J. Differential Equations 248 (2010) 328–341

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Remarks on the blowup criteria for Oldroyd models

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ARTICLE INFO

Article history:
Received 8 April 2009
Revised 16 June 2009
Available online 24 July 2009

Keywords:
Losing *a priori* estimate
Blowup criteria
Oldroyd model

ABSTRACT

We provide a new method to prove and improve the Chemin–Masmoudi criterion for viscoelastic systems of Oldroyd type in [J.Y. Chemin, N. Masmoudi, About lifespan of regular solutions of equations related to viscoelastic fluids, SIAM J. Math. Anal. 33 (1) (2001) 84–112] in two space dimensions. Our method is much easier than the one based on the well-known *losing a priori estimate* and is expected to be easily adopted to other problems involving the losing *a priori* estimate.

1. Introduction

In this paper, we are going to study the non-blowup criteria of solutions of a type of incompressible non-Newtonian fluid flows described by the Oldroyd-B model in the whole 2-D space:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \nu \Delta v + \mu_1 \nabla \cdot \tau, \\ \partial_t \tau + v \cdot \nabla \tau + a\tau = Q(\tau, \nabla v) + \mu_2 D(v), \\ \nabla \cdot v = 0, \end{cases} \tag{1.1}$$

where v is the velocity field, τ is the non-Newtonian part of the stress tensor and p is the pressure. The constants ν (the viscosity of the fluid), a (the reciprocal of the relaxation time), μ_1 and μ_2

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.07.011

(determined by the dynamical viscosity of the fluid, the retardation time and a) are assumed to be non-negative. The bilinear term Q has the following form:

$$Q(\tau, \nabla v) = W(v)\tau - \tau W(v) + b\big(D(v)\tau + \tau D(v)\big). \tag{1.2}$$

Here $b \in [-1, 1]$ is a constant, $D(v) = \frac{\nabla v + (\nabla v)^t}{2}$ is the deformation tensor and $W(v) = \frac{\nabla v - (\nabla v)^t}{2}$ is the vorticity tensor. Fluids of this type have both elastic properties and viscous properties. More discussions and the derivation of Oldroyd-B model (1.1) can be found in Oldroyd [22] or Chemin and Masmoudi [5].

There has been a lot of work on the existence theory of Oldroyd model [5,8–10,14,17]. In particular, the following theorem is established by Chemin and Masmoudi in [5]:

Theorem *(Chemin and Masmoudi). In two space dimensions, the solutions to the Oldroyd model* (1.1) *with smooth initial data do not develop singularities for $t \leqslant T$ provided that*

$$\int_0^T \big\|\tau(t,\cdot)\big\|_{L^\infty} + |b|\big\|\tau(t,\cdot)\big\|_{L^2}^2 \, dt < \infty. \tag{1.3}$$

To establish the blowup criterion (1.3), the authors in [5] use a losing *a priori* estimate for solutions of transport equations which was developed by Bahouri and Chemin [1] and used later on by a lot of authors (for example, see [5–7,16,18,19,21] and the references therein). Our purpose of this paper is to provide a simple method which avoids using the complicated losing *a priori* estimate and to improve the blowup criterion (1.3) for Oldroyd model (1.1) established by Chemin and Masmoudi [5]. To best illustrate our ideas and for simplicity, we will take $a = 0$ and $\nu = \mu_1 = \mu_2 = b = 1$ throughout this paper. More precisely, we study the following system

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \Delta v + \nabla \cdot \tau, \\ \partial_t \tau + v \cdot \nabla \tau = \nabla v \tau + \tau (\nabla v)^t + D(v), \\ \nabla \cdot v = 0. \end{cases} \tag{1.4}$$

We point out here that the results in this paper are obviously true for general constants $a, \mu_1, \mu_2 \geqslant 0$, $\nu, b > 0$ from our proofs.

Our main result concerning system (1.4) is:

Theorem 1.1. *Assume that (v, τ) is a local smooth solution to the Oldroyd model* (1.4) *on* $[0, T)$ *and* $\|v(0,\cdot)\|_{L^2 \cap \dot C^{1+\alpha}(\mathbb{R}^2)} + \|\tau(0,\cdot)\|_{L^1 \cap \dot C^{\alpha}(\mathbb{R}^2)} < \infty$ *for some $\alpha \in (0, 1)$. Then one has*

$$\big\|v(t,\cdot)\big\|_{\dot C^{1+\alpha}} + \big\|\tau(t,\cdot)\big\|_{\dot C^{\alpha}} < \infty$$

for all $0 \leqslant t \leqslant T$ provided that

$$\big\|\tau(t,\cdot)\big\|_{L^1_T(\mathrm{BMO})} < \infty \quad \text{and} \quad \|\tau\|_{L^\infty_T(L^1)} < \infty. \tag{1.5}$$

Remark 1.2. This result is in the spirit of the Beale–Kato–Majda [2] non-blowup criterion for 3-D Euler equations. There were many subsequent results improving the criterion, see for instance [12,13,23]. In particular our result still holds if we replace BMO with the Besov space $B^0_{\infty,\infty}$ used in H. Kozono, T. Ogawa, and Y. Taniuchi [12] or if we replace the condition by the one introduced in Planchon [23]. In other words, the first condition in (1.5) in the above theorem can be weakened to

$$\int_0^T \|\tau\|_{B^0_{\infty,\infty}} = \int_0^T \sup_q \|\Delta_q \tau(t,\cdot)\|_{L^\infty}\, dt < \infty,$$

or to

$$\lim_{\delta\to 0} \sup_q \int_{T-\delta}^T \|\Delta_q \tau(t,\cdot)\|_{L^\infty}\, dt < \epsilon$$

for some sufficiently small $\epsilon > 0$. The second condition in (1.5) can be replaced by

$$\|\tau\|_{L^2_T(L^2)} < \infty,$$

which was used in [5].

Remark 1.3. It is easy to check that smooth solutions to (1.4) enjoy the following energy law:

$$\int_{\mathbb{R}^2} |v(t,\cdot)|^2 + \operatorname{tr}\tau(t,\cdot)\, dx + \int_0^t \int_{\mathbb{R}^2} |\nabla v|^2\, dx\, ds = \int_{\mathbb{R}^2} |v(0,\cdot)|^2 + \operatorname{tr}\tau(0,\cdot)\, dx, \tag{1.6}$$

which means that

$$v \in L^\infty_T(L^2) \cap L^2_T(\dot{H}^1) \tag{1.7}$$

for all $T > 0$ under the second condition of (1.5). The *a priori* estimate (1.7) will be important to apply Lemma 3.1.

Finally, it is well known that if $A = 2\tau + I$ is a positive definite symmetric matrix at $t = 0$ (which is actually the physical case), then this property is conserved for later times. Indeed, A satisfies the equation

$$\partial_t A + v \cdot \nabla A = \nabla v A + A(\nabla v)^t.$$

Also, if at $t = 0$, we have $\det(A(0)) > 1$ and A is positive definite, then this will also hold for later times (see [11]). In particular this implies that $\operatorname{tr}(\tau) > 0$ (or one has $-1 < \operatorname{tr}(\tau) \leqslant 0$, which contradicts with $\det(A) > 1$). Hence, we have the following corollary where we also use the improved criterion of Planchon.

Corollary 1.4. *There exists an* $\epsilon > 0$, *such that if* (v,τ) *is a local smooth solution to the Oldroyd model* (1.4) *on* $[0,T)$, $\|v(0,\cdot)\|_{L^2\cap\dot{C}^{1+\alpha}(\mathbb{R}^2)} + \|\tau(0,\cdot)\|_{L^1\cap\dot{C}^\alpha(\mathbb{R}^2)} < \infty$ *for some* $\alpha \in (0,1)$ *and that* $\det(I + 2\tau(0)) > 1$, $A = I + 2\tau(0)$ *is positive definite symmetric, then one has*

$$\|v(t,\cdot)\|_{\dot{C}^{1+\alpha}} + \|\tau(t,\cdot)\|_{\dot{C}^\alpha} < \infty$$

for all $0 \leqslant t \leqslant T$ *provided that*

$$\lim_{\delta\to 0} \sup_q \int_{T-\delta}^T \|\Delta_q \tau(t,\cdot)\|_{L^\infty}\, dt < \epsilon. \tag{1.8}$$

Our proof is based on careful Hölder estimates of heat and transport equations and the standard Littlewood–Paley theory, which is much easier than the extensively used losing *a priori* estimates (for example, see [1,5–7,16]). In fact, the main innovation of this paper is that our analysis may be viewed as a replacement of the losing *a priori* estimate. Our method is expected to be easily adopted to other problems via the losing *a priori* estimate. Moreover, our criterion slightly improves the one established by Chemin and Masmoudi (see [5]).

Finally, let us make a remark on MHD:

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = \nu\Delta v + \nabla\cdot(H\times H), \\ \partial_t H + v\cdot\nabla H = H\cdot\nabla v, \\ \nabla\cdot v = \nabla\cdot H = 0, \end{cases} \tag{1.9}$$

where H denotes the magnetic field. A direct corollary of Theorem 1.1 for MHD (1.9) is the following:

Corollary 1.5. *Assume that* (v, H) *is a local smooth solution to MHD* (1.9) *on* $[0, T)$ *and* $\|v(0,\cdot)\|_{L^2\cap\dot C^{1+\alpha}} + \|H(0,\cdot)\|_{L^2\cap\dot C^{\alpha}} < \infty$ *for some* $\alpha \in (0,1)$. *Then one has*

$$\|v(t,\cdot)\|_{\dot C^{1+\alpha}} + \|H(t,\cdot)\|_{\dot C^{\alpha}} < \infty$$

for all $0 \leqslant t \leqslant T$ *provided that*

$$\int_0^T \|(H\times H)(t,\cdot)\|_{\mathrm{BMO}}\,dt < \infty. \tag{1.10}$$

The proof of this corollary is given in Section 4. Unfortunately, at present we are not able to improve (1.10) as

$$\int_0^T \|H(t,\cdot)\|_{\mathrm{BMO}}^2\,dt < \infty,$$

and this is still an open problem.

The paper is organized as follows: Section 2 is devoted to recalling some basic properties of Littlewood–Paley theory and proving two interpolation inequalities. The proof of Theorem 1.1 is given in Section 3. In the last section we sketch the proof of Corollary 1.5.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^2)$, its Fourier transform $\mathcal{F}f = \hat f$ (inverse Fourier transform $\mathcal{F}^{-1}g = \check g$, respectively) is defined by $\hat f(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$ ($\check g(x) = \int e^{ix\cdot\xi} g(\xi)\,d\xi$, respectively). Now let us recall the Littlewood–Paley decomposition (see [3,4]). Choose two non-negative radial functions $\psi, \phi \in \mathcal{S}(\mathbb{R}^2)$, supported respectively in $B = \{\xi \in \mathbb{R}^2\colon |\xi| \leqslant \frac{4}{3}\}$ and $C = \{\xi \in \mathbb{R}^2\colon \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\psi(\xi) + \sum_{j\geqslant 0}\phi\left(\frac{\xi}{2^j}\right) = 1 \quad \text{for } \xi \in R^2, \qquad \sum_{-\infty\leqslant j\leqslant\infty}\phi\left(\frac{\xi}{2^j}\right) = 1 \quad \text{for } \xi \in R^2\setminus\{0\}.$$

The frequency localization operator is defined by

$$\Delta_q f = \int_{\mathbb{R}^2} \check{\phi}(y) f\big(x - 2^{-q}y\big)\,dy, \qquad S_q f = \int_{\mathbb{R}^2} \check{\psi}(y) f\big(x - 2^{-q}y\big)\,dy. \tag{2.1}$$

The following lemma is well known (for example, see [4]).

Lemma 2.1. *For $s \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$ and integer q, one has*

$$\begin{cases} c2^{qs}\|\Delta_q f\|_{L^p} \leqslant \big\|\nabla^s \Delta_q f\big\|_{L^p} \leqslant C2^{qs}\|\Delta_q f\|_{L^p}, \\ \big\||\nabla|^s S_q f\big\|_{L^p} \leqslant C2^{qs}\|f\|_{L^p}, \\ ce^{-C2^{2q}t}\|\Delta_q f\|_{L^\infty} \leqslant \big\|e^{t\Delta}\Delta_q f\big\|_{L^\infty} \leqslant Ce^{-c2^{2q}t}\|\Delta_q f\|_{L^\infty}. \end{cases} \tag{2.2}$$

Here C and c are positive constants independent of s, p and q.

We also need the following lemma (see also [20,21] where similar estimates were used).

Lemma 2.2. *Assume that $\beta > 0$. Then there exists a positive constant $C > 0$ such that*

$$\begin{cases} \|f\|_{L^\infty} \leqslant C\big(1 + \|f\|_{L^2} + \|f\|_{\mathrm{BMO}} \ln\big(e + \|f\|_{\dot{C}^\beta}\big)\big), \\ \displaystyle\int_0^T \big\|\nabla g(s,\cdot)\big\|_{L^\infty}\,ds \leqslant C\Bigg(1 + \int_0^T \big\|g(s,\cdot)\big\|_{L^2}\,ds \\ \displaystyle\qquad + \sup_q \int_0^T \big\|\Delta_q \nabla g(s,\cdot)\big\|_{L^\infty}\,ds \ln\Bigg(e + \int_0^T \big\|\nabla g(s,\cdot)\big\|_{\dot{C}^\beta}\,ds\Bigg)\Bigg). \end{cases} \tag{2.3}$$

Proof. The first inequality is well known. For example, see [2,13,15]. To prove the second inequality, we use the Littlewood–Paley theory to compute that

$$\begin{aligned} \int_0^T \big\|\nabla g(s,\cdot)\big\|_{L^\infty}\,ds &\leqslant C\int_0^T \Big\|\sum_{q\leqslant 0}\nabla\Delta_q g(s,\cdot)\Big\|_{L^\infty}\,ds + CN\max_{1\leqslant q\leqslant N}\int_0^T \big\|\Delta_q\nabla g(s,\cdot)\big\|_{L^\infty}\,ds \\ &\quad + \int_0^T \sum_{q\geqslant N+1} 2^{-\beta q}2^{\beta q}\big\|\Delta_q\nabla g(s,\cdot)\big\|_{L^\infty}\,ds \\ &\leqslant C\Bigg(\int_0^T \big\|g(s,\cdot)\big\|_{L^2}\,ds + \sup_{1\leqslant q\leqslant N}\int_0^T N\big\|\Delta_q\nabla g(s,\cdot)\big\|_{L^\infty}\,ds \\ &\quad + 2^{-\beta N}\int_0^T \big\|\nabla g(s,\cdot)\big\|_{\dot{C}^\beta}\,ds\Bigg). \end{aligned}$$

Then the second inequality in Lemma 2.2 follows by choosing

$$N = \frac{1}{\beta}\log_2\Bigg(e + \int_0^T \big\|\nabla g(s,\cdot)\big\|_{\dot{C}^\beta}\Bigg) \leqslant C\ln\Bigg(e + \int_0^T \big\|\nabla g(s,\cdot)\big\|_{\dot{C}^\beta}\,ds\Bigg). \qquad \square$$

3. Blowup criteria for Oldroyd-B model

This section is devoted to establishing the blowup criterion for the Oldroyd-B model (1.1) and proving Theorem 1.1. Our analysis is based on careful Hölder estimates of heat and transport equations and the standard Littlewood–Paley theory, which is much easier than the extensively used losing *a priori* estimates (for example, see [1,5–7,16]). Moreover, our criterion slightly improves the one established by Chemin and Masmoudi (see [5]). We divide our proof into two steps. The first step is focused on establishing some *a priori* estimates for 2-D Navier–Stokes equations. Then we establish Hölder estimates for the velocity field v and the stress tensor τ in the second step.

Step 1. The a priori estimates for 2-D Navier–Stokes equations. We need the following lemma which is basically established by Chemin and Masmoudi in [5]. For completeness, the proof will be also sketched here.

Lemma 3.1 *(Chemin–Masmoudi). Let v be a solution of the Navier–Stokes equations with initial data in L^2 and an external force $f \in \widetilde{L}^1_T(C^{-1}) \cap L^2_T(H^{-1})$:*

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \Delta v + f, \\ \nabla \cdot v = 0, \\ v(0,x) = v_0(x). \end{cases} \tag{3.1}$$

Then we have the following a priori estimate:

$$\begin{aligned} \|v\|_{\widetilde{L}^1_T(C^1)} \leqslant C\Bigg(&\sup_q \|\Delta_q v_0\|_{L^2}\big(1 - \exp\{-c2^{2q}T\}\big) + \big(\|v_0\|_{L^2} + \|f\|_{L^2_T(\dot{H}^{-1})}\big)\|\nabla v\|^2_{L^2_T(L^2)} \\ &+ \sup_q \int_0^T \big\|2^{-q}\Delta_q f(s)\big\|_{L^\infty}\, ds\Bigg). \end{aligned} \tag{3.2}$$

Proof. First of all, applying the operator Δ_q to the 2-D Navier–Stokes equations (3.1) and then using Lemma 2.1 and the standard energy estimate, we deduce that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta_q v\|^2_{L^2} + c2^{2q}\|\Delta_q v\|^2_{L^2} &\leqslant \big\|2^q \Delta_q v\big\|_{L^2}\big(\big\|2^{-q}\Delta_q f\big\|_{L^2} + \big\|\Delta_q(v \otimes v)\big\|_{L^2}\big) \\ &\leqslant c2^{2q}\|\Delta_q v\|^2_{L^2} + C\big(\big\|2^{-q}\Delta_q f\big\|^2_{L^2} + \big\|\Delta_q(v \otimes v)\big\|^2_{L^2}\big). \end{aligned}$$

Integrating with respect to time and summing over q, we get

$$\begin{aligned} \sum_q \|\Delta_q v\|^2_{L^\infty_T(L^2)} &\leqslant \|v_0\|^2_{L^2} + C\big(\|f\|^2_{L^2_T(\dot{H}^{-1})} + \|v \otimes v\|^2_{L^2_T(L^2)}\big) \\ &\leqslant \|v_0\|^2_{L^2} + C\big(\|f\|^2_{L^2_T(\dot{H}^{-1})} + \|v\|^2_{L^\infty_T(L^2)}\|\nabla v\|^2_{L^2_T(L^2)}\big), \end{aligned}$$

where we used the standard interpolation inequality $\|v\|^2_{L^4} \leqslant C\|v\|_{L^2}\|\nabla v\|_{L^2}$. Recalling the basic energy estimate

$$\|v\|^2_{L^\infty_T(L^2)} + \|\nabla v\|^2_{L^2_T(L^2)} \leqslant \|v_0\|^2_{L^2} + \|f\|^2_{L^2_T(\dot{H}^{-1})}, \tag{3.3}$$

one has

$$\sum_q \|\Delta_q v\|^2_{L^\infty_T(L^2)} \leqslant C\big(\|v_0\|^2_{L^2} + \|f\|^2_{L^2_T(\dot H^{-1})}\big)\big(1 + \|v_0\|^2_{L^2} + \|f\|^2_{L^2_T(\dot H^{-1})}\big). \tag{3.4}$$

Next, let us apply Δ_q to (3.1) and use Lemma 2.1 to estimate

$$\big\|\Delta_q v(t)\big\|_{L^\infty} \leqslant C\|\Delta_q v_0\|_{L^\infty} e^{-c2^{2q}t} + \int_0^t \big(\big\|\Delta_q f(s)\big\|_{L^\infty} + \big\|\Delta_q \nabla\cdot(v\otimes v)(s)\big\|_{L^\infty}\big) e^{-c2^{2q}(t-s)}\, ds,$$

which yields

$$\begin{aligned}
\|v\|_{\widetilde L^1_T(C^1)} &\leqslant C\sup_q \int_0^T \|\Delta_q v_0\|_{L^\infty} 2^q e^{-c2^{2q}t}\, dt \\
&\quad + C\sup_q \int_0^T \int_0^t \big\|\Delta_q f(s)\big\|_{L^\infty} 2^q e^{-c2^{2q}(t-s)}\, ds\, dt \\
&\quad + C\sup_q \int_0^T \int_0^t \big\|\Delta_q \nabla\cdot(v\otimes v)(s)\big\|_{L^\infty} 2^q e^{-c2^{2q}(t-s)}\, ds\, dt \\
&\leqslant C\sup_q \|\Delta_q v_0\|_{L^2}\big(1 - e^{-c2^{2q}T}\big) \\
&\quad + C\sup_q \int_0^T \big\|\Delta_q(v\otimes v)(s)\big\|_{L^\infty}\, ds + \|f\|_{\widetilde L^1_T(C^{-1})}.
\end{aligned} \tag{3.5}$$

Using the Bony's decomposition, one can write

$$\begin{aligned}
\big\|\Delta_q(v\otimes v)(s)\big\|_{L^\infty} &= \sum_{|p-r|\leqslant 1} \big\|\Delta_q(\Delta_p v\otimes \Delta_r v)(s)\big\|_{L^\infty} + \sum_{p-r\geqslant 2} \big\|\Delta_q(\Delta_p v\otimes \Delta_r v)(s)\big\|_{L^\infty} \\
&\quad + \sum_{r-p\geqslant 2} \big\|\Delta_q(\Delta_p v\otimes \Delta_r v)(s)\big\|_{L^\infty}.
\end{aligned}$$

A straightforward computation gives

$$\begin{aligned}
&\int_0^T \sum_{|p-r|\leqslant 1} \big\|\Delta_q(\Delta_p v\otimes \Delta_r v)(s)\big\|_{L^\infty}\, ds \\
&\quad \leqslant C\int_0^T \sum_{|p-r|\leqslant 1} 2^q \big\|\Delta_q(\Delta_p v\otimes \Delta_r v)(s)\big\|_{L^2}\, ds
\end{aligned}$$

$$\leqslant C\int_0^T \sum_{|p-r|\leqslant 1,\, p\geqslant q-3} 2^{q-\frac{p+r}{2}} \|2^p\Delta_p v\|_{L^\infty}^{\frac12} \|\Delta_r v\|_{L^2}^{\frac12} \|\Delta_p v\|_{L^\infty}^{\frac12} \|2^r\Delta_r v\|_{L^2}^{\frac12}\,ds$$

$$\leqslant C\int_0^T \sum_{|p-r|\leqslant 1,\, p\geqslant q-3} 2^{q-\frac{p+r}{2}} \|2^p\Delta_p v\|_{L^\infty}^{\frac12} \|\Delta_r v\|_{L^2}^{\frac12} \|2^p\Delta_p v\|_{L^2}^{\frac12} \|2^r\Delta_r v\|_{L^2}^{\frac12}\,ds$$

$$\leqslant C\|v\|_{L^\infty_T(L^2)}^{\frac12} \|\nabla v\|_{L^2_T(L^2)} \|v\|_{\widetilde{L}^1_T(C^1)}^{\frac12}.$$

Similarly, one has

$$\int_0^T \Bigg(\sum_{p-r\geqslant 2} \big\|\Delta_q(\Delta_p v\otimes\Delta_r v)(s)\big\|_{L^\infty} + \sum_{r-p\geqslant 2} \big\|\Delta_q(\Delta_p v\otimes\Delta_r v)(s)\big\|_{L^\infty}\Bigg)\,ds$$

$$\leqslant C\int_0^T \sum_{p-r\geqslant 2,\, |p-q|\leqslant 2} \|\Delta_p v\|_{L^\infty}\|\Delta_r v\|_{L^\infty}\,ds$$

$$\leqslant C\int_0^T \sum_{p-r\geqslant 2,\, |p-q|\leqslant 2} \|2^p\Delta_p v\|_{L^\infty}^{\frac12} \|2^p\Delta_p v\|_{L^2}^{\frac12} 2^{r-\frac{p}{2}} \|\Delta_r v\|_{L^2}\,ds$$

$$\leqslant C\|v\|_{L^\infty_T(L^2)}^{\frac12} \|\nabla v\|_{L^2_T(L^2)} \|v\|_{\widetilde{L}^1_T(C^1)}^{\frac12}.$$

Using the above two estimates, one can improve (3.5) as

$$\|v\|_{\widetilde{L}^1_T(C^1)} \leqslant C\Big(\sup_q \|\Delta_q v_0\|_{L^2}\big(1-e^{-c2^{2q}T}\big) + \|v\|_{L^\infty_T(L^2)}\|\nabla v\|_{L^2_T(L^2)}^2 + \|f\|_{\widetilde{L}^1_T(C^{-1})}\Big).$$

Consequently, one can deduce (3.2) from the basic energy estimate (3.3) and the above inequality. □

Now let us assume that $f\in L^1_T(\dot{C}^{-1})\cap L^2_T(H^{-1})$. By Lemma 3.1, it is easy to see that

$$\|v\|_{\widetilde{L}^1_{[t_0,T]}(C^1)} \leqslant C\Bigg(\sup_q \big\|\Delta_q v(t_0)\big\|_{L^2}\big(1-\exp\{-c2^{2q}(T-t_0)\}\big) + \int_{t_0}^T \sup_q \big\|2^{-q}\Delta_q f(s)\big\|_{L^\infty}\,ds + \big(\big\|v(t_0)\big\|_{L^2} + \|f\|_{L^2_{[t_0,T]}(\dot{H}^{-1})}\big)\|\nabla v\|_{L^2_{[t_0,T]}(L^2)}^2\Bigg) \tag{3.6}$$

holds for any $t_0\in[0,T)$. By (3.4), one can choose some q_0 such that

$$\sup_{q>q_0}\|\Delta_q v\|_{L^\infty_{[t_0,T]}(L^2)}^2 \leqslant \frac{\epsilon}{4C}.$$

Furthermore, by the basic energy estimate (3.3), one can choose some $t_1\in[t_0,T)$ such that

$$\begin{aligned}
&\sup_{t_1\leqslant t\leqslant T}\sup_{q\leqslant q_0}\big\|\Delta_q v(t)\big\|_{L^2}\big(1-\exp\big\{-c2^{2q}(T-t)\big\}\big)\\
&\quad\leqslant \sup_{t_1\leqslant t\leqslant T}\big\|v(t)\big\|_{L^2}2c2^{2q_0}(T-t_1)\\
&\quad\leqslant C2^{2q_0}\big(\|v_0\|_{L^2}+\|f\|_{L^2_{[0,T]}(\dot H^{-1})}\big)(T-t_1)\leqslant\frac{\epsilon}{4C}.
\end{aligned}$$

Consequently, one has

$$\sup_{t_1\leqslant t\leqslant T}\sup_{q}\big\|\Delta_q v(t)\big\|_{L^2}\big(1-\exp\big\{-c2^{2q}(T-t)\big\}\big)\leqslant\frac{\epsilon}{2C}. \tag{3.7}$$

On the other hand, it is obvious that one can choose some $t_2\in[t_1,T)$ such that

$$\Big(\sup_{t_2\leqslant t\leqslant T}\big\|v(t)\big\|_{L^2}+\|f\|_{L^2_{[t_2,T]}(\dot H^{-1})}\Big)\|\nabla v\|^2_{L^2_{[t_2,T]}(L^2)}+\int_{t_2}^{T}\sup_q\big\|2^{-q}\Delta_q f(s)\big\|_{L^\infty}\,ds\leqslant\frac{\epsilon}{2C}. \tag{3.8}$$

Combining (3.7) and (3.8) with (3.6), one arrives at

$$\|v\|_{\widetilde L^1_{[t_2,T]}(C^1)}\leqslant\epsilon. \tag{3.9}$$

Step 2. Hölder estimate for v and τ. First of all, by (3.9) and the assumption (1.5), one can choose $t_\star\in[t_2,T)$ such that

$$\|v\|_{\widetilde L^1_{[t_\star,T]}(C^1)}\leqslant\epsilon,\qquad \|\tau\|_{L^1_{[t_\star,T]}(\mathrm{BMO})}\leqslant\epsilon. \tag{3.10}$$

For $0\leqslant t<T$, define

$$A(t)=\sup_{0\leqslant s<t}\big\|v(t,\cdot)\big\|_{\dot C^{1+\alpha}},\qquad B(t)=\sup_{0\leqslant s<t}\big\|\tau(t,\cdot)\big\|_{\dot C^{\alpha}}.$$

We are about to estimate $A(t)$ and $B(t)$ for $0\leqslant t<T$. For this purpose, let us apply Δ_q to the Oldroyd-B system (1.4) to get

$$\begin{cases}\partial_t\Delta_q v-\Delta\Delta_q v+\nabla\Delta_q p=\nabla\cdot\Delta_q(\tau-v\otimes v),\\ \partial_t\Delta_q\tau+v\cdot\nabla\Delta_q\tau=\Delta_q\big(\nabla v\tau+\tau(\nabla v)^t+D(v)\big)+[v\cdot\nabla,\Delta_q]\tau.\end{cases} \tag{3.11}$$

Let us first estimate $\|v(t,\cdot)\|_{\dot C^{1+\alpha}}$. By the first equation in (3.11) and Lemma 2.1, one has

$$\|\Delta_q v\|_{L^\infty}\leqslant Ce^{-c2^{2q}t}\big\|\Delta_q v(0)\big\|_{L^\infty}+\int_0^t e^{-c2^{2q}(t-s)}\big\|\nabla\cdot\Delta_q(\tau-v\otimes v)\big\|_{L^\infty}(s)\,ds. \tag{3.12}$$

Multiplying $2^{q(1+\alpha)}$ to both sides of (3.12), we have

$$\begin{aligned}\|\Delta_q v(t,\cdot)\|_{\dot C^{1+\alpha}} &\leqslant C\|v(0,\cdot)\|_{\dot C^{1+\alpha}} + C\int_0^t 2^{2q}e^{-c2^{2q}(t-s)}\|\Delta_q\tau\|_{\dot C^\alpha}\,ds\\ &\quad + C\int_0^t 2^{\frac{3}{2}q}e^{-c2^{2q}(t-s)}\|(v\otimes v)(s,\cdot)\|_{\dot C^{1/2+\alpha}}\,ds\\ &\leqslant C\big(\|v(0,\cdot)\|_{\dot C^{1+\alpha}} + B(t)\big) + C\left(\int_0^t \|v\|_{L^4}^4\|v\|_{\dot C^{1+\alpha}}^4\,ds\right)^{\frac{1}{4}},\end{aligned}$$

where we have used Hölder inequality and the fact that $\|v\otimes v\|_{\dot C^{1/2+\alpha}} \leqslant C\|v\|_{L^4}\|v\|_{\dot C^{1+\alpha}}$. Consequently, there holds

$$\begin{aligned}\|v(t,\cdot)\|_{\dot C^{1+\alpha}}^4 &\leqslant C\big(\|v(0,\cdot)\|_{\dot C^{1+\alpha}} + B(t)\big)^4 + C\int_0^t \|v\|_{L^4}^4\|v\|_{\dot C^{1+\alpha}}^4\,ds\\ &\leqslant C\big(\|v(0,\cdot)\|_{\dot C^{1+\alpha}} + B(\tilde t)\big)^4 + C\int_0^t \|v(s,\cdot)\|_{L^2}^2\|\nabla v(s,\cdot)\|_{L^2}^2\|v(s,\cdot)\|_{\dot C^{1+\alpha}}^4\,ds\end{aligned}$$

for any fixed $\tilde t$: $0\leqslant \tilde t < T$ and $t\leqslant \tilde t < T$. Here we used the fact that $B(t)$ is nondecreasing. Consequently, Gronwall's inequality gives that

$$\begin{aligned}A(\tilde t)^4 &= \sup_{0\leqslant t<\tilde t}\|v(t,\cdot)\|_{\dot C^{1+\alpha}}^4\\ &\leqslant C\big(\|v(0,\cdot)\|_{\dot C^{1+\alpha}} + B(\tilde t)\big)^4\exp\left\{C\int_0^t \|v(s,\cdot)\|_{L^2}^2\|\nabla v(s,\cdot)\|_{L^2}^2\,ds\right\}.\end{aligned}$$

Since $\tilde t\in[0,T)$ is arbitrary, using the basic energy inequality (1.7), we in fact have

$$A(t)\leqslant C\big(\|v(0,\cdot)\|_{\dot C^{1+\alpha}} + B(t)\big),\quad 0\leqslant t<T. \tag{3.13}$$

Next, by the second equation in (3.11), we have

$$\begin{aligned}\|\Delta_q\tau(t,\cdot)\|_{L^\infty} &\leqslant \|\Delta_q\tau(0,\cdot)\|_{L^\infty} + \int_0^t\big(2^q\|\Delta_q v(s,\cdot)\|_{L^\infty} + \|\Delta_q\big(\nabla v\tau + \tau(\nabla v)^t\big)(s,\cdot)\|_{L^\infty}\\ &\quad + \|[v\cdot\nabla,\Delta_q]\tau(s,\cdot)\|_{L^\infty}\big)\,ds,\end{aligned}$$

which implies that

$$\begin{aligned}\|\Delta_q\tau(t,\cdot)\|_{\dot C^\alpha} &\leqslant C\|\tau(0,\cdot)\|_{\dot C^\alpha} + \int_0^t\big(\|v\|_{\dot C^{1+\alpha}} + \|\nabla v\|_{L^\infty}\|\tau\|_{\dot C^\alpha} + \|\tau\|_{L^\infty}\|v\|_{\dot C^{1+\alpha}}\\ &\quad + 2^{\alpha q}\|[v\cdot\nabla,\Delta_q]\tau(s,\cdot)\|_{L^\infty}\big)\,ds.\end{aligned} \tag{3.14}$$

By Bony's decomposition, one has

$$\begin{aligned}[v\cdot\nabla,\Delta_q]\tau &= \sum_{|p-q'|\leqslant 1}[\Delta_p v\cdot\nabla,\Delta_q]\Delta_{q'}\tau + \sum_{p\leqslant q'-2}[\Delta_p v\cdot\nabla,\Delta_q]\Delta_{q'}\tau \\ &\quad + \sum_{p\leqslant q'-2}[\Delta_{q'} v\cdot\nabla,\Delta_q]\Delta_p\tau \\ &= \sum_{|q'-q|\leqslant 2}\big([S_{q'-1}v\cdot\nabla,\Delta_q]\Delta_{q'}\tau + [\Delta_{q'}v\cdot\nabla,\Delta_q]S_{q'-1}\tau\big) \\ &\quad + \sum_{|p-q'|\leqslant 1}[\Delta_p v\cdot\nabla,\Delta_q]\Delta_{q'}\tau.\end{aligned}$$

Noting that

$$[S_{q'-1}v,\Delta_q]f = \int h(y)\big[(S_{q'-1}v)(x) - (S_{q'-1}v)\big(x-2^{-q}y\big)\big]f\big(x-2^{-q}y\big)\,dy,$$

one has

$$\big\|[S_{q'-1}v,\Delta_q]f\big\|_{L^\infty} \leqslant C2^{-q}\|\nabla S_{q'-1}v\|_{L^\infty}\|f\|_{L^\infty}.$$

Consequently, we have

$$\begin{aligned}&\sum_{|q'-q|\leqslant 2}\int_0^t \big(2^{\alpha q}\big\|[S_{q'-1}v\cdot\nabla,\Delta_q]\Delta_{q'}\tau(s,\cdot)\big\|_{L^\infty}\big)\,ds \\ &\quad\leqslant \sum_{|q'-q|\leqslant 2}\int_0^t 2^{\alpha q}2^{-q}\|\nabla S_{q'-1}v\|_{L^\infty}\|\Delta_{q'}\nabla\tau\|_{L^\infty}(s,\cdot)\,ds \\ &\quad\leqslant C\sum_{|q'-q|\leqslant 2}\int_0^t \|\nabla S_{q'-1}v\|_{L^\infty}\big[2^{\alpha q'}\|\Delta_{q'}\tau\|_{L^\infty}\big](s,\cdot)\,ds \\ &\quad\leqslant C\int_0^t \|\nabla v\|_{L^\infty}\|\tau\|_{\dot C^\alpha}\,ds.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\sum_{|q'-q|\leqslant 2}\int_0^t \big(2^{\alpha q}\big\|[\Delta_{q'}v\cdot\nabla,\Delta_q]S_{q'-1}\tau(s,\cdot)\big\|_{L^\infty}\big)\,ds \\ &\quad\leqslant \sum_{|q'-q|\leqslant 2}\int_0^t 2^{\alpha q}2^{-q}\|\Delta_{q'}\nabla v\|_{L^\infty}\|S_{q'-1}\nabla\tau\|_{L^\infty}(s,\cdot)\,ds \\ &\quad\leqslant C\int_0^t \|\tau\|_{L^\infty}\|v\|_{\dot C^{1+\alpha}}\,ds.\end{aligned}$$

At last, one computes that

$$\begin{aligned}
&\sum_{|p-q'|\leqslant 1}\int_0^t \big(2^{\alpha q}\big\|[\Delta_p v\cdot\nabla,\Delta_q]\Delta_{q'}\tau(s,\cdot)\big\|_{L^\infty}\big)\,ds\\
&\quad\leqslant \sum_{p,q'\backsim q}\int_0^t \big(2^{(1+\alpha)q}\big\|[\Delta_p v,\Delta_q]\Delta_{q'}\tau(s,\cdot)\big\|_{L^\infty}\big)\,ds\\
&\qquad+\sum_{p,q'\geqslant q+2}\int_0^t \big(2^{(1+\alpha)q}\big\|[\Delta_p v,\Delta_q]\Delta_{q'}\tau(s,\cdot)\big\|_{L^\infty}\big)\,ds\\
&\quad\leqslant C\int_0^t \|\tau\|_{L^\infty}\|v\|_{\dot C^{1+\alpha}}\,ds.
\end{aligned}$$

The above inequalities yield an improvement of (3.14):

$$\big\|\tau(t,\cdot)\big\|_{\dot C^\alpha}\leqslant C\big\|\tau(0,\cdot)\big\|_{\dot C^\alpha}+C\int_0^t\big(\|\nabla v\|_{L^\infty}+\|\tau\|_{L^\infty}\big)\big(\big\|\tau(s,\cdot)\big\|_{\dot C^\alpha}+\big\|v(s,\cdot)\big\|_{\dot C^{1+\alpha}}\big)\,ds. \tag{3.15}$$

Now let us insert (3.13) into (3.15) to get

$$\begin{aligned}
B(t)&=\sup_{0\leqslant s<t}\big\|\tau(t,\cdot)\big\|_{\dot C^\alpha}\\
&\leqslant C\big\|\tau(0,\cdot)\big\|_{\dot C^\alpha}+C\int_0^t\big(\|\nabla v\|_{L^\infty}+\|\tau\|_{L^\infty}\big)\big(\big\|v(0,\cdot)\big\|_{\dot C^{1+\alpha}}+B(s)\big)\,ds.
\end{aligned}$$

Noting that by the inequalities in Lemma 2.2, we can estimate

$$\begin{aligned}
&\int_0^t\big(\|\nabla v\|_{L^\infty}+\|\tau\|_{L^\infty}\big)\,ds\\
&\quad\leqslant\int_0^{t_\star}\big(\|\nabla v\|_{L^\infty}+\|\tau\|_{L^\infty}\big)\,ds+C\int_{t_\star}^t\big(1+\|v\|_{L^2}+\|\tau\|_{L^2}\big)\,ds\\
&\qquad+C\sup_q\int_{t_\star}^t\|\nabla\Delta_q v\|_{L^\infty}\,ds\ln\left(e+\int_0^t\|v\|_{\dot C^{1+\alpha}}\,ds\right)+C\int_{t_\star}^t\|\tau\|_{\mathrm{BMO}}\ln\big(e+\|\tau\|_{\dot C^\alpha}\big)\,ds\\
&\quad\leqslant C_\star+C\epsilon\ln\big[e+Ct\big(\big\|v(0,\cdot)\big\|_{\dot C^{1+\alpha}}+B(t)\big)\big]+C\epsilon\ln\big(e+B(t)\big)\\
&\quad\leqslant C_\star+C\epsilon\ln\big(e+\big\|v(0,\cdot)\big\|_{\dot C^{1+\alpha}}+B(t)\big).
\end{aligned}$$

Here $C_\star$ is a positive constant depending on the solution (v,τ) on $[0,t_\star]$. Consequently, we have

$$B(t) \leqslant C_\star\big(1 + \|\tau(0,\cdot)\|_{\dot C^\alpha} + \|v(0,\cdot)\|_{\dot C^{1+\alpha}}\big) + C\int_0^t \big(\|\nabla v\|_{L^\infty} + \|\tau\|_{L^\infty}\big)B(s)\,ds.$$

Then Gronwall's inequality yields that

$$\begin{aligned} &e + \|v(0,\cdot)\|_{\dot C^{1+\alpha}} + B(t) \\ &\quad\leqslant C_\star\big(e + \|\tau(0,\cdot)\|_{\dot C^\alpha} + \|v(0,\cdot)\|_{\dot C^{1+\alpha}}\big)\exp\left\{C\int_{t_\star}^t \big(\|\nabla v\|_{L^\infty} + \|\tau\|_{L^\infty}\big)\,ds\right\} \\ &\quad\leqslant C_\star\big(e + \|\tau(0,\cdot)\|_{\dot C^\alpha} + \|v(0,\cdot)\|_{\dot C^{1+\alpha}}\big)e^{C_\star + C\epsilon\ln(e+\|v(0,\cdot)\|_{\dot C^{1+\alpha}}+B(t))} \\ &\quad\leqslant C_\star\big(e + \|\tau(0,\cdot)\|_{\dot C^\alpha} + \|v(0,\cdot)\|_{\dot C^{1+\alpha}}\big)\big(e + \|v(0,\cdot)\|_{\dot C^{1+\alpha}} + B(t)\big)^{C\epsilon}. \end{aligned}$$

From the above inequalities and (3.13), we have

$$A(t) + B(t) \leqslant C_\star\big(1 + \|\tau(0,\cdot)\|_{\dot C^\alpha} + \|v(0,\cdot)\|_{\dot C^{1+\alpha}}\big)^2 \tag{3.16}$$

by choosing $\epsilon = \frac{1}{2C}$.

4. Proof of Corollary 1.5

In fact, it is easy to see that the tensor $H \otimes H$ satisfies the following transport equation:

$$\partial_t(H\otimes H) + v\cdot\nabla(H\otimes H) = \nabla v(H\otimes H) + (H\otimes H)(\nabla v)^t. \tag{4.1}$$

Hence, the tensor $H\otimes H - \frac{1}{2}I$ plays the role of τ. (However, it seems not being able to directly apply Theorem 1.1 to get Corollary 1.5.) The rest part of the proof of Corollary 1.5 is similar as that of Theorem 1.1.

In fact, by the assumption $\|v(0,\cdot)\|_{L^2\cap\dot C^{1+\alpha}} + \|H(0,\cdot)\|_{L^2\cap\dot C^\alpha} < \infty$, one can easily derive that

$$\|v(0,\cdot)\|_{L^2\cap\dot C^{1+\alpha}(\mathbb{R}^2)} + \|H\otimes H(0,\cdot)\|_{L^1\cap\dot C^\alpha(\mathbb{R}^2)} < \infty. \tag{4.2}$$

Moreover, one has the following energy law

$$\|(v,H)\|_{L^\infty_T(L^2)} + 2\|\nabla v\|_{L^2_T(L^2)} = \|(v,H)(0,\cdot)\|_{L^2}, \tag{4.3}$$

which gives that

$$\|H\otimes H\|_{L^\infty_T(L^1)} < \infty. \tag{4.4}$$

Having (4.2), (4.3) and (4.4) in hand, and noticing the assumption (1.10) and the transport equation (4.1) for $H\otimes H$, one has

$$\|v(t,\cdot)\|_{\dot C^{1+\alpha}} + \|H\otimes H(t,\cdot)\|_{\dot C^\alpha} < \infty$$

by exactly the same manner as in Section 3. Coming back to the transport equation for H in (1.9), we have

$$\|H\|_{\dot{C}^\alpha} < \infty$$

in a standard manner. This completes the proof of Corollary 1.5.

Acknowledgments

Zhen Lei was in part supported by NSFC (grant No. 10801029), a special Postdoc Science Foundation of China (grant No. 200801175) and SGST 09DZ2272900. Nader Masmoudi was in part supported by NSF under Grant DMS-0703145. Yi Zhou is partially supported by the NSFC (grant No. 10728101), the 973 project of the Ministry of Science and Technology of China, the Doctoral Program Foundation of the Ministry of Education of China, the "111" project (B08018) and SGST 09DZ2272900.

References

[1] H. Bahouri, J.Y. Chemin, Equations de transport relatives à des champs de vecteurs non-lipschitziens et méchanique des fluides, Arch. Ration. Mech. Anal. 127 (1994) 159–182.
[2] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys. 94 (1984) 61–66.
[3] J.M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. École Norm. Sup. (4) 14 (2) (1981) 209–246.
[4] J.Y. Chemin, Perfect Incompressible Fluids, Oxford Lecture Ser. Math. Appl., vol. 14, The Clarendon Press/Oxford Univ. Press, New York, 1998.
[5] J.Y. Chemin, N. Masmoudi, About lifespan of regular solutions of equations related to viscoelastic fluids, SIAM J. Math. Anal. 33 (1) (2001) 84–112.
[6] F. Colombini, N. Lerner, Hyperbolic operators with non-Lipschitz coefficients, Duke Math. J. 77 (1995) 657–698.
[7] P. Constantin, N. Masmoudi, Global well posedness for a Smoluchowski equation coupled with Navier–Stokes equations in 2d, Comm. Math. Phys. 278 (1) (2008) 179–191.
[8] C. Guillopé, J.C. Saut, Existence results for the flow of viscoelastic fluids with a differential constitutive law, Nonlinear Anal. 15 (1990) 849–869.
[9] C. Guillopé, J.C. Saut, Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type, RAIRO Model. Math. Anal. Numer. 24 (1990) 369–401.
[10] E. Fernandez-Cara, F. Guillén, R.R. Ortega, Existence et unicité de solution forte locale en temps pour des fluides non newtoniens de type Oldroyd (version Ls–Lr), C. R. Acad. Sci. Paris Sér. I Math. 319 (1994) 411–416.
[11] D. Hu, T. Lelièvre, New entropy estimates for Oldroyd-B and related models, Commun. Math. Sci. 5 (4) (2007) 909–916.
[12] H. Kozono, T. Ogawa, Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations, Math. Z. 242 (2) (2002) 251–278.
[13] H. Kozono, Y. Taniuchi, Limiting case of the Sobolev inequality in BMO with application to the Euler equations, Comm. Math. Phys. 214 (1) (2000) 191–200.
[14] Zhen Lei, Global existence of classical solutions for some Oldroyd-B model via the incompressible limit, Chinese Ann. Math. Ser. B 27 (5) (2006) 565–580.
[15] Zhen Lei, Yi Zhou, BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity, DCDS-A, in press.
[16] F.-H. Lin, P. Zhang, Z. Zhang, On the global existence of smooth solution to the 2-d FENE dumbbell model, Comm. Math. Phys. 277 (2008) 531–553.
[17] P.-L. Lions, N. Masmoudi, Global solutions for some Oldroyd models of non-Newtonian flows, Chinese Ann. Math. Ser. B 21 (2000) 131–146.
[18] Pierre-Louis Lions, Nader Masmoudi, Global existence of weak solutions to some micro–macro models, C. R. Math. Acad. Sci. Paris 345 (1) (2007) 15–20.
[19] Nader Masmoudi, Well-posedness for the FENE dumbbell model of polymeric flows, Comm. Pure Appl. Math. 61 (12) (2008) 1685–1714.
[20] Nader Masmoudi, Global well posedness for the Maxwell–Navier–Stokes system in 2d, preprint, 2009.
[21] Nader Masmoudi, Ping Zhang, Zhifei Zhang, Global well-posedness for 2D polymeric fluid models and growth estimate, Phys. D 237 (10–12) (2008) 1663–1675.
[22] J.G. Oldroyd, Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids, Proc. R. Soc. Lond. Ser. A 245 (1958) 278–297.
[23] F. Planchon, An extension of the Beale–Kato–Majda criterion for the Euler equations, Comm. Math. Phys. 232 (2) (2003) 319–326.