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# Global well posedness for the Maxwell–Navier–Stokes system in 2D

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**Abstract**

We prove global existence of regular solutions to the full MHD system (or more precisely the Maxwell–Navier–Stokes system) in 2D. We also provide an exponential growth estimate for the $H^s$ norm of the solution when the time goes to infinity.


**Résumé**

Nous montrons l'existence globale de solutions régulières pour le système complet de la MHD (plus précisément le système de Maxwell–Navier–Stokes) en 2D. Nous donnons aussi une estimation de croissance sur les normes $H^s$ en temps grand.


*MSC:* 35Q30; 82C31; 76A05

*Keywords:* Navier–Stokes equations; Maxwell equations; MHD; Global well-posedness

## 1. Introduction

We consider a coupled system of equations consisting of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. The coupling comes from the Lorentz force in the fluid equation and the electric current in the Maxwell equations. We refer to Davidson [11] and Biskamp [4] for some physical introduction to magnetohydrodynamics.

### *1.1. The model*

We consider the full MHD system, namely:

$$\begin{cases} \frac{\partial v}{\partial t} + v \cdot \nabla v - \nu \Delta v + \nabla p = j \times B & \text{in } \Omega \times (0,T), \\ \frac{\partial E}{\partial t} - \operatorname{curl} B = -j & \text{in } \Omega \times (0,T), \\ \frac{\partial B}{\partial t} + \operatorname{curl} E = 0 & \text{in } \Omega \times (0,T), \\ \operatorname{div} B = \operatorname{div} v = 0 & \text{in } \Omega \times (0,T), \\ \quad j = \sigma(E + v \times B), \end{cases} \tag{1}$$

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written in the whole space $\Omega = \mathbb{R}^2$. We also supplement (1) with the following initial condition

$$v(t=0) = v_0, \qquad B(t=0) = B_0, \qquad E(t=0) = E_0. \tag{2}$$

Here, $v$ is the velocity of the fluid. The fluid is assumed to be incompressible, electrically conducting and non-magnetic. It can be a liquid metal, a hot ionized plasma .... The pressure is $p$, $\nu$ is the viscosity, $j$ is the electric current which is given by Ohm's law, $\sigma$ is the electric conductivity, $E$ is the electric field, $B$ is the magnetic field and $j \times B$ is the Lorentz force. For simplicity, we will take $\nu = \sigma = 1$. Here, $v, E, B, j$ are defined on $\Omega = \mathbb{R}^2$ and take their values in $\mathbb{R}^3$. This justifies the use of the cross product $v \times B$ and $j \times B$.

The system (1) has the following energy identity:

$$\frac{1}{2}\partial_t\big[\|v\|_{L^2}^2 + \|B\|_{L^2}^2 + \|E\|_{L^2}^2\big] + \|j\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = 0, \tag{3}$$

which is similar to the energy identity for the Navier–Stokes system. It translates the dissipation of energy by the viscosity and the electric resistivity. Hence, one can hope to extend the Leray theory of global weak solutions to this system and prove global existence of solutions with $v \in L^\infty(0,T;L^2) \cap L^2(0,T;\dot{H}^1)$ and $(E,B) \in L^\infty(0,T;L^2)$. However, due to the hyperbolic nature of the Maxwell equation, we do not get compactness of $B$ and hence passing to the limit in the product $j \times B$ seems to be a difficult problem. Moreover, ideas based on the propagation of compactness used in [24] and [25] seem not to work here. Hence, proving global existence of weak solutions to the (1) system in the energy space seems to us a very important open problem in both 2 and 3 dimensions.

Before stating our main results, let us explain a little bit the relevance of the model. The first equation in (1) is the Navier–Stokes equation for incompressible flows with a Lorentz force term. We recall that the Lorentz force on a charge $q$ which has the velocity $v$ is given by $q(E + v \times B)$. Hence, the force on a macroscopic fluid element is the sum of forces acting on its individual particles $\delta q E + \delta j \times B$ where $\delta q$ is the net charge and $\delta j$ is the electric current carried by the fluid element. Quasi-neutrality implies that $\delta q \simeq 0$ and hence the force terms becomes $\delta j \times B$. The second equation in (1) is the Ampere–Maxwell equation which includes here the displacement current $\partial_t E$. In most MHD models (see for instance the textbooks [28,11,4,14]) this term is neglected either using the quasi-neutrality assumption or using the fact that it is much smaller than the two other terms. Moreover, most of the mathematical works dealing with existence, uniqueness [13,30,1,2], blow-up criterion [6,20], regularity criteria [15,33,32], long time behavior or inviscid and non-resistive limits [12] or related problems [22] also make this assumption and hence remove the displacement current from the Ampere–Maxwell. Mathematically, this hypothesis makes the problem easier since it transforms the Maxwell's system which is hyperbolic into a parabolic equation for the magnetic field $B$. Here, we would like to keep this term. It preserves the hyperbolic nature of the Maxwell equation. The third equation of (1) is the Faraday's law. The forth equation is the divergence free condition for $v$ and $B$ and finally, the fifth equation is the Ohm's law which states that the electric current is proportional to the electric field measured in a frame moving with the local velocity of the conductor. This explains the extra term $v \times B$.

### *1.2. Statement of the result*

Even though, we were not able to prove the existence of global weak solutions in the energy space, we can prove, in 2 space dimension, the global existence of solutions such that $v_0 \in L^2$ and $E_0, B_0 \in H^s(\mathbb{R}^2)$ for some $0 < s < 1$. The proof uses the conservation of the energy as well as a logarithmic estimate to bound the $L^\infty$ norm of $v$ in terms of the $H^1$ norm of $v$ and a logarithmic term involving the $H^r$ norm of $v$ for some $r > 1$, see (31). This estimate yields an exponential growth of the $H^s$ norms. One can compare this growth estimate with the double exponential growth estimate of the $H^s$ norms in the 2D incompressible Euler system. Indeed, for 2D incompressible Euler system, one gets a double exponential growth which is due to the estimate of $\|\nabla v\|_{L^\infty}$ using $\|\operatorname{curl}(v)\|_{L^\infty}$ and a logarithmic term involving $\|v\|_{H^r}$ for some $r > 2$, see (35).

One can say that the use of a logarithmic estimate is due to the failure of the embedding of $H^1$ in $L^\infty$ in our case, whereas it is due to the failure of the boundedness of the Riesz transforms in $L^\infty$ for the 2D incompressible Euler. There are two other major differences between (31) and (35). The first one is the presence of a square root in our estimate (31). The second one is the fact that (31) is integrated in time. The combination of these two facts allows us to get an exponential growth of the $H^s$ norms instead of a double exponential growth as in the 2D incompressible Euler. We prove the following global existence result.

**Theorem 1.1.** *Take $0 < s < 1$, $v_0 \in L^2(\mathbb{R}^2)$ and $E_0, B_0 \in H^s(\mathbb{R}^2)$. Then, there exists a unique global solution $(v, E, B)$ of* (1) *such that for all $T > 0$, $v \in C([0,T); L^2) \cap L^2(0,T; \dot{H}^1)$ and $E, B \in C([0,T); H^s)$. Moreover, $j \in L^2(0,T; L^2) \cap L^1(0,T; H^s)$ and $v \in L^1(0,T; H^{s'+1})$ for each $0 < s' < \min(2s, 1)$. In addition, the energy identity* (3) *holds and we have the following double exponential growth estimate for all $t > 0$:*

$$\|v\|_{L^1(0,t;H^{s'+1})} + \big\|(E,B)(t)\big\|_{H^s} \leqslant \big(1 + \big\|(E_0,B_0)\big\|_{H^s}\big)e^{C_0(t+1)}, \tag{4}$$

*where $C_0 = C[\|v_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + 1]$ for some constant $C$.*

Besides, we also prove propagation of regularity, namely the global existence of more regular solutions if the initial data is more regular. Here, we still have some difference with respect to incompressible Euler. Indeed, for incompressible Euler, one can propagate regularity by only controlling $\nabla v$ in $L^1_T L^\infty$ and hence one can use the same estimate (35) with $r$ being the regularity one wants to propagate as long as $r > 2$. Hence the proof is done in one step and yields a double exponential growth for all the $H^s$ norms if the initial data is regular enough. In our case, this cannot be done directly since we have also to estimate $B$ in $L^1_T L^\infty$ if we want to propagate high regularities. We can estimate $B$ in $L^1_T L^\infty$ with a double exponential growth in time. Hence, when we put it in a Gronwall lemma it yields a triple exponential growth of the $H^s$ norms. Actually, we provide a different method based on some induction on the regularity to get an exponential growth.

**Theorem 1.2.** *Take $1 \leqslant s$ and $0 \leqslant s_0$, such that $s_0 \leqslant s < s_0 + 2$, $v_0 \in H^{s_0}(\mathbb{R}^2)$ and $E_0, B_0 \in H^s(\mathbb{R}^2)$. Then, the solution constructed in Theorem* 1.1 *is such that for all $T > 0$, $E, B \in C([0,T); H^s)$ and $v \in C([0,T); H^{s_0}) \cap L^1(0,T; H^{s_0'})$ for each $s_0' < s_0 + 2$ and we have the following growth estimate:*

$$\big\|v(t)\big\|_{H^{s_0}} + \|v\|_{L^1(0,t;H^{s_0'})} + \big\|(E,B)(t)\big\|_{H^s} \leqslant De^{C_0 t}, \tag{5}$$

*where $D = D_{(s,s_0)}$ is a function of $\|(E_0,B_0)\|_{H^s}$ and $\|v_0\|_{H^{s_0}}$.*

In the next Section 1.3, we give some preliminaries about Besov spaces and some parabolic regularity estimates. In Section 2, we prove some a priori estimate and derive the growth bound (4). In Section 3, we prove Theorem 1.1 by using a Galerkin approximation. Section 4 deals with the propagation of regularity and the proof of Theorem 1.2.

### *1.3. Preliminaries*

We recall here the Littlewood–Paley decomposition of a function. We define $\mathcal{C}$ to be the ring of center 0, of small radius 1/2 and great radius 2. There exist two nonnegative radial functions $\chi$ and $\varphi$ belonging respectively to $\mathcal{D}(B(0,1))$ and to $\mathcal{D}(\mathcal{C})$ so that

$$\chi(\xi) + \sum_{q \geqslant 0} \varphi\big(2^{-q}\xi\big) = 1, \tag{6}$$

$$|p - q| \geqslant 2 \quad \Rightarrow \quad \operatorname{Supp}\varphi\big(2^{-q}\cdot\big) \cap \operatorname{Supp}\varphi\big(2^{-p}\cdot\big) = \emptyset. \tag{7}$$

For instance, one can take $\chi \in \mathcal{D}(B(0,1))$ such that $\chi \equiv 1$ on $B(0,1/2)$ and take,

$$\varphi(\xi) = \chi(\xi/2) - \chi(\xi).$$

Then, we are able to define the Littlewood–Paley decomposition. Let us denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^d$. Let $h$, $\tilde{h}$, $\Delta_q$, $S_q$ ($q \in \mathbb{Z}$) be defined as follows:

$$h = \mathcal{F}^{-1}\varphi \quad \text{and} \quad \tilde{h} = \mathcal{F}^{-1}\chi,$$

$$\Delta_q u = \mathcal{F}^{-1}\big(\varphi\big(2^{-q}\xi\big)\mathcal{F}u\big) = 2^{qd}\int h\big(2^q y\big)u(x-y)\,dy,$$

$$S_q u = \mathcal{F}^{-1}\big(\chi\big(2^{-q}\xi\big)\mathcal{F}u\big) = 2^{qd}\int \tilde{h}\big(2^q y\big)u(x-y)\,dy.$$

We point out that $S_q u = \sum_{q' \leqslant q-1,\, q' \in \mathbb{Z}} \Delta_{q'} u$. We use the para-product decomposition of Bony [5]:

$$uv = T_u v + T_v u + R(u,v),$$

where

$$T_u v = \sum_{q\in\mathbb{Z}} S_{q-1}u\Delta_q v \quad \text{and} \quad R(u,v) = \sum_{|q-q'|\leqslant 1} \Delta_{q'}u\Delta_q v.$$

We define the inhomogeneous and homogeneous Besov spaces by

**Definition 1.3.** Let $s$ be a real number, $p$ and $r$ two real numbers greater than 1. Then we define the following norm,

$$\|u\|_{B^s_{p,r}} \stackrel{\text{def}}{=} \|S_0 u\|_{L^p} + \left\| \left(2^{qs}\|\Delta_q u\|_{L^p}\right)_{q\in\mathbb{N}} \right\|_{\ell^r(\mathbb{N})},$$

and the following semi-norm

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \left\| \left(2^{qs}\|\Delta_q u\|_{L^p}\right)_{q\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}.$$

**Definition 1.4.**

- Let $s$ be a real number, $p$ and $r$ two real numbers greater than 1. We denote by $B^s_{p,r}$ the space of tempered distributions $u$ such that $\|u\|_{B^s_{p,r}}$ is finite.
- If $s < d/p$ or $s = d/p$ and $r = 1$ we define the homogeneous Besov space $\dot{B}^s_{p,r}$ as the closure of compactly supported smooth functions for the norm $\|\cdot\|_{\dot{B}^s_{p,r}}$.

We refer to [7,9] for the proof of the following results and for the multiplication law in Besov spaces.

**Lemma 1.5.**

$$\|\Delta_q u\|_{L^b} \leqslant 2^{d(\frac{1}{a}-\frac{1}{b})q}\|\Delta_q u\|_{L^a} \quad \textit{for } b \geqslant a \geqslant 1,$$
$$\left\|e^{t\Delta}\Delta_q u\right\|_{L^b} \leqslant C2^{-ct2^{2q}}\|\Delta_q u\|_{L^b} \quad \textit{for } t \geqslant 0.$$

The following corollary is straightforward.

**Corollary 1.6.** *If $b \geqslant a \geqslant 1$, then, we have the following continuous embeddings*:

$$B^s_{a,r} \subset B^{s-d(\frac{1}{a}-\frac{1}{b})}_{b,r}.$$

When $a = r = 2$, we denote $\dot{H}^s = \dot{B}^s_{2,2}$ and $H^s = B^s_{2,2}$ the classical homogeneous and inhomogeneous Sobolev spaces. In the sequel, we will mostly deal with the Sobolev space $H^s$. We have the following product estimates:

**Corollary 1.7.** *For $0 < s < 1$, we have*:

$$\|jF\|_{\dot{H}^{s-1}} \leqslant C\|j\|_{L^2}\|F\|_{H^s}, \tag{8}$$
$$\|uF\|_{H^s} \leqslant C\left(\|u\|_{L^\infty} + \|u\|_{H^1}\right)\|F\|_{H^s}, \tag{9}$$
$$\|uv\|_{H^{2s-1}} \leqslant C\|u\|_{H^s}\|v\|_{H^s}. \tag{10}$$

For the proof of this corollary, we use the para-product decomposition of Bony [5] and write for instance $jF = T_j F + R(j,F) + T_F j$ and then we use standard estimate for $T$ and $R$ (we refer for instance to Chemin [7]).

When dealing with functions which depend on $t$ and $x$, the Littlewood–Paley decomposition will only apply to the $x$ variable. We will also use the notation $L^p_T(B^s_{a,r})$ to denote the space $L^p(0,T;B^s_{a,r})$ of functions $f$ such that for almost all $t \in (0,T)$, $f(t) \in B^s_{a,r}$ and $\|f(t)\|_{B^s_{a,r}} \in L^p(0,T)$. We will also sometimes drop the $T$ and denote it $L^p B^s_{a,r}$ if no ambiguity can occur. We also define the following spaces.

**Definition 1.8.** Let $p, r$ and $a$ be in $[1,\infty]$ and $s$ in $\mathbb{R}$; the space $\tilde{L}^p_T(\dot{B}^s_{a,r})$ is the space of distributions $u$ such that

$$\|u\|_{\tilde{L}^p(0,T;\dot{B}^s_{a,r})} \stackrel{\text{def}}{=} \big\|2^{qs}\|\Delta_q u\|_{L^p_T(L^a)}\big\|_{\ell^r(\mathbb{Z})} < \infty,$$

and $\tilde{L}^p_T(B^s_{a,r})$ is the space of distributions $u$ such that

$$\|u\|_{\tilde{L}^p(0,T;B^s_{a,r})} \stackrel{\text{def}}{=} \|S_0 u\|_{L^p_T L^a} + \big\|2^{qs}\|\Delta_q u\|_{L^p_T(L^a)}\big\|_{\ell^r(\mathbb{N})} < \infty.$$

Notice here that the integration in time is taken before the summation in $\ell^r$. This type of spaces where used by Chemin and Lerner [8]. We recall that for $p \geqslant r$, we have:

$$\|u\|_{L^p(0,T;B^s_{a,r})} \leqslant C\|u\|_{\tilde{L}^p(0,T;B^s_{a,r})}, \tag{11}$$

and that if $p \leqslant r$, then the opposite inequality holds, namely

$$\|u\|_{\tilde{L}^p(0,T;B^s_{a,r})} \leqslant C\|u\|_{L^p(0,T;B^s_{a,r})}. \tag{12}$$

Moreover, if $p \leqslant r$ and $s' < s$, then we can lose some regularity and get:

$$\|u\|_{L^p(0,T;B^{s'}_{a,r})} \leqslant C\|u\|_{\tilde{L}^p(0,T;B^s_{a,r})}. \tag{13}$$

We will use this space mostly with $p = 1$. Also, in the sequel, $C$ will denote any constant which may change from one line to the other and we will use the notation $A \lesssim B$ for $A \leqslant CB$ for some constant $C$.

We will use the following lemma giving parabolic regularity:

**Lemma 1.9.** *If $u$ solves*

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f, \\ u(t=0) = u_0, \end{cases} \tag{14}$$

*on some time interval $(0,T)$ then for $p \geqslant r \geqslant 1$ and $s \in \mathbb{R}$, $s > -1$, we have*:

$$\|u\|_{C([0,T);\dot{H}^{s+2-\frac{2}{r}}) \cap \tilde{L}^p_T \dot{H}^{s+2+\frac{2}{p}-\frac{2}{r}}} \leqslant C\big(\|f\|_{\tilde{L}^r_T \dot{H}^s} + \|u_0\|_{\dot{H}^{s+2-\frac{2}{r}}}\big). \tag{15}$$

*In particular, if $f = 0$, we have*:

$$\|u\|_{\tilde{L}^\infty_T \dot{H}^s \cap \tilde{L}^1_T \dot{H}^{s+2}} \leqslant C\|u_0\|_{\dot{H}^s}. \tag{16}$$

*We also have a similar result in the inhomogeneous case with a constant $C_T$ which may depend on $T$, namely*

$$\|u\|_{\tilde{L}^p_T H^{s+2+\frac{2}{p}-\frac{2}{r}}} \leqslant C_T\big(\|f\|_{\tilde{L}^r_T H^s} + \|u_0\|_{H^{s+2-\frac{2}{r}}}\big). \tag{17}$$

*Actually, one can take $C_T = C\max(1,T)$.*

We only give a sketch of the proof. Applying $\Delta_q$ to (14), taking the $L^2$ norm and applying Lemma 1.5, we get:

$$\|\Delta_q u\|_{L^2}(t) \lesssim \|\Delta_q u_0\|_{L^2} e^{-c2^{2q}t} + \int_0^t e^{-c2^{2q}(t-s)}\|\Delta_q f\|_{L^2}\,ds. \tag{18}$$

Taking the $L^p$ norm in time and using convolution estimates, we get

$$\begin{aligned}\|\Delta_q u\|_{L^p_T L^2} &\lesssim 2^{\frac{2q}{p}}\|\Delta_q u_0\|_{L^2} + \big\|e^{-2^{2q}t}1_{t>0}\big\|_{L^\alpha}\|\Delta_q f\|_{L^r L^2} \\ &\lesssim 2^{\frac{-2q}{p}}\|\Delta_q u_0\|_{L^2} + C2^{\frac{-2q}{\alpha}}\|\Delta_q f\|_{L^r L^2},\end{aligned}$$

where $\frac{1}{p} + 1 = \frac{1}{\alpha} + \frac{1}{r}$. Hence, multiplying by $2^{q(s+2+\frac{2}{p}-\frac{2}{r})}$ and taking the $\ell^2$ norm over $q \in \mathbb{Z}$, we get the result in the homogeneous case.

The proof in the inhomogeneous case follows the same lines and is left to the reader. One has to sum for $q \geqslant 1$ and deal with low frequency separately. We only mention that the reason we get a constant $C_T$ which depends on $T$ comes from the low frequencies. One can check that, for the inhomogeneous spaces, the only case where the constant $C_T$ can be taken independent of $T$ is when $p = \infty$ and $r = 1$. We also point out that the results stated in this lemma also holds in Besov spaces, namely replacing $\dot{H}^s$ by $\dot{B}^s_{a,r}$ for $a, r \in [1, \infty]$.

The next lemma gives a regularity result for the Maxwell equation.

**Lemma 1.10.** *If* $(E, B)$ *solves*

$$\begin{cases} \partial_t E - \operatorname{curl} B = -j, \\ \partial_t B + \operatorname{curl} E = 0, \\ E(t = 0) = E_0, \quad B(t = 0) = B_0, \end{cases} \tag{19}$$

*on some time interval* $(0, T)$ *then for* $s$ *in* $\mathbb{R}$, *we have*:

$$\big\|(E, B)\big\|_{C([0,T);H^s)} \leqslant \big\|(E_0, B_0)\big\|_{H^s} + \|j\|_{L^1(0,T;H^s)}. \tag{20}$$

The proof is very simple, we can use the Duhamel formula and write $F = e^{tL} F_0 + \int_0^t e^{(t-s)L} f(s)\,ds$ where $F = (E, B)$, $L$ is the operator define by $L(E, B) = (\operatorname{curl} B, -\operatorname{curl} E)$ and $f(s) = (j(s), 0)$. It is then clear that $e^{tL}$ defines an isometry on $H^s$ and hence the claim follows.

**Remark 1.11.** One can get some decay from the Maxwell equation if we split the electric current and include the $-E$ coming from Ohm's law in the definition of the operator $L$, namely if we take,

$$\begin{cases} \partial_t E - \operatorname{curl} B + E = j_1, \\ \partial_t B + \operatorname{curl} E = 0, \\ E(t = 0) = E_0, \quad B(t = 0) = B_0, \end{cases} \tag{21}$$

then (20) still holds with $j$ replaced by $j_1$. Moreover one can prove that $e^{tL'}$ where $L'(E, B) = (\operatorname{curl} B - E, -\operatorname{curl} E)$ satisfies some decay estimate for $t > 0$. We will come back to this in a forthcoming work to prove global existence for small data and to study long time behavior. In particular, (20) can be improved by replacing $\|j\|_{L^1_T H^s}$ by $\|j_1\|_{L^1_T H^s}$ on the left-hand side.

## 2. A priori estimates

Multiplying the first equation of (1) by $v$, the second one by $E$ and the third one by $B$ and integrating by parts, the energy estimate reads

$$\frac{1}{2}\partial_t\big[\|v\|_{L^2}^2 + \|B\|_{L^2}^2 + \|E\|_{L^2}^2\big] + \|j\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = 0,$$

and hence

$$\frac{1}{2}\big[\|v\|_{L^2}^2 + \|B\|_{L^2}^2 + \|E\|_{L^2}^2\big](t) + \int_0^t \|j\|_{L^2}^2 + \|\nabla v\|_{L^2}^2$$
$$= \frac{1}{2}\big[\|v_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 + \|E_0\|_{L^2}^2\big] = C_0. \tag{22}$$

This, formally, yields the bounds $v \in L^\infty(0, T; L^2) \cap L^2(0, T; \dot{H}^1)$, $E, B \in L^\infty(0, T; L^2)$ and $j \in L^2(0, T; L^2)$. Here and below $C_0$ will denote any constant of the form $C[\|v_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + 1]$, where $C$ may change from one line to the other.

Moreover, applying $\Delta_q$ to the Navier–Stokes equation and denoting $v_q = \Delta_q v$, we get:

$$\partial_t v_q + \Delta_q(v.\nabla v) - \nu \Delta v_q + \nabla p_q = \Delta_q(j \times B). \tag{23}$$

We also denote $v_H = v - S_1(v)$ the high frequency part of $v$. From Lemma 1.9, we deduce that

$$\begin{aligned}\|v_H\|_{\tilde{L}^1_T\dot{H}^{s+1}} &\leqslant C\min(1,T)\big\|v(0)\big\|_{L^2} + C\big\|(j\times B)\big\|_{\tilde{L}^1_T\dot{H}^{s-1}} + C\big\|(v.\nabla v)\big\|_{\tilde{L}^1_T\dot{H}^{s-1}}\\ &\leqslant C\min(1,T)\big\|v(0)\big\|_{L^2} + C\big\|(j\times B)\big\|_{L^1_T\dot{H}^{s-1}} + C\|v\|_{L^2H^1}\|\nabla v\|_{L^2_TL^2},\end{aligned}\tag{24}$$

where we used Corollary 1.7 and the fact that $0<s<1$. We remark here that we could have used $v$ instead of $v_H$ in (24) modulo replacing $\tilde{L}^1_T\dot{H}^{s+1}$ by $\tilde{L}^1_T(\dot{H}^{s+1}+\dot{H}^2)$ to take into account the bad decay of the low frequency part. Since, we only need (24) for high frequency, we elect to use $v_H$ here.

Using (8), we deduce that for $T>0$, we have:

$$\big\|(j\times B)\big\|_{\dot{H}^{s-1}} \leqslant C\|j\|_{L^2}\|B\|_{H^s}.\tag{25}$$

Hence, we deduce for $0<s_1<s$, that we have:

$$\|v_H\|_{L^1_T\dot{H}^{s_1+1}} \leqslant \|v_H\|_{\tilde{L}^1_T\dot{H}^{s+1}} \leqslant C\min(1,T)\big\|v(0)\big\|_{L^2} + C_0\big(1+T^{1/2}\big) + C_0^{1/2}T^{1/2}\|B\|_{L^\infty H^s},\tag{26}$$

where we have used that

$$\|v\|_{L^2H^1} \leqslant T^{1/2}\|v\|_{L^\infty L^2} + \|\nabla v\|_{L^2L^2} \leqslant C_0^{1/2}\big(T^{1/2}+1\big).\tag{27}$$

This yields a bound for $v\in L^1L^\infty$. If we denote $F=(E,B)$, then we get from Lemma 1.10 (see also Remark 1.11) that for $t>0$:

$$\|F\|_{L^\infty_t H^s} \leqslant \|F_0\|_{H^s} + \|v\times B\|_{L^1_tH^s}.\tag{28}$$

Moreover, using that

$$\|v\times B\|_{H^s} \leqslant C\big(\|v\|_{L^\infty}\|F\|_{H^s} + \|v\|_{H^1}\|F\|_{H^s}\big),\tag{29}$$

we deduce from Gronwall lemma that

$$\big\|F(t)\big\|_{H^s} \leqslant \|F_0\|_{H^s} e^{C_0(T^{1/2}+T)+C\int_0^t\|v\|_{L^\infty}},\tag{30}$$

for $0<t<T$. Since the Sobolev embedding of $H^1$ in $L^\infty$ fails in dimension 2, we estimate $\|u\|_{L^\infty}$ using $H^1$ and a logarithmic correction in $\dot{H}^{s+1}$. Indeed, we have:

$$v = S_1v + \sum_{q=1}^N \Delta_q v + \sum_{q>N}\Delta_q v,$$

where $N$ is an integer that will be fixed later. We consider $s_2$ and $s_1$ such that $0<s_2<s_1<s$. Applying Lemma 1.5 we deduce that

$$\begin{aligned}\int_0^T \|v\|_{L^\infty} &\leqslant C\int_0^T \|v\|_{L^2} + \sum_{q=1}^N 2^q\|\Delta_q v\|_{L^2} + 2^{-Ns_2}\sum_{q>N}2^{q(s_2+1)}\|\Delta_q v\|_{L^2}\\ &\leqslant C\|v\|_{L^1L^2} + CN^{1/2}\|v\|_{L^1\dot{H}^1} + C2^{-Ns_2}\|v_H\|_{L^1\dot{B}^{s_2+1}_{2,1}}\\ &\leqslant C\|v\|_{L^1L^2} + CN^{1/2}T^{1/2}\|v\|_{L^2\dot{H}^1} + C2^{-Ns_2}\|v_H\|_{L^1\dot{H}^{s_1+1}}.\end{aligned}$$

We optimize in $N$, by taking $N$ of the order $\frac{1}{s_2\log(2)}\log(e+\frac{\|v\|_{L^1H^{s_1+1}}}{T^{1/2}\|v\|_{L^2\dot{H}^1}})$. Hence,

$$\|v\|_{L^1L^\infty} \leqslant C_0^{1/2}T + CT^{1/2}\|v\|_{L^2\dot{H}^1}\log^{1/2}\left(e+\frac{\|v\|_{L^1\dot{H}^{s_1+1}}}{T^{1/2}\|v\|_{L^2\dot{H}^1}}\right).\tag{31}$$

Hence, we have:

$$\begin{aligned}\log\big(e+\|F\|_{L^\infty H^s}\big) &\leqslant \log\big(e+\|F_0\|_{H^s}\big) + C_0^{1/2}\big(T^{1/2}+T\big) + C\|v\|_{L^1L^\infty}\\ &\leqslant \log\big(e+\|F_0\|_{H^s}\big) + C_0^{1/2}\big(T^{1/2}+T\big) + CT^{1/2}\|v\|_{L^2\dot{H}^1}\log^{1/2}\left(e+\frac{\|v\|_{L^1\dot{H}^{s_1+1}}}{T^{1/2}\|v\|_{L^2\dot{H}^1}}\right)\\ &\leqslant \log\big(e+\|F_0\|_{H^s}\big) + C_0^{1/2}\big(T^{1/2}+T\big)\\ &\quad + CT^{1/2}\|v\|_{L^2\dot{H}^1}\log^{1/2}\left(e+\frac{C_0(\min(1,T)+T^{1/2}) + C_0T^{1/2}\|B\|_{L^\infty H^s}}{T^{1/2}\|v\|_{L^2\dot{H}^1}}\right).\end{aligned}$$

Then, we use that the function $a \to a\,\log(e + \frac{C}{a})$ is increasing in $a$ to deduce that there exists a $C_0$ such that for all $T > 0$, we have:

$$\sup_{0\leqslant t\leqslant T} \log^{1/2}\bigl(e + \|F\|_{L^\infty H^s}\bigr) \leqslant \log^{1/2}\bigl(e + \|F_0\|_{H^s}\bigr) + C_0^{1/2} T^{1/2}. \tag{32}$$

Therefore, there exists a constant $D_0 = C\|F_0\|_{H^s}$ and constant $C_0$ such that for all $T > 0$, we have:

$$\log^{1/2}\bigl(e + \bigl\|F(T)\bigr\|_{L^\infty H^s}\bigr) \leqslant C\log\bigl(e + \|F_0\|_{H^s}\bigr) + C_0 T, \tag{33}$$

which yields the desired bound,

$$\bigl\|F(T)\bigr\|_{H^s} \leqslant C\bigl(e + \|F_0\|_{H^s}\bigr)e^{C_0 T}. \tag{34}$$

**Remark 2.1.** (1) The a priori estimate given here is reminiscent of the growth estimate of the $H^s$ norms in the incompressible Euler system in 2D. There, the following estimate is used

$$\|\nabla v\|_{L^\infty} \leqslant C\bigl\|\mathrm{curl}(v)\bigr\|_{L^\infty} \log\left(e + \frac{\|v\|_{H^s}}{\|\mathrm{curl}(v)\|_{L^\infty}}\right) \tag{35}$$

for $s > 2$ and is combined with the conservation of the $L^\infty$ norm of the vorticity $\mathrm{curl}(v)$. Here we are only interested in an estimate of $v$ in $L^1_T L^\infty$ since it is needed to control $v \times B$ in $H^s$. This is done using (31) combined with the energy inequality to control $L^2_T \dot H^1$. Also, (35) is used by Beale, Kato and Majda [3] to give a non-blow up criterion of 3D incompressible Euler (see also [18,29]).

(2) In the previous argument and in particular in (31), it was important to have the $T^{1/2}$ and the $\log^{1/2}$ to prove the exponential growth. Actually, if instead of $\log^{1/2}$, we had a log in (31), we would have gotten a double exponential growth as in the incompressible Euler system in 2D.

(3) There are also similarities with an other logarithmic estimate, namely

$$\|\nabla v\|_{L^1(t_1,t_2;L^\infty)} \leqslant C\|v\|_{L^1(t_1,t_2;L^2)} + C\|\nabla v\|_{\tilde L^1(t_1,t_2;L^\infty)} \log\bigl(e + \|v\|_{L^1(t_1,t_2;C^{1+\alpha})}\bigr), \tag{36}$$

for $\alpha > 0$. This estimate (and similar ones) were used extensively in many mathematical results about Oldroyd B model and polymeric flows in 2D (see [27,10,26,21,19]). In particular in [27] a double exponential growth of $H^s$ norms was proved for some 2D polymeric fluid models.

(4) The inequality (31) was also used in [17,16] to deal with the Klein–Gordon equation in 2D with exponential nonlinearity. The problem there was also the fact that $H^1$ is not embedded in $L^\infty$. There, it was important to get a sharp constant in the inequality.

## 3. Proof of Theorem 1.1

In this section, we prove the existence and uniqueness of Theorem 1.1.

### 3.1. Existence of solutions

The existence of a solution $(u, E, B)$ which solves (1) follows from the a priori estimates proved in the last section. We shall use the very classical Friedrich's method (also called Galerkin method in the periodic case) which consists in approximating the system (1) by a cutoff in the frequency system. For this, let us define the operator $J_n$ by:

$$J_n a \stackrel{\mathrm{def}}{=} \mathcal{F}^{-1}\bigl(\mathbf{1}_{B(0,n)}(\xi)\hat u(\xi)\bigr),$$

where $\mathcal{F}$ denotes the Fourier transform in the space variables. Let us consider the approximate system:

$$\begin{cases} \partial_t v_n + J_n(J_n v_n \cdot \nabla J_n v_n) - \nu\Delta J_n v_n + \nabla p_n = J_n(j_n \times J_n B_n) & \text{in } \Omega \times (0,T), \\ \partial_t E_n - \mathrm{curl}\, J_n B_n = -j_n & \text{in } \Omega \times (0,T), \\ \partial_t B_n + \mathrm{curl}\, J_n E_n = 0 & \text{in } \Omega \times (0,T), \\ \mathrm{div}\, B_n = \mathrm{div}\, v_n = 0 & \text{in } \Omega \times (0,T), \\ \quad j_n = \sigma J_n(E_n + J_n v_n \times J_n B_n) \end{cases} \tag{37}$$

with the initial data,

$$v_n(t=0)=J_n(v_0),\qquad B_n(t=0)=J_n(B_0),\qquad E_n(t=0)=J_n(E_0). \tag{38}$$

The above system appears as a system of ordinary differential equations on $L^2$. So, the usual Cauchy–Lipschitz theorem implies the existence of a strictly positive maximal time $T_n$ such that a unique solution exists which is continuous in time with value in $L^2$. But, as $J_n^2=J_n$, we claim that $J_n(v_n,B_n,E_n)$ is also a solution, so uniqueness implies that $J_n(v_n,B_n,E_n)=(v_n,B_n,E_n)$ and hence, one can remove all the $J_n$ in front of $v_n$, $B_n$ and $E_n$ keeping only those in front of nonlinear terms:

$$\begin{cases} \partial_t v_n + J_n(v_n\cdot\nabla v_n) - \nu\Delta v_n + \nabla p_n = J_n(j_n\times B_n) & \text{in } \Omega\times(0,T),\\ \partial_t E_n - \operatorname{curl} B_n = -j_n & \text{in } \Omega\times(0,T),\\ \partial_t B_n + \operatorname{curl} E_n = 0 & \text{in } \Omega\times(0,T),\\ \operatorname{div} B_n = \operatorname{div} v_n = 0 & \text{in } \Omega\times(0,T),\\ \quad j_n = \sigma J_n(E_n + v_n\times B_n). \end{cases} \tag{39}$$

The main goal is to prove that $T_n$ can be taken to be equal to $+\infty$ and that we have some local in time estimate which are uniform in $n$. Then, one can pass to the limit and recover a solution of the initial system (1).

As $J_n$ is a Fourier multiplier, it commutes with constant coefficient differentiations and hence, the energy estimate (22) still holds:

$$\begin{aligned} &\frac{1}{2}\big[\|v_n\|_{L^2}^2+\|B_n\|_{L^2}^2+\|E_n\|_{L^2}^2\big](t)+\int_0^t \|j_n\|_{L^2}^2+\|\nabla v_n\|_{L^2}^2\\ &\quad=\frac{1}{2}\big[\big\|J_n(v_0)\big\|_{L^2}^2+\big\|J_n(B_0)\big\|_{L^2}^2+\big\|J_n(E_0)\big\|_{L^2}^2\big]\leqslant C_0. \end{aligned} \tag{40}$$

This implies that the $L^2$ norm of $(v_n,B_n,E_n)$ is controlled and hence, $T_n=+\infty$. Moreover, the estimates performed in the previous section apply in the same way to the system (39) and hence the a priori estimates derived there still hold (with bounds which are independent of $n$), namely we have:

$$\big\|F_n(t)\big\|_{H^s}\leqslant C\big(e+\|F_0\|_{H^s}\big)e^{C_0 t}, \tag{41}$$

and

$$\|v_n\|_{L^1_T H^{s_1+1}}\leqslant C\big(e+\|F_0\|_{H^s}\big)e^{C_0 T}. \tag{42}$$

Moreover, we also have that for all $T>0$, there exists a constant $C_T$ such that

$$\|\partial_t v_n\|_{L^2(0,T;H^{-1})}\leqslant C_T \quad\text{and}\quad \|\partial_t F_n\|_{L^2(0,T;H^{-1})}\leqslant C_T. \tag{43}$$

Hence, extracting a subsequence, standard compactness arguments allow us to pass to the limit in (39). This yields the existence of a solution $(v,B,E)$ to (1) (see for instance [31,23]) with the initial data (2).

### *3.2. Uniqueness of solutions*

Here, we prove the uniqueness of solutions to (1) in $L^\infty(0,T;L^2)\cap L^2(0,T;\dot H^1)\times L^\infty(0,T;H^s)\times L^\infty(0,T;H^s)$. Actually, we prove here a uniqueness result slightly stronger than the one stated in the theorem since we do not require the continuity in time. This actually is a very small improvement since one can get the continuity just from the fact that $(v_i,E_i,B_i)$ solves the system. Take $(v_1,E_1,B_1)$ and $(v_2,E_2,B_2)$ two solutions of (1) with the same initial condition (2) and such that for $i=1,2$, we have $v_i\in L^\infty(0,T;L^2)\cap L^2(0,T;\dot H^1)$ and $E_i,B_i\in L^\infty(0,T;H^s)$.

We start by applying the regularity theory for the Navier–Stokes system. Indeed, $v_i$ solves the Navier–Stokes system:

$$\begin{cases} \frac{\partial v}{\partial t}+v\cdot\nabla v-\nu\Delta v+\nabla p=f_i & \text{in } \Omega\times(0,T),\\ \operatorname{div} v=0 & \text{in } \Omega\times(0,T), \end{cases} \tag{44}$$

with the initial data $v_i(t=0)=v_0$ and the force $f_i=(E_i+v_i\times B_i)\times B_i$. Product rules in Sobolev spaces show that $f_i\in L^2H^{s'-1}$ for each $s'<2s$. Moreover, the Navier–Stokes equation (44) has a unique solution $v_i$ in the energy space $L^\infty([0,T);L^2)\cap L^2([0,T);\dot H^1)$ and this solution also satisfies the fact that $v_i\in C([0,T);L^2)\cap L^1H^{s'+1}$ for $s'<\min(2s,1)$.

We denote $v=v_2-v_1$, $E=E_2-E_1$, $j=j_2-j_1$ and $B=B_2-B_1$. We have:

$$\begin{cases} \frac{\partial v}{\partial t}+v_2\cdot\nabla v+v\cdot\nabla v_1-\nu\Delta v+\nabla p=j\times B_2+j_1\times B & \text{in } \Omega\times(0,T),\\ \frac{\partial E}{\partial t}-\operatorname{curl} B=j & \text{in } \Omega\times(0,T),\\ \frac{\partial B}{\partial t}+\operatorname{curl} E=0 & \text{in } \Omega\times(0,T),\\ \operatorname{div} B=\operatorname{div} v=0 & \text{in } \Omega\times(0,T),\\ \quad j=E+v_1\times B+v\times B_2. \end{cases} \tag{45}$$

We notice that due to the fact that we do not have the energy estimate for the difference of two solutions, we need slightly different spaces for the uniqueness proof. We denote $X=L^\infty(0,T;L^2)\cap L^2(0,T;H^1)\cap L^q(0,T;H^{1+\frac{s}{4}})$ where $\frac{1}{q}=\frac12(1+\frac{s}{2})$. We also denote $Y=X\times L^\infty(0,T;H^s)\times L^\infty(0,T;H^s)$.

We will also use that $X\subset L^{2q_1}(0,T;H^{\frac{1+s}{2}})$ $\frac{1}{2q_1}=\frac{1+s}{4}$.

Applying Lemma 1.9, we get:

$$\|v\|_X\leqslant \big\|j\times B_2+j_1\times B-\nabla(v_2\otimes v+v\otimes v_1)\big\|_{L^{q_1}H^{s-1}} \tag{46}$$
$$\leqslant C\|v_2\|_{L^{2q_1}H^{\frac12(s+1)}}\|v\|_{L^{2q_1}H^{\frac12(s+1)}}+\|v\|_{L^{2q_1}H^{\frac12(s+1)}}\|v_1\|_{L^{2q_1}H^{\frac12(s+1)}} \tag{47}$$
$$+T^{\frac{1}{q_1}-\frac12}\|j_1\|_{L^2L^2}\|B\|_{L^\infty H^s}+T^{\frac{1}{q_1}-\frac1q}\|v_1\|_{L^qH^{1+\frac{s}{4}}}\|B\|_{L^\infty H^s}\|B_2\|_{L^\infty H^s} \tag{48}$$
$$+T^{\frac{1}{q_1}-\frac1q}\|v\|_{L^qH^{1+\frac{s}{4}}}\|B_2\|_{L^\infty H^s}\|B_2\|_{L^\infty H^s}+T^{\frac{1}{q_1}}\|E\|_{L^\infty H^s}\|B_2\|_{L^\infty H^s}, \tag{49}$$

where we have used that $s-1+2-\frac{2}{q_1}=0$, that $q_1<q<2$ and that $s-1+2+\frac{2}{q}-\frac{2}{q_1}=1+\frac{s}{2}<1+\frac{s}{4}$.

Moreover, we have:

$$\|F\|_{L^\infty H^s}\leqslant C\|j\|_{L^1H^s} \tag{50}$$
$$\leqslant CT\|E\|_{L^\infty H^s}+CT^{1-\frac{1}{q_1}}\|v\times B_2+v_1\times B\|_{L^{q_1}H^s} \tag{51}$$
$$\leqslant CT\|E\|_{L^\infty H^s}+CT^{1-\frac{1}{q_1}}\big(\|v\|_{L^{q_1}H^{1+\frac{s}{4}}}\|B_2\|_{L^\infty H^s}+\|v_1\|_{L^{q_1}H^{1+\frac{s}{4}}}\|B\|_{L^\infty H^s}\big). \tag{52}$$

Choosing $T$ small enough, we get that

$$\|F\|_{L^\infty H^s}+\|v\|_X\leqslant\frac12\big(\|F\|_{L^\infty H^s}+\|v\|_X\big), \tag{53}$$

hence, $v=0$ and $F=0$ which yields the uniqueness of the solution on a small time interval. One can then repeat the argument and get the uniqueness on the whole real line.

## 4. Propagation of regularity

In this section, we prove Theorem 1.2. We will only present the a priori estimate since the existence of solutions satisfying these estimates can be proved using the same proof as in the previous section. Recall that usually, this can be done if we have an estimate of $\nabla u$ in $L^1L^\infty$. Notice that the solutions constructed in Theorem 1.1 do not necessary satisfy this estimate.

Since, $1<s$, we can apply the result of Theorem 1.1 with some $s'<1$ close from 1. It is enough to take $s'>3/4$ for instance. Hence, we get a solution $(v,E,B)\in X\times L^\infty(0,T;H^{s'})\times L^\infty(0,T;H^{s'})$.

Therefore, $v$ solves the Navier–Stokes system with an initial data $v_0\in H^{s_0}$ and a force term $j\times B$ in $L^{q_1}H^{2s'-1}$. Regularity results for the Navier–Stokes system yield that $v\in L^1H^{\min(s_0',5/2)}\cap L^\infty H^{\min(s_0,1/2)}$ where $s_0'$ satisfies $\max(s,2)<s_0'<s_0+2$.

In the sequel, we will only present a priori estimates on $(v, E, B)$, namely in the next calculations, we will assume that $(v, E, B)$ are regular enough to perform the multiplications and estimates. Of course, one has to write these estimate on the approximate system (39) and then pass to the limit to deduce that the solution satisfies these bounds.

We first give a simple proof of propagation of regularity but with a triple exponential growth. Then, we modify the argument to get the exponential bound (4).

First, we take $s''$ such that $1 < s'' < \min(s, 3/2)$ and we use that $\|v \times B\|_{H^{s''}} \lesssim \|v\|_{H^{s''}} \|B\|_{H^{s''}}$ to deduce from Lemma 1.10 that

$$\left\|F(t)\right\|_{H^{s''}} \leqslant \|F_0\|_{H^{s''}} + C\int_0^t \|v\|_{H^{s''}} \|F\|_{H^{s''}}. \tag{54}$$

Hence, by Gronwall lemma and Sobolev embedding, we have:

$$\left\|F(t)\right\|_{L^\infty} \lesssim \left\|F(t)\right\|_{H^{s''}} \leqslant \|F_0\|_{H^{s''}} e^{C\|v\|_{L^1_t H^{s''}}}. \tag{55}$$

This gives a double exponential growth for $\|F(t)\|_{H^{s''}}$ and for $\|F(t)\|_{L^\infty}$.

Then, we use the following rough estimates for the Maxwell and Navier–Stokes equations:

$$\left\|F(t)\right\|_{H^s} \leqslant \|F_0\|_{H^s} + C\int_0^t \|v\|_{L^\infty} \|F\|_{H^s} + \|v\|_{H^s} \|F\|_{L^\infty}, \tag{56}$$

$$\left\|v(t)\right\|_{H^s} \leqslant \|v_0\|_{H^s} + C\int_0^t \|F\|_{L^\infty} \|F\|_{H^s} + \|v\|_{L^\infty} \|F\|_{L^\infty} \|F\|_{H^s} + \|v\|_{H^s} \bigl(\|F\|^2_{L^\infty} + \|v\|_{L^\infty}\bigr). \tag{57}$$

Hence, if we denote $g(t) = \|F(t)\|_{H^s} + \|v(t)\|_{H^s}$, we get:

$$g(t) \leqslant g(0) + C\int_0^t \bigl(1 + \|v\|_{L^\infty} + \|F\|_{L^\infty}\bigr)^2 g(\tau)\tau \tag{58}$$

and by Gronwall lemma, we get that

$$g(t) \leqslant g(0) e^{\int_0^t (1+\|v\|_{L^\infty} + \|F\|_{L^\infty})^2\, d\tau} \tag{59}$$

which gives a triple exponential growth since $\|F(t)\|_{L^\infty}$ has a double exponential growth.

Now, we want to modify this argument to get an exponential growth of the $H^s$ norm of $F$. We have to argue by induction on the regularity. From the previous estimates, we see that if $s' < 1$, then the exponential growth follows from (34). We make the following induction assumption. We denote $\alpha = s - s_0$. Hence $0 \leqslant \alpha < 2$. Let $\varepsilon > 0$ be such that $\alpha + 2\varepsilon < 2$. For $k \leqslant s$, the property $(P_k)$ stands for the following bounds:

$$(P_k) \quad \begin{cases} \|F(t)\|_{H^k} \leqslant D_k\, e^{C_k t}, \\ \|v(t)\|_{H^{(k-\alpha)_+}} + \|v(t)\|_{L^1_t H^{k+2\varepsilon}} \leqslant D_k\, e^{C_k t}, \end{cases} \tag{60}$$

where $D_k$ is a function of $\|F_0\|_{H^k}$ and $\|v_0\|_{H^{(k-\alpha)_+}}$ and $(k-\alpha)_+ = \max(k-\alpha, 0)$ and $C_k$ depends only on the energy $C_0$ and on $k$.

It is clear that $(P_k)$ holds for $k = s'$ for any $s' < 1$. Indeed, (34) yields the first bound of (60). Moreover, regularity results for the Navier–Stokes system give the second estimate of $(P_k)$. Indeed, $j \times B$ can be estimated in $L^1 H^{2s'-1}$. Hence, one has to take $s'$ close to 1 to get the regularity in $(P_k)$. In the sequel, we start the induction from some $k = s'$ close to one such that $k + \varepsilon > 1$.

Now, we would like to prove that $(P_k)$ yields $(P_{k+\varepsilon})$ as long as $k + \varepsilon \leqslant s$. We have:

$$\left\|F(t)\right\|_{H^{k+\varepsilon}} \leqslant \|F_0\|_{H^{k+\varepsilon}} + C\int_0^t \|v\|_{L^\infty} \|F\|_{H^{k+\varepsilon}} + \|v\|_{H^{k+2\varepsilon}} \|F\|_{H^{1-\varepsilon}}, \tag{61}$$

where we have used that $\|vF\|_{H^{k+\varepsilon}} \leqslant C\|v\|_{L^\infty} \|F\|_{H^{k+\varepsilon}} + C\|F\|_{H^{1-\varepsilon}} \|v\|_{H^{k+2\varepsilon}}$.

Hence, by Gronwall lemma, we get that

$$\begin{aligned}\|F(t)\|_{H^{k+\varepsilon}} &\leqslant \left(\|F_0\|_{H^{k+\varepsilon}} + \int_0^t \|v\|_{H^{k+2\varepsilon}}\|F\|_{H^{1-\varepsilon}}\right)e^{C\int_0^t \|v\|_{L^\infty}} \\ &\leqslant D_{k+\varepsilon}e^{C_{k+\varepsilon}t}.\end{aligned}$$

Moreover, Lemma 1.9 yields that

$$\begin{aligned}&\|v(t)\|_{H^{(k-\alpha+\varepsilon)_+}} + \|v(t)\|_{L^1_t H^{k+3\varepsilon}} \\ &\quad\leqslant C\big(\|v_0\|_{H^{(k-\alpha+\varepsilon)_+}} + \|(E+v\times B)\times B\|_{L^1 H^{(k-\alpha+\varepsilon)_+}} + \|\nabla(v\otimes v)\|_{L^1 H^{(k-\alpha+\varepsilon)_+}}\big) \\ &\quad\leqslant C\|v_0\|_{H^{(k-\alpha+\varepsilon)_+}} + C\int_0^t \|F\|^2_{H^{(k+\varepsilon)}}\big(1+\|v\|_{H^{k+2\varepsilon}}\big) + \|v\|_{H^{(k-\alpha)_+}}\|v\|_{H^{k+2\varepsilon}} \\ &\quad\leqslant D_{k+\varepsilon}e^{C_{k+\varepsilon}t}.\end{aligned}$$

This ends the proof of Theorem 1.2.

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