proposition 3.13, the fixed support property and interpolation argument, to the convergence of the sequence $(f(v_n))$ to 0 in $L^1([0, T]; L^2(\mathbb{R}^2))$.

Concerning the second term J_n , we will show that

$$(66) \quad J_n \leq \varepsilon_n \|w_n\|_{\text{ST}(I)},$$

where $\varepsilon_n \rightarrow 0$.

Using Hölder inequality, we infer that

$$J_n = \|f'(v_n) w_n\|_{L^1([0, T]; L^2(\mathbb{R}^2))} \leq \|w_n\|_{L^{1+\frac{1}{\eta}}([0, T]; L^{2+\frac{2}{\eta}}(\mathbb{R}^2))} \|f'(v_n)\|_{L^{1+\eta}([0, T]; L^{2+2\eta}(\mathbb{R}^2))}$$

Arguing exactly in the same manner as for I_n , we prove that for $\eta \leq \eta_0$ small enough the sequence $(f'(v_n))$ is bounded in $L^{1+\eta}([0, T]; L^{2(1+\eta)}(\mathbb{R}^2))$ and converges to 0 in measure which ensures its convergence to 0 in $L^1([0, T]; L^2(\mathbb{R}^2))$. Hence the sequence $(f'(v_n))$ converges to 0 in $L^{1+\eta}([0, T]; L^{2+2\eta}(\mathbb{R}^2))$, for $\eta < \eta_0$, by interpolation argument. This completes the proof of (66) under the Strichartz estimate (53).

For the last (more difficult) term we will establish that

$$(67) \quad K_n \leq \varepsilon_n \|w_n\|_{\text{ST}(I)}^2, \quad \varepsilon_n \rightarrow 0,$$

provided that

$$(68) \quad \limsup_{n \rightarrow \infty} \|w_n\|_{L^\infty([0,T];H^1)} \leq \frac{1 - L\sqrt{4\pi}}{2}.$$

By Hölder inequality, Strichartz estimate and convexity argument, we infer that

$$\begin{aligned} K_n &\leq \|w_n^2\|_{L^{1+\frac{1}{\eta}}([0,T];L^{2+\frac{2}{\eta}}(\mathbb{R}^2))} \|f''(v_n + \theta_n w_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta}(\mathbb{R}^2))} \\ &\leq \|w_n\|_{\text{ST}(T)}^2 \left(\|f''(v_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta}(\mathbb{R}^2))} + \|f''(u_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta}(\mathbb{R}^2))} \right). \end{aligned}$$

According to the previous step, we are then led to prove that for η small enough

$$(69) \quad \|f''(u_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta}(\mathbb{R}^2))} \rightarrow 0.$$

Arguing exactly as in the subcritical case, it suffices to establish that the sequence $(f''(u_n))$ is bounded in $L^{1+\eta_0}([0,T];L^{2+2\eta_0}(\mathbb{R}^2))$ for some $\eta_0 > 0$. Let us first point out that the assumption (68) implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|u_n\|_{L^\infty([0,T];\mathcal{L})} &\leq \limsup_{n \rightarrow \infty} \|v_n\|_{L^\infty([0,T];\mathcal{L})} + \limsup_{n \rightarrow \infty} \|w_n\|_{L^\infty([0,T];\mathcal{L})} \\ &\leq L + \frac{1}{\sqrt{4\pi}} \|w_n\|_{L^\infty([0,T];H^1)} \\ &\leq \frac{1}{2} \left(L + \frac{1}{\sqrt{4\pi}} \right) < \frac{1}{\sqrt{4\pi}}. \end{aligned}$$

This ensures thanks to Remark 3.15 the boundedness of the sequence (u_n) in $L^4([0,T],\mathcal{C}^{1/4})$ which leads to (69) in a similar way than above. Now we are in position to end of the proof of Theorem 3.5. According to (66)- (67), we can rewrite (65) as follows

$$(70) \quad \mathbf{X}_n(T) \lesssim I_n + \varepsilon_n \mathbf{X}_n(T)^2,$$

where $\mathbf{X}_n(T) := \|w_n\|_{\text{ST}([0,T])}$. In view of Lemma 3.9, we deduce that

$$\mathbf{X}_n(T) \lesssim \varepsilon_n.$$

This leads to the desired result under (68). To remove the assumption (68), we use classical arguments. More precisely, let us set

$$(71) \quad T^* := \sup \left\{ 0 \leq t \leq T; \quad \limsup_{n \rightarrow \infty} \|w_n\|_{L^\infty([0,t];H^1)} \leq \nu \right\},$$

where $\nu := \frac{1-L\sqrt{4\pi}}{2}$. Since $w_n(0) = 0$, we have $T^* > 0$. Assume that $T^* < T$ and apply the same arguments as above, we deduce that $\mathbf{X}_n(T^*) \rightarrow 0$. By continuity this implies that $\limsup_{n \rightarrow \infty} \|w_n\|_{L^\infty([0,T^*+\epsilon];H^1)} \leq \nu$ for some ϵ small enough. Obviously, this contradicts the definition of T^* and hence $T^* = T$.

4. APPENDIX

4.1. Appendix A: Some known results on Sobolev embedding.

Lemma 4.1. *$H^1(\mathbb{R}^2)$ is embedded into $L^p(\mathbb{R}^2)$ for all $2 \leq p < \infty$ but not in $L^\infty(\mathbb{R}^2)$.*

Proof of Lemma 4.1. Using Littlewood-Paley decomposition and Bernstein inequalities (see for instance [11]), we infer that

$$\begin{aligned} \|v\|_{L^p} &\leq \sum_{j \geq -1} \|\Delta_j v\|_{L^p}, \\ &\leq C \sum_{j \geq -1} 2^{-\frac{2j}{p}} 2^j \|\Delta_j v\|_{L^2}. \end{aligned}$$

Taking advantage of Schwartz inequality, we deduce that

$$\|v\|_{L^p} \leq C \left(\sum_{j \geq -1} 2^{-\frac{4j}{p}} \right)^{\frac{1}{2}} \|v\|_{H^1} \leq C_p \|v\|_{H^1},$$

which achieves the proof of the embedding for $2 \leq p < \infty$. However, $H^1(\mathbb{R}^2)$ is not included in $L^\infty(\mathbb{R}^2)$. To be convenience, it suffices to consider the function u defined by

$$u(x) = \varphi(x) \log(-\log|x|)$$

for some smooth function φ supported in $B(0, 1)$ with value 1 near 0. \square

It will be useful to notice, that in the radial case, we have the following estimate which implies the control of the L^∞ -norm far away from the origin.

Lemma 4.2. *Let $u \in H_{rad}^1(\mathbb{R}^2)$ and $1 \leq p < \infty$. Then*

$$|u(x)| \leq \frac{C_p}{r^{\frac{2}{2+p}}} \|u\|_{L^p}^{\frac{p}{p+2}} \|\nabla u\|_{L^2}^{\frac{2}{p+2}},$$

with $r = |x|$. In particular

$$(72) \quad |u(x)| \leq \frac{C_2}{r^{\frac{1}{2}}} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \leq \frac{C_2}{r^{\frac{1}{2}}} \|u\|_{H^1}.$$

Proof of Lemma 4.2. By density, it suffices to consider smooth compactly supported functions. Let us then consider $u(x) = \varphi(r)$, with $\varphi \in \mathcal{D}([0, \infty[)$. Obviously, we have

$$\varphi(r)^{\frac{p}{2}+1} = -\frac{p+2}{2} \int_r^\infty \varphi'(s) \varphi^{\frac{p}{2}}(s) ds.$$

Hence

$$\begin{aligned} |\varphi(r)|^{\frac{p}{2}+1} &\leq \frac{p+2}{2r} \int_r^\infty |\varphi'(s)| |\varphi(s)|^{\frac{p}{2}} s ds, \\ &\leq \frac{p+2}{2r} \|\nabla u\|_{L^2} \|u\|_{L^p}^{\frac{p}{2}}. \end{aligned}$$

This achieves the proof of the lemma. \square

Remark 4.3. *In the general case, the embedding of $H^1(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ is not compact. This observation can be illustrated by the following example: $u_n(x) = \varphi(x + x_n)$ with $0 \neq \varphi \in \mathcal{D}$ and $|x_n| \rightarrow \infty$. However, by virtue of Rellich-Kondrachov's theorem, this embedding is compact in the case of $H_K^1(\mathbb{R}^2)$ the subset of functions of $H^1(\mathbb{R}^2)$ supported in the compact K . Moreover, in the radial case, the following compactness result holds.*

Lemma 4.4. *Let $2 < p < \infty$. The embedding $H_{rad}^1(\mathbb{R}^2)$ in $L^p(\mathbb{R}^2)$ is compact.*

Proof of Lemma 4.4. The proof is quite standard and can be found in many references (see for example [5, 24, 45]). We sketch it here for the sake of completeness. (u_n) being a sequence in $H_{rad}^1(\mathbb{R}^2)$ which converges weakly to $u \in H_{rad}^1(\mathbb{R}^2)$, let us set $v_n := u_n - u$. The problem is then reduced to the proof of the fact that $\|v_n\|_{L^p}$ tends to zero. On the one hand, using the above Lemma, we get for any $R > 0$,

$$\int_{|x|>R} |v_n(x)|^p dx = \int_{|x|>R} |v_n(x)|^{p-2} |v_n(x)|^2 dx \leq C R^{-\frac{p-2}{2}}.$$

On the other hand, we know by Rellich-Kondrachov's theorem that the injection $H^1(|x| \leq R)$ into $L^p(|x| \leq R)$ is compact. This ends the proof. \square

Remark 4.5. $H_{rad}^1(\mathbb{R}^2)$ is not compactly embedded in $L^2(\mathbb{R}^2)$. To see this, it suffices to consider the family $u_n(x) = \frac{1}{\alpha_n} e^{-|\frac{x}{\alpha_n}|^2}$ where (α_n) is a sequence of nonnegative real numbers tending to infinity. One can easily show that (u_n) is bounded in H^1 but cannot have a subsequence converging strongly in L^2 .

4.2. Appendix B: Some additional properties on Orlicz spaces. Here we recall some well known properties of Orlicz spaces. For a complete presentation and more details, we refer the reader to [39]. The first result that we state here deals with the connection between Orlicz spaces and Lebesgue spaces L^1 and L^∞ .

Proposition 4.6. *We have*

- a) $(L^\phi, \|\cdot\|_{L^\phi})$ is a Banach space.
- b) $L^1 \cap L^\infty \subset L^\phi \subset L^1 + L^\infty$.
- c) If $T : L^1 \rightarrow L^1$ with norm M_1 and $T : L^\infty \rightarrow L^\infty$ with norm M_∞ , then $T : L^\phi \rightarrow L^\phi$ with norm $\leq C(\phi) \sup(M_1, M_\infty)$.

The following result concerns the behavior of Orlicz norm against convergence of sequences.

Lemma 4.7. *We have the following properties*

- a) *Lower semi-continuity:*

$$u_n \rightarrow u \quad \text{a.e.} \quad \implies \quad \|u\|_{\mathcal{L}} \leq \liminf \|u_n\|_{\mathcal{L}}.$$

- b) *Monotonicity:*

$$|u_1| \leq |u_2| \quad \text{a.e.} \quad \implies \quad \|u_1\|_{\mathcal{L}} \leq \|u_2\|_{\mathcal{L}}.$$

- c) *Strong Fatou property:*

$$0 \leq u_n \nearrow u \quad \text{a.e.} \quad \implies \quad \|u_n\|_{\mathcal{L}} \nearrow \|u\|_{\mathcal{L}}.$$

Let us now stress that besides the topology induced by its norm, the Orlicz space \mathcal{L} is equipped with one other topology, namely the mean topology. More precisely,

Definition 4.8. *A sequence (u_n) in \mathcal{L} is said to be mean (or modular) convergent to $u \in \mathcal{L}$, if*

$$\int \phi(u_n - u) dx \longrightarrow 0.$$

It is said strongly (or norm) convergent to $u \in \mathcal{L}$, if

$$\|u_n - u\|_{\mathcal{L}} \longrightarrow 0.$$

Clearly there is no equivalence between these convergence notions. Precisely, the strong convergence implies the modular convergence but the converse is false as shown by taking the Lions's functions f_α .

To end this subsection, let us mention that our Orlicz space \mathcal{L} behaves like L^2 for functions in $H^1 \cap L^\infty$.

Proposition 4.9. *For every $\mu > 0$ and every function u in $H^1 \cap L^\infty$, we have*

$$(73) \quad \frac{1}{\sqrt{\kappa}} \|u\|_{L^2} \leq \|u\|_{\mathcal{L}} \leq \mu + \frac{e^{\frac{\|u\|_{L^\infty}^2}{2\mu^2}}}{\sqrt{\kappa}} \|u\|_{L^2}.$$

Proof. The left hand side of (73) is obvious. The second inequality follows immediately from the following simple observation

$$\left\{ \lambda \geq \mu + \frac{e^{\frac{\|u\|_{L^\infty}^2}{2\mu^2}}}{\sqrt{\kappa}} \|u\|_{L^2} \right\} \subset \left\{ \lambda > 0; \quad \int \left(e^{\frac{|u(x)|^2}{\lambda^2}} - 1 \right) dx \leq \kappa \right\}.$$

Indeed, assuming $\lambda \geq \mu + \frac{e^{\frac{\|u\|_{L^\infty}^2}{2\mu^2}}}{\sqrt{\kappa}}$, we get

$$\begin{aligned} \int \left(e^{\frac{|u(x)|^2}{\lambda^2}} - 1 \right) dx &\leq \int \frac{|u(x)|^2}{\lambda^2} e^{\frac{|u(x)|^2}{\lambda^2}} dx \\ &\leq \frac{e^{\frac{\|u\|_{L^\infty}^2}{\mu^2}}}{\lambda^2} \|u\|_{L^2}^2 \\ &\leq \kappa. \end{aligned}$$

□

4.3. Appendix C: BMO and \mathcal{L} . Now, we shall discuss the connection between the Orlicz space \mathcal{L} and BMO. At first, let us recall the following well known embeddings

$$H^1 \hookrightarrow \text{BMO} \cap L^2, \quad L^\infty \hookrightarrow \text{BMO} \hookrightarrow \text{B}_{\infty, \infty}^0, \quad H^1 \hookrightarrow \mathcal{L} \hookrightarrow \bigcap_{2 \leq p < \infty} L^p.$$

However, there is no comparison between \mathcal{L} and BMO in the following sense.

Proposition 4.10. *We have*

$$\mathcal{L} \not\hookrightarrow \text{BMO} \cap L^2 \quad \text{and} \quad \text{BMO} \cap L^2 \not\hookrightarrow \mathcal{L}.$$

Proof of Proposition 4.10. Let us consider $g_\alpha(r, \theta) = f_\alpha(r) e^{i\theta}$ and $B_\alpha = B(0, e^{-\frac{\alpha}{2}})$. Clearly we have

$$\int_{B_\alpha} g_\alpha = 0.$$

Moreover

$$\begin{aligned} \frac{1}{|B_\alpha|} \int_{B_\alpha} |g_\alpha| &= 2e^\alpha \int_0^{e^{-\alpha}} \sqrt{\frac{\alpha}{2\pi}} r \, dr + \int_{e^{-\alpha}}^{e^{-\frac{\alpha}{2}}} -\frac{\log r}{\sqrt{2\pi\alpha}} r \, dr \\ &= \frac{\sqrt{\alpha}}{2\sqrt{2\pi}} + \frac{1 - e^{-\alpha}}{2\sqrt{2\pi\alpha}}. \end{aligned}$$

Hence

$$\|g_\alpha\|_{\text{BMO}} \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty.$$

Since $\|g_\alpha\|_{\mathcal{L}} = \|f_\alpha\|_{\mathcal{L}} \rightarrow \frac{1}{\sqrt{4\pi}}$, we deduce that

$$\mathcal{L} \not\hookrightarrow \text{BMO} \cap L^2.$$

To show that $\text{BMO} \cap L^2$ is not embedded in \mathcal{L} , we shall use the following **sharp** inequality (see [26])

$$(74) \quad \|u\|_{L^q} \leq C q \|u\|_{\text{BMO} \cap L^2}, \quad q \geq 2,$$

together with the fact that (for $u \neq 0$),

$$(75) \quad \int_{\mathbb{R}^2} \left(e^{\frac{|u(x)|^2}{\|u\|_{\mathcal{L}}^2}} - 1 \right) dx \leq \kappa.$$

Indeed, let us suppose that $\text{BMO} \cap L^2$ is embedded in \mathcal{L} . Then, for any integer $q \geq 1$,

$$\|u\|_{L^{2q}} \leq \kappa^{1/2q} (q!)^{1/2q} \|u\|_{\mathcal{L}} \leq C \kappa^{1/2q} (q!)^{1/2q} \|u\|_{\text{BMO} \cap L^2}$$

which contradicts (74) since

$$(q!)^{1/2q} \sim e^{-1/2} \sqrt{q},$$

where \sim is used to indicate that the ratio of the two sides goes to 1 as q goes to ∞ . \square

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