Energy convergence for singular limits of Zakharov type systems

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Abstract

We prove existence and uniqueness of solutions to the Klein-Gordon-Zakharov system in the energy space $H^1 \times L^2$ on some time interval which is uniform with respect to two large parameters c and α. These two parameter correspond to the plasma frequency and the sound speed. In the simultaneous high-frequency and subsonic limit, we recover the nonlinear Schrödinger system at the limit. We are also able to say more when we take the limits seperately.

Contents

1. Introduction

The Klein-Gordon-Zakharov system describes the interaction between Langmuir waves and ion sound waves in a plasma (see Dendy [9] and Bellan [3]). It can be derived from the two-fluid Euler-Maxwell system (see Sulem and Sulem [24], Colin and Colin [8] and Texier [26, 27] for some rigorous derivations). In this paper, we derive uniform bounds for the energy norms for the Klein-Gordon-Zakharov and Zakharov systems with two large parameters which correspond to the plasma frequency and the sound speed. We prove strong convergence of the solution in the energy space as the parameters tend to infinity.

We start with the (rescaled) Klein-Gordon-Zakharov system for (E,n) with two parameters (c,α) (see [19, Introduction] for the rescaling). We also refer to Dendy [9] and Bellan [3] for the physical relevance of the model

$$\begin{aligned} &c^{-2}\ddot{E} - \Delta E + c^2 E = -nE, \quad E : \mathbb{R}^{1+3} \to \mathbb{R}^3, \\ &\alpha^{-2}\ddot{n} - \Delta n = \Delta |E|^2, \quad n : \mathbb{R}^{1+3} \to \mathbb{R} \end{aligned} \tag{1.1}$$

where $E : \mathbb{R}^{1+3} \to \mathbb{R}$ is the electric field, $n : \mathbb{R}^{1+3} \to \mathbb{R}$ is the density fluctuation of ions, c^2 is the plasma frequency and α the ion sound speed. (1.1) has the following conserved energy

$$\mathcal{E} = \int c^2|E|^2 + |\nabla E|^2 + c^{-2}|\dot{E}|^2 + \frac{1}{2}||\alpha\nabla|^{-1}\dot{n}|^2 + \frac{1}{2}|n|^2 + n|E|^2 dx. \tag{1.2}$$

Notice that this energy is not uniformly bounded when c goes to infinity. First we consider the simultaneous high-frequency and subsonic limit $(c,\alpha) \to \infty$ from the above system to the nonlinear Schrödinger equation (NLS):

$$2i\dot{u} - \Delta u = |u|^2 u, \quad u = (u_1, u_2) : \mathbb{R}^{1+3} \to \mathbb{C}^3 \times \mathbb{C}^3. \tag{1.3}$$

More precisely E and n can be approximated by

$$E \sim e^{ic^2 t} u_1 + e^{-ic^2 t}\overline{u_2}, \quad n \sim -|u|^2. \tag{1.4}$$

We have to assume that $\sup(\alpha/c) < 1$, which is physically natural since c^2/α^2 is the same order as the mass ratio of the ions and the electrons. In [19] we have shown the convergence in $H^s \times H^{s-1}$ for $s > 3/2$. In this paper we extend this to the energy space $H^1 \times L^2$.

First, we recall that local well-posedness for (1.1) in the energy space $(E,n) \in H^1 \times L^2$ was performed by Ozawa, Tsutaya and Tsutsumi [21] when $\alpha \neq c$. We also point out that (1.1) does not have the null form structure as in Klainerman and Machedon [15] and this suggests that when $\alpha = c$ the system (1.1) may be locally ill-posed in $H^1 \times L^2$ (cf. the counter example of Lindblad [17] for similar equations). Here, we are only interested in the case $\alpha < c$.

The main difficulty in this limit is, regardless of regularity, the existence of a resonant frequency

$$|\xi| = \frac{2c^2}{c^2 - \alpha^2}\alpha, \tag{1.5}$$

around which the quadratic interactions in (1.1) cannot be controlled if each function is approximated by the free solution [19, Theorem 10.1]. To overcome it in [19], we employed a modified energy localized around the resonant frequency (1.5). The condition $s > 3/2$ was a natural requirement in controlling the error terms of the localization, and it seems extremely difficult to lower s down to 1 by that argument.

Here we observe that in the special but most important case $s = 1$, we do not need the localization, estimating the whole functions in the modified energy. Some error terms still remain because the energy diverges in the high-frequency limit $c \to \infty$, but they can be bounded by non-resonant bilinear estimates with some loss of regularity. Interestingly, those norms with regularity loss are essentially the same as what we can afford by iterative estimates, i.e. 1 loss for the $X^{s,b}$ norms and 1/2 loss for the Strichartz norm. There arises an additional complication due to the failure of the Sobolev embedding

$$L_t^2(H_6^{1/2}(\mathbb{R}^3)) \not\subset L_t^2(L^\infty(\mathbb{R}^3)), \tag{1.6}$$

where the left hand side is the Strichartz norm with 1/2 loss for the Schrödinger equation. It is also related to the failure of the endpoint Strichartz $L_t^2(L^\infty(\mathbb{R}^3))$ for the wave equation (see [13]). This difficulty is overcome by taking into account the better decay of the non-resonant frequencies in the Strichartz norm. We also point out that the proof given here only works for $s = 1$ and that the case $1 < s \le 3/2$ remains open.

If we consider the limits $c \to \infty$ and $\alpha \to \infty$ separately, then the above difficulties are decoupled, and hence much simpler proofs become available. Indeed, we can prove the convergence in the high-frequency limit $c \to \infty$ from the Klein-Gordon-Zakharov (1.1) to the Zakharov system for (u, n):

$$\begin{aligned} &2i\dot{u} - \Delta u = -nu, \quad u : \mathbb{R}^{1+3} \to \mathbb{C}^3 \times \mathbb{C}^3, \\ &\alpha^{-2}\ddot{n} - \Delta n = \Delta|u|^2, \quad n : \mathbb{R}^{1+3} \to \mathbb{R}, \end{aligned} \tag{1.7}$$

by the iterative argument in the energy space $H^1 \times L^2$. Also, the proof we present here works the same for any $s > 1$. This is because the resonant frequency (1.5) is bounded in this limit, so that we can treat it as low frequency or regular part. However, we encounter another difficulty when E has frequency $\gg c$ and n has much smaller one, due to a regularity gap in the Strichartz estimate between the wave and the Schrödinger equation. We exploit the smallness of the resonant frequency set to overcome it. The above convergence in the case $E \sim e^{ic^2 t}u$, namely only one mode of oscillation is present, has been previously proved by [4] in $H^s \times H^{s-1}$ with $s > 7/2$.

The limit $\alpha \to \infty$ from the Zakharov system (1.7) to the nonlinear Schrödinger equation (1.3) is even easier. In fact, we can prove the convergence in the energy space H^1, just by the energy conservation and the Sobolev embedding. This is because the conserved energy is uniformly bounded. Although the nonlinear part of energy is not positive and can be bigger than the linear part, we can control it by less regular norms on a uniform short time interval. The convergence in this limit has been proved in [23, 1, 22, 14], assuming at least H^5 uniform bound on the initial data. The argument in [19] works well giving convergence in H^s for $s > 3/2$. Our proof in this paper seems the simplest among them. However, the case $1 < s \leq 3/2$ remains open.

The method used here applies also to the vectorial Zakharov system

$$\begin{aligned} &2i\dot{u} - \nabla\nabla\cdot u + \beta\nabla\times\nabla\times u = -nu, \quad u : \mathbb{R}^{1+3} \to \mathbb{C}^3, \\ &\alpha^{-2}\ddot{n} - \Delta n = \Delta|u|^2, \quad n : \mathbb{R}^{1+3} \to \mathbb{R} \end{aligned} \tag{1.8}$$

The simultaneous limit $(\alpha, \beta) \to \infty$ will be investigated in a forthcoming paper.

The rest of paper is organized as follows. In Section 2, we collect preparatory materials, mainly on the $X^{s,b}$ spaces and the Strichartz norms. Sections 3 and 4 are devoted to the limit from the Klein-Gordon-Zakharov to the nonlinear Schrödinger. First we prove uniform bounds in Section 3, then we prove the convergence in Section 4. In Section 5, we deal with the limit from the Klein-Gordon-Zakharov to the Zakharov. In Section 6, we study the limit from the Zakharov to the nonlinear Schrödinger.

The main results are Theorems 3.1, 5.1 and 6.1.

2. Preliminaries

In this section, we give some notations and basic estimates used throughout this paper. In the first subsection, we introduce Fourier multipliers, the Littlewood-Paley decomposition, and the Besov spaces. Next we recall the Strichartz estimate for the Klein-Gordon equation, introducing some notations for the mixed norms. In the third subsection, we introduce the $X^{s,b}$ space, related operators and formulation of the integral equations, together with the basic linear estimates and an interpolation inequality.

First we introduce general notations. For any real numbers a, b and any number or vector c, we denote

$$\min(a, b) = a \wedge b, \quad \max(a, b) = a \vee b, \quad \langle c\rangle = \sqrt{1 + |c|^2}. \tag{2.1}$$

We define the real-valued inner products by

$$\begin{aligned} &\langle a, b\rangle := \mathrm{Re}(a\bar{b}), \quad \langle f \mid g\rangle_x := \int_{\mathbb{R}^d} \langle f(x), g(x)\rangle dx, \\ &\langle u \mid v\rangle_{t,x} := \int_{\mathbb{R}} \langle u(t) \mid v(t)\rangle_x dt. \end{aligned} \tag{2.2}$$

For any set A, we denote its characteristic function by the same letter A:

$$A(x) = \begin{cases} 1 & (x \in A) \\ 0 & (x \notin A) \end{cases} \tag{2.3}$$

Given any Banach function space Z on $\mathbb{R}^{1+d}$ which is L^p in time, we denote for any space-time function $u(t,x)$ and for any $T > 0$,

$$\|u\|_{Z(0,T)} := \|(0,T)u\|_Z. \tag{2.4}$$

When X is a Banach space, w-X denotes the same space X endowed with the weak topology.

2.1. **Fourier multipliers.** For any measurable function $f : \mathbb{R}^d \to \mathbb{C}$, we define the Fourier multiplier by $f(i\nabla) := \mathcal{F}^{-1} f(\xi)\mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^d$. For the Klein-Gordon, we will use the following special multipliers:

$$I_c := \langle \nabla/c\rangle^{-1}, \quad \Delta_c := -2\omega_c(\nabla), \quad \omega_c(\xi) := c^2(\langle \xi/c\rangle - 1). \tag{2.5}$$

Next we introduce the Littlewood-Paley decomposition. Fix a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ satisfying

$$\chi(t) = \begin{cases} 1 & |t| \le 4/3 \\ 0 & |t| \ge 5/3 \end{cases} \tag{2.6}$$

We denote frequency localization for any function $u(t,x)$, $\varphi(\tau,\xi)$ and $\delta > 0$ by

$$\begin{aligned} &P_{\varphi(\tau,\xi)\le\delta} u := \begin{cases} \mathcal{F}_{t,x}^{-1}\chi(\varphi(\tau,\xi)/\delta)\mathcal{F}_{t,x}u, & (\delta > 1/2) \\ 0 & (\delta \le 1/2) \end{cases}, \\ &P_{\varphi(\tau,\xi)>\delta} u := u - P_{\varphi(\tau,\xi)\le\delta} u, \end{aligned} \tag{2.7}$$

where $\mathcal{F}_{t,x}$ is the space-time Fourier transform for the variables $(t,x) \mapsto (\tau,\xi)$. The above convention $P_{\varphi\le 1/2} u \equiv 0$ is just for convenience in treating small frequency. For the spatial frequency localization, we abbreviate

$$f_{\le a} := f_{|\xi|\le a}, \quad f_{>a} := f_{|\xi|>a}, \quad f_a := f_{\le a} - f_{\le a/2}, \tag{2.8}$$

and for space-time localization, we use the notation

$$f_{a,b} := (f_{|\tau|\le a} - f_{|\tau|\le a/2})_b, \tag{2.9}$$

where $a, b > 0$ will be mainly chosen from the dyadic frequencies defined by

$$\mathbb{D} := \{2^z \mid z = 0, 1, 2 \dots\}. \tag{2.10}$$

The inhomogeneous Besov spaces are defined by

$$\|f\|_{B^\sigma_{q,r}(\mathbb{R}^d)} = \left\| \|j^\sigma f_j\|_{L^q_x(\mathbb{R}^d)} \right\|_{\ell^r_j(\mathbb{D})}, \tag{2.11}$$

and the Sobolev space by $H^\sigma = B^\sigma_{2,2}$. We will also use the mixed space

$$\|u\|_{B^b_{2,r}(\mathbb{R};H^s(\mathbb{R}^d))} = \left\| \|j^b k^s u_{j,k}\|_{L^2_{t,x}(\mathbb{R}^{1+d})} \right\|_{\ell^r_j \ell^2_k(\mathbb{D}^2)}. \tag{2.12}$$

By the Fourier support property, we have for any space functions u, v, w,

$$\langle uv \,|\, w \rangle_x = \sum_{(j,k,l)\in\mathcal{T}} \langle u_j v_k \,|\, w_l \rangle_x, \tag{2.13}$$

where $\mathcal{T} \subset \mathbb{D}^3$ such that for any $(j,k,l) \in \mathcal{T}$, either $j \lesssim k \sim l$, $k \lesssim l \sim j$, or $l \lesssim j \sim k$ holds.

2.2. **Strichartz norms.** The Strichartz estimate for $e^{-it\Delta_c/2}$ on $\mathbb{R}^3$ can be written as follows (see [11]). For any $\theta \in [0,1]$, $p \in [2,\infty]$, $r \in [1,\infty]$ and $s \in \mathbb{R}$, we define $\mathrm{St}^{s,p}_{\theta,r}$ by the norm

$$\|u\|_{\mathrm{St}^{s,p}_{\theta,r}} := \left\| k^{s+\frac{1}{p}(\theta-1)} \|I_c^{\frac{1}{p}\left(1+\frac{2\theta}{3}\right)} u_k\|_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^3))} \right\|_{\ell^r_k}, \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{p} + \frac{\theta}{3p}. \tag{2.14}$$

Then we have, if $(p,\theta) \neq (2,0)$,

$$\|e^{-it\Delta_c/2}\varphi\|_{\mathrm{St}^{s,p}_{\theta,r}} \le C_{\theta,p}\|\varphi\|_{B^s_{2,r}}. \tag{2.15}$$

$\theta = 0$ corresponds to the Strichartz estimate for the wave equation, and $\theta = 1$ without I_c is for the Schrödinger equation. We will use mostly $r = 2$, omitting it as $\mathrm{St}^{s,p}_\theta = \mathrm{St}^{s,p}_{\theta,2}$.

2.3. $X^{s,b}$ **space.** In this subsection, we give some general setting and estimates for the $X^{s,b}$ spaces (see [5, 25]). Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. We consider equations of the form

$$i\dot{u} + \omega(i\nabla)u = f, \tag{2.16}$$

where $f(t,x)$ is a given function. The $X^{s,b,r}$ space for this equation is defined by

$$\begin{aligned} X^{s,b,r} &= \{e^{it\omega(i\nabla)}v(t) \mid v \in B^b_{2,r}(\mathbb{R}; H^s_x(\mathbb{R}^d))\}, \\ \|u\|_{X^{s,b,r}} &= \|e^{-it\omega(i\nabla)}u(t)\|_{B^b_{2,r}H^s_x} \end{aligned} \tag{2.17}$$

for any $(s,b,r) \in \mathbb{R}^2 \times [1,\infty]$. We denote $X^{s,b} := X^{s,b,2}$. The space $X^{s,b,r}$ with $r \neq 2$ will be used only for the limit from the Klein-Gordon-Zakharov to the Zakharov, where we need the critical spaces $b = \pm 1/2$.

We will use the duality for $1 \le r < \infty$,

$$(X^{s,b,r})^* = X^{-s,-b,r/(r-1)}. \tag{2.18}$$

The above equation (2.16) is solved for $0 < t < T$ ($0 < T \lesssim 1$) by

$$u(t) = e^{it\omega(i\nabla)}[\chi(t)u(0) + \mathcal{I}_T e^{-it\omega(i\nabla)} f(t)], \tag{2.19}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is the same function as in (2.6), and the time operator $\mathcal{I}_T$ is defined by

$$(\mathcal{I}_T f)(t) = \int_{\substack{0<s<t \\ t-T<s<T}} f(s)ds = \int_0^T ((0,T)f)(t-s)ds. \tag{2.20}$$

Thus $u(t,x)$ in (2.19) is defined on the whole $t \in \mathbb{R}$, which is convenient to estimate the $X^{s,b}$ norms. In the following estimates, implicit constants do not depend on T for $0 < T \lesssim 1$. The fundamental (well-known) property of the $X^{s,b}$ space is the following.

Lemma 2.1. *Let $s \in \mathbb{R}$ and $r \in [1,\infty]$. For any $b \in \mathbb{R}$, we have*

$$\|e^{it\omega(i\nabla)}\chi(t)\varphi\|_{X^{s,b,r}} \lesssim \|\varphi\|_{H^s}. \tag{2.21}$$

For any $b \in (-1/2, 1/2)$ and any $\theta \in [0,1]$, we have

$$\|e^{it\omega(i\nabla)}\mathcal{I}_T e^{-it\omega(i\nabla)} f\|_{X^{s,b+\theta}} \lesssim T^{1-\theta}\|f\|_{X^{s,b}}. \tag{2.22}$$

In the critical case $b = 1/2$, we have

$$\|e^{it\omega(i\nabla)}\mathcal{I}_T e^{-it\omega(i\nabla)} f\|_{X^{s,1/2,\infty}\cap(L^\infty\cap C)(H^s)} \lesssim \|f\|_{X^{s,-1/2,1}}. \tag{2.23}$$

Moreover, let $P : \mathbb{R}^d \to [0,\infty]$ and assume that V is a Banach function space on $\mathbb{R}^d$ satisfying the space-time estimate

$$\|P(i\nabla)e^{it\omega(i\nabla)} f\|_{L^q_t V_x} \lesssim \|f\|_{H^s_x}, \tag{2.24}$$

for some $q \geq 2$. Then we have for any $b > 1/2$,

$$\|P(i\nabla)u\|_{L^q_t V_x} \lesssim \|u\|_{X^{s,b}}. \tag{2.25}$$

Remark 2.2. $P(i\nabla)$ will be either identity or some frequency cut-off in the later use.

Proof. The first inequality is trivial by the definition of $X^{s,b,r}$. The second one is proved as follows. Since the function $(0,T)(t)$ is uniformly bounded in $L^\infty \cap B^{1/2}_{2,\infty}$, we deduce that cut-off by $(0,T)$ is bounded in $H^b(\mathbb{R})$ if $-1/2 < b < 1/2$. Hence we have, denoting $g = e^{-it\omega(i\nabla)} f$, and using the second identity of (2.20),

$$\|\mathcal{I}_T g\|_{H^b_t(\mathbb{R})} \leq \int_0^T \|((0,T)g)(t-s)\|_{H^b_t(\mathbb{R})} ds \leq T\|(0,T)g\|_{H^b_t(\mathbb{R})} \lesssim T\|g\|_{H^b_t(\mathbb{R})} \tag{2.26}$$

for $-1/2 < b < 1/2$. In addition, we have

$$\partial_t \mathcal{I}_T g(t) = (0,T)(t) \cdot g(t) - (T,2T)(t) \cdot g(t-T), \tag{2.27}$$

and so $\mathcal{I}_T$ is bounded $H^b \to H^{b+1}$ for $-1/2 < b < 1/2$. By the complex interpolation and the definition of $X^{s,b}$, we obtain (2.22). The $B^{1/2}_{2,\infty}$ bound of (2.23) is proved in the same way, since the cut-off operator is bounded $B^{-1/2}_{2,1} \to B^{-1/2}_{2,\infty}$. The L^∞ bound is derived by regarding $\mathcal{I}_T$ at each fixed t as the duality coupling for $B^{1/2}_{2,\infty}$ and $B^{-1/2}_{2,1}$. Then the continuity follows from the standard density argument.

For the last inequality, we use the trace argument

$$\|P(i\nabla)u\|_{L^q_t V_x} \lesssim \|P(i\nabla)e^{itH}e^{-i\tau H}u(\tau)\|_{L^q_t L^\infty_\tau V_x}, \tag{2.28}$$

and then the Sobolev embedding $H^b(\mathbb{R}) \subset L^\infty(\mathbb{R})$ and the Minkowski inequality $H^b_\tau L^q_t \subset L^q_t H^b_\tau$ to bound the right hand side by

$$\|e^{-i\tau H}P(i\nabla)e^{itH}u(\tau)\|_{H^b_\tau L^q_t V_x} \lesssim \|e^{-i\tau H}u(\tau)\|_{H^b_\tau H^s_x} = \|u\|_{X^{s,b}}. \tag{2.29}$$

□

Remark 2.3. The definition of extension operator ρ_T in [19, (2.33)] was incorrect. It should be defined as above, namely twisted by the evolution operator, such as

$$\rho_T u(t) = e^{-it\Delta_c/2}\chi(t)e^{it_T\Delta_c/2}u(t_T). \tag{2.30}$$

Next we give an interpolation inequality connecting the $X^{s,b}$ and the energy spaces yielding the Strichartz bound. An estimate of this type was first derived in [16] with a small loss of regularity, which has been removed in [20]. Their estimates are covered by the following lemma with the special choice $(b_0, b_1) = (1, 0)$, together with the trivial embedding $L^2(0,T) \subset T^{1/2}L^\infty(0,T)$.

Lemma 2.4. *Let $\omega : \mathbb{R}^d \to \mathbb{R}$ and $P, Q, R : \mathbb{R}^d \to [0, \infty]$ be measurable functions satisfying*

$$Q(\xi) \gtrsim Q_j := \sup_{\eta \in \operatorname{supp}\varphi_j} Q(\eta), \quad R(\xi) \gtrsim R_j = \sup_{\eta \in \operatorname{supp}\varphi_j} R(\eta) \tag{2.31}$$

uniformly for all $\xi \in \operatorname{supp}\varphi_j$ and all $j \in \mathbb{D}$. Let V be a Banach function space on $\mathbb{R}^d$ satisfying the following estimates

$$\|P(i\nabla)e^{itH}f\|_{L^2_t V_x} \lesssim \|f\|_{H^s}, \quad \|f\|_V^2 \lesssim \sum_{j\in\mathbb{D}} \|f_j\|_V^2. \tag{2.32}$$

Let $s_0, s_1, s \in \mathbb{R}$, $b_0, b_1 \in [0,1]$, $b_0 \neq b_1$, $\theta \in (0,1)$ and assume

$$s = (1-\theta)s_0 + \theta s_1, \quad 1/2 = (1-\theta)b_0 + \theta b_1. \tag{2.33}$$

Then we have

$$\|(PQ^{1-\theta}R^\theta)(i\nabla)u(t)\|_{L^2_t V_x} \lesssim \|Q(i\nabla)u\|^{1-\theta}_{X^{s_0,b_0}} \|R(i\nabla)u\|^\theta_{X^{s_1,b_1}}. \tag{2.34}$$

Proof. Let $f = (PQ^{1-\theta}R^{\theta})(i\nabla)u$. We use the trace argument

$$\|f_k\|_{L_t^2V_x} \lesssim \|e^{itH}e^{-i\tau H}f_k(\tau)\|_{L_t^2L_\tau^\infty V_x}. \tag{2.35}$$

for $\forall k \in \mathbb{D}$. By the real interpolation (and the fact that $b_0 \neq b_1$) and the Sobolev embedding, we have $(H^{b_0}, H^{b_1})_{\theta,1} \subset B^{1/2}_{2,1} \subset L^\infty$. Hence we have for any g,

$$\|g\|_{L_\tau^\infty V_x} \lesssim \|g\|^{1-\theta}_{H_\tau^{b_0}V_x}\|g\|^{\theta}_{H_\tau^{b_1}V_x}. \tag{2.36}$$

Applying this and the Hölder inequality to the above estimate, and changing the order of integration for τ and t, we obtain

$$\|f_k\|_{L_t^2V_x} \lesssim \|e^{itH}e^{-i\tau H}f_k(\tau)\|^{1-\theta}_{H_\tau^{b_0}L_t^2V_x}\|e^{itH}e^{-i\tau H}f_k(\tau)\|^{\theta}_{H_\tau^{b_1}L_t^2V_x} \tag{2.37}$$

The right hand side is bounded by using (2.32) as

$$\begin{aligned}&\lesssim \|e^{-i\tau H}(Q^{1-\theta}R^{\theta})(i\nabla)u_k(\tau)\|^{1-\theta}_{H_\tau^{b_0}H_x^s}\|e^{-i\tau H}(Q^{1-\theta}R^{\theta})(i\nabla)u_k(\tau)\|^{\theta}_{H_\tau^{b_1}H_x^s}\\ &\lesssim Q_k^{1-\theta}R_k^{\theta}k^{(s-s_0)(1-\theta)}k^{(s-s_1)\theta}\|e^{-i\tau H}u_k(\tau)\|^{1-\theta}_{H_\tau^{b_0}H_x^{s_0}}\|e^{-i\tau H}u_k(\tau)\|^{\theta}_{H_\tau^{b_1}H_x^{s_1}},\end{aligned} \tag{2.38}$$

where we used (2.31). Taking ℓ^2 summation for $k \in \mathbb{D}$, and using the relation $s = (1-\theta)s_0 + \theta s_1$, we arrive at the desired estimate. □

By reiterating interpolation, we immediately obtain

Corollary 2.5. *Assume* (2.32) *and* $|P| \lesssim 1$. *Then we have*

$$\|P(i\nabla)u\|_{L^2((H^s,V)_{\theta,2})} \lesssim \|u\|_{X^{s,\theta/2}}, \tag{2.39}$$

for $\theta \in (0,1)$, *where* $(\cdot,\cdot)_{\theta,r}$ *denotes the real interpolation space.*

Proof. See [18, Lemma 2.2], which was written in the dual form in the case $H = |\nabla|$, but the proof applies to the general case. Alternatively, we may use the above lemma and the reiteration theorem to get

$$\begin{aligned}X^{s,\theta/2} &= (X^{s,0}, X^{s,1})_{\theta/2,2} = (X^{s,0}, (X^{s,0}, X^{s,1})_{1/2,1})_{\theta,2}\\ &\overset{P(i\nabla)}{\to} (L^2H^s, L^2V)_{\theta,2} = L^2((H^s,V)_{\theta,2}).\end{aligned} \tag{2.40}$$

□

3. Uniform bounds for the Klein-Gordon-Zakharov

In this and the next sections, we consider simultaneous high-frequency and subsonic limit $(c,\alpha) \to \infty$ from the Klein-Gordon-Zakharov system (1.1) to the nonlinear Schrödinger equation (1.3). We recall that the local existence of a unique solution to (1.1) was proved in [21, Theorem 1.1] in the $X^{1,b} \times Y^{0,b}$ space for any initial data in the energy space. The main result in this case is the following.

Theorem 3.1. *Let $0 < \gamma < 1$ and consider the limit $(c, \alpha) \to \infty$ under the condition $\alpha \le \gamma c$. For each (c, α), let $(E^{c,\alpha}, n^{c,\alpha})$ be a solution of (1.1) given by [21], and denote its maximal existence time by $T^{c,\alpha}$. Assume that its initial data satisfy for some $(\varphi, \psi) \in H^1$*

$$\begin{aligned} &(E^{c,\alpha}(0), c^{-2} I_c \dot{E}^{c,\alpha}(0)) \to (\varphi, \psi) \text{ in } H^1, \\ &(n^{c,\alpha}(0), |\alpha\nabla|^{-1} \dot{n}^{c,\alpha}(0)) \text{ bounded in } L^2, \end{aligned} \tag{3.1}$$

and that the latter has uniform decay for high frequency, namely,

$$\lim_{R\to\infty} \limsup_{(c,\alpha)\to\infty} \|(n^{c,\alpha}(0), |\alpha\nabla|^{-1} \dot{n}^{c,\alpha}(0))_{>R}\|_{L^2} = 0. \tag{3.2}$$

Let $\mathbb{E}^\infty := (\mathbb{E}^\infty_+, \mathbb{E}^\infty_-)$ be the solution of (1.3) with the initial condition

$$\mathbb{E}^\infty(0) = \frac{1}{2}(\varphi - i\psi, \overline{\varphi} - i\overline{\psi}), \tag{3.3}$$

and T^∞ be the maximal existence time. Then we have $\liminf T^{c,\alpha} \ge T^\infty$, and for all $0 < T < T^\infty$,

$$\begin{aligned} &E^{c,\alpha} - (e^{ic^2 t} \mathbb{E}^\infty_+ + e^{-ic^2 t} \overline{\mathbb{E}^\infty_-}) \to 0 \text{ in } C([0,T]; H^1), \\ &c^{-2} I_c \dot{E}^{c,\alpha} - i(e^{ic^2 t} \mathbb{E}^\infty_+ - e^{-ic^2 t} \overline{\mathbb{E}^\infty_-}) \to 0 \text{ in } C([0,T]; H^1), \\ &n^{c,\alpha} + |\mathbb{E}^\infty|^2 - n^{c,\alpha}_f \to 0 \text{ in } C([0,T]; L^2), \\ &|\alpha\nabla|^{-1}(\dot{n}^{c,\alpha} - \dot{n}^{c,\alpha}_f) \to 0 \text{ in } C([0,T]; L^2), \end{aligned} \tag{3.4}$$

where $n^{c,\alpha}_f$ is the free wave defined by

$$\begin{cases} \alpha^{-2} \ddot{n}^{c,\alpha}_f - \Delta n^{c,\alpha}_f = 0, \\ n^{c,\alpha}_f(0) = n^{c,\alpha}(0) + |\mathbb{E}^\infty(0)|^2, \quad \dot{n}^{c,\alpha}_f(0) = \dot{n}^{c,\alpha}(0). \end{cases} \tag{3.5}$$

Remark 3.2. The uniform decay for high frequency (3.2) is satisfied if the data stay in a compact subset of L^2, but it also allows some part of the data to escape to the spatial infinity by translation and/or by dispersion, for example.

The main part of proof is uniform bound on the energy norm before taking the limit, and for notational ease we will suppress the superscript (c, α). Since the original energy is diverging as $c \to \infty$, we introduce a modified energy, eliminating the oscillation $e^{\pm ic^2 t}$. Then the time derivative of the modified energy has oscillatory error terms, which can be bounded by using the $X^{s,b}$ norms of the L^2 regularity and the Strichartz norm on E of the $H^{1/2}$ regularity.

Those auxiliary norms are bounded in return by using the uniform bound of energy. Here we use the interpolation inequality (2.34) to bound the Strichartz norm with 1/2 loss by the $X^{s,b}$ norm with 1 loss and the energy. Since the Strichartz norm $L^2_t B^{1/2}_{6,2}$ suffers from the logarithmic loss due to the failure of the Sobolev embedding into L^∞, we have to recover summability for the non-resonant frequency.

Once the uniform bound is derived, the strong convergence in the limit is proved by using weak compactness and convergence of the modified energy. We carry it out in the next section.

The rest of this section is organized as follows. In the first subsection, we set up the integral equation and the function spaces for the proof. In subsection 3.3, we derive the main estimates on the $X^{s,b}$ norm, the Strichartz norm and the energy norm, respectively in Lemmas 3.5, 3.6 and 3.7. Finally in subsection 3.4, we put those estimates together, and derive a uniform bound for small T and large c.

3.1. **Integral equation and function spaces.** As in [19], we rewrite (1.1) into the first order system by introducing new variables $\mathbb{E}, N$:

$$\mathbb{E} = (\mathbb{E}_1, \mathbb{E}_2) := \frac{e^{-ic^2t}}{2}(1 - ic^{-2}I_c\partial_t)(E^{c,\alpha}, \overline{E^{c,\alpha}}), \quad N := n^{c,\alpha} - i|\alpha\nabla|^{-1}\dot{n}^{c,\alpha}. \tag{3.6}$$

Remark that $\mathbb{E}$ and N depend on (c, α). The important thing is that the implicit constants in the estimates are always independent of (c, α). The original variables are given by

$$\begin{aligned} &E^{c,\alpha} = e^{ic^2t}\mathbb{E}_1 + e^{-ic^2t}\overline{\mathbb{E}_2}, \quad \dot{E}^{c,\alpha} = ic^2 I_c^{-1}(e^{ic^2t}\mathbb{E}_1 - e^{-ic^2t}\overline{\mathbb{E}_2}), \\ &n^{c,\alpha} = \operatorname{Re} N, \quad \dot{n}^{c,\alpha} = -\operatorname{Im}(|\alpha\nabla|N), \end{aligned} \tag{3.7}$$

and the equations are transformed into

$$\begin{aligned} &2i\dot{\mathbb{E}} - \Delta_c\mathbb{E} = I_c n(\mathbb{E} + \mathbb{E}^*), \quad n = \operatorname{Re} N, \\ &i\dot{N} + |\alpha\nabla|N = -|\alpha\nabla|\langle \mathbb{E}, \mathbb{E} + \mathbb{E}^*\rangle, \end{aligned} \tag{3.8}$$

where we denote $\mathbb{E}^* := e^{-2ic^2t}(\overline{\mathbb{E}_2}, \overline{\mathbb{E}_1})$.

Next we introduce notations for space-time norms, for which we will derive uniform bounds. First we fix parameters $\mu, \nu \in (0, 1/2)$ such that

$$\max(1/3 + \nu, 1/2 - \nu) < \mu < \frac{1 - \nu}{2}. \tag{3.9}$$

For instance, we can take $(\mu, \nu) = (21/48, 1/12)$. We denote the energy space by

$$\mathcal{H} := L_t^\infty H_x^1 \times L_t^\infty L_x^2. \tag{3.10}$$

We denote the $X^{s,b}$ spaces for $\mathbb{E}$ and N respectively by

$$X^{s,b} := e^{-it\Delta_c/2} H_t^b H_x^s, \quad Y^{s,b} := e^{it|\alpha\nabla|} H_t^b H_x^s. \tag{3.11}$$

Then we introduce the following specific spaces

$$\mathcal{X}' := I_c^{1-\nu} X^{0,1} + I_c c^{-2\nu} X^{0,1-\nu}, \quad \mathcal{X} := \mathcal{X}' \times I_c^{-\nu}\alpha Y^{0,1}. \tag{3.12}$$

For the Strichartz norm, we fix $\theta \in (0, 1/2)$ such that

$$\mu + \nu > 1/2 + \theta/3, \tag{3.13}$$

which is possible by (3.9). Using the notation in subsection 2.2, we define *

$$\mathcal{M} := I_c^{\mu}(\mathrm{St}_{1,1}^{1/2,2} \cap \mathrm{St}_{\theta,1}^{1/2,2}). \tag{3.14}$$

Here we chose ℓ^1 for the frequency to have the Sobolev embedding into L_x^∞:

$$\|I_c^\nu \mathbb{E}\|_{L^2L^\infty} \lesssim \|I_c^\nu \mathbb{E}\|_{L^2B^0_{\infty,1}} \lesssim \|I_c^\nu \mathbb{E}\|_{L^2B^{\theta/2}_{6/\theta,1}} \lesssim \|\mathbb{E}\|_{\mathcal{M}}, \tag{3.15}$$

where we used that $\ell_k^1 L_t^2 \subset L_t^2 \ell_k^1$ and (3.13).

To use the frequency localization of E and N, we have to extend them to the whole space time. This will create some technical problems since we also need a precise dependence on T in the estimates. Using the notation in subsection 2.3, we transform the equation (3.8) on $t \in (0,T)$ into the following integral equations which hold for $t \in \mathbb{R}$. Define a map $\Phi : (\mathbb{E}, N) \mapsto (\mathbb{E}^\sharp, N^\sharp)$ by

$$\begin{aligned} \mathbb{E}^\sharp(t) &:= e^{-it\Delta_c/2}\left[\chi(t)\mathbb{E}(0) + \frac{i}{2}\mathcal{I}_T e^{it\Delta_c/2} I_c n(\mathbb{E}+\mathbb{E}^*)\right] =: \mathbb{E}^0 + \mathbb{E}^1, \\ N^\sharp(t) &:= e^{it|\alpha\nabla|}\left[\chi(t)N(0) + i\mathcal{I}_T e^{-it|\alpha\nabla|}|\alpha\nabla|\langle \mathbb{E}, \mathbb{E}+\mathbb{E}^*\rangle\right] =: N^0 + N^1. \end{aligned} \tag{3.16}$$

Then we have

Lemma 3.3. *Let $(\mathbb{E}, N) \in C([0,T]; w\text{-}(H^1 \times L^2))$ and let $(\mathbb{E}^\sharp, N^\sharp) = \Phi(\mathbb{E}, N)$, given by* (3.16)*. Then we have $(\mathbb{E}^\sharp, N^\sharp) \in C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^3))$ and*

$$\|(\mathbb{E}^\sharp, N^\sharp)\|_{\mathcal{H}} \lesssim \|(\mathbb{E}^\sharp, N^\sharp)\|_{\mathcal{H}(0,T)}. \tag{3.17}$$

Moreover,

(i) *If $(\mathbb{E}, N)$ is a weak solution of* (3.8) *on $(0,T)$, then $(\mathbb{E}^\sharp, N^\sharp) = (\mathbb{E}, N)$ on $(0,T)$, and so $(\mathbb{E}^\sharp, N^\sharp) = \Phi(\mathbb{E}^\sharp, N^\sharp)$ on $\mathbb{R}$.*

(ii) *If $\mathbb{E} \in \mathrm{St}_0^{1,p}(0,T)$ for all $p > 2$, then $(\mathbb{E}^\sharp, N^\sharp) \in \mathcal{X}$ and $\mathbb{E}^\sharp \in \mathcal{M}$.*

In particular, if $(\mathbb{E}, N)$ is a solution of (3.8) given by [21], then $\mathbb{E} \in X^{1,1/2+} \subset \mathrm{St}_0^{1,p}(0,T)$ by (2.25), and so all the above conclusions hold.

Proof. (3.17) follows from the identity

$$g(t) := e^{it\Delta_c/2}\mathbb{E}^1(t) = \begin{cases} g(T) - g(t-T) & (T < t < 2T), \\ 0 & (t \le 0, \text{ or } 2T \le t) \end{cases}. \tag{3.18}$$

(i) is obvious, since the right hand side of (3.16) is the Duhamel formula for (3.8) on $(0,T)$, and it depends only on the values on $(0,T)$. For (ii) we have on $(0,T)$,

$$\begin{aligned} \|I_c^{-\varepsilon} N\mathbb{E}\|_{L^2L^2} &\lesssim c^\varepsilon \|N\mathbb{E}\|_{L^2L^{p'}} \\ &\lesssim c^\varepsilon \|N\|_{L^\infty L^2}\|\mathbb{E}\|_{L^2L^{3/\varepsilon}} \lesssim c^\varepsilon T^{\varepsilon/3}\|N\|_{L^\infty L^2}\|\mathbb{E}\|_{\mathrm{St}_0^{1,p}}, \end{aligned} \tag{3.19}$$

*There is no inclusion between St_1 and St_θ. We will mostly use St_θ, but St_1 is also needed for (3.34), which is used to bound the quadratic term in the derivative of the modified energy.

where $1/p = 1/2 - \varepsilon/3 = 1 - 1/p'$. Similarly, we have

$$\|I_c^{-\varepsilon}|\alpha\nabla|(\mathbb{E})^2\|_{L^2L^2} \lesssim \alpha c^{\varepsilon} T^{\varepsilon/3}\|\mathbb{E}\|_{L^\infty H^1}\|\mathbb{E}\|_{\mathrm{St}_0^{1,p}}. \tag{3.20}$$

Hence by choosing $\varepsilon = \nu/2$ and using (2.22), we deduce that $(\mathbb{E}^\sharp, N^\sharp) \in \mathcal{X}$. Then Lemma 2.4 implies that $I_c^{(\nu-1)/2}\mathbb{E}^\sharp \in \mathrm{St}_a^{1/2+\nu/4,2}$ for any $a \in (0,1]$. Since $\mu < (1-\nu)/2$ and $\nu > 0$, we get $\mathbb{E}^\sharp \in \mathcal{M}$. □

3.2. **Resonant frequency and nonresonant interactions.** We denote the non-resonant frequencies by

$$\mathbb{D}_X := \{j \in \mathbb{D} \mid |\log(j/M)| > 5\}, \quad M := \frac{2c^2}{c^2-\alpha^2}\alpha, \tag{3.21}$$

where the resonant frequency M is determined by the equation $\alpha M = \omega_c(M)$. Since $\alpha/c < 1$, we have $M \sim \alpha$. As in [19], we estimate interactions of the form $\langle \mathrm{Re}(N)E \mid F\rangle_{t,x}$ for $N \in Y^{s,b}$ and $E, F \in X^{s,b}$, splitting each function with respect to the distance from each characteristic surfaces. We define

$$\begin{aligned} &N^C = P_{|\tau-\alpha|\xi||\le\delta}N,\ E^C = P_{|\tau-\omega_c(\xi)|\le\delta}E,\ E^{*C} = P_{|\tau+\omega_c(\xi)+2c^2|\le\delta}E^*,\\ &N^F = P_{|\tau-\alpha|\xi||>\delta}N,\ E^F = P_{|\tau-\omega_c(\xi)|>\delta}E,\ E^{*F} = P_{|\tau+\omega_c(\xi)+2c^2|>\delta}E^*,\end{aligned} \tag{3.22}$$

where $\delta > 0$ will be mostly determined according to Lemma 3.4. We decompose F in the same way as E. We denote $n^F := \mathrm{Re}(N^F)$, $n^C := \mathrm{Re}(N^C)$. Notice also that $\mathbb{E}^{*C} = \mathbb{E}^{C*} = e^{-2ic^2t}(\overline{\mathbb{E}_2^C}, \overline{\mathbb{E}_1^C})$. In addition, for any interval $I \subset \mathbb{R}$, we denote

$$I^C = P_{|\tau|\le\delta}I, \quad I = I^C + I^F. \tag{3.23}$$

The nonresonance property is expressed in the following way.

Lemma 3.4. *Let $\alpha/c \le \gamma < 1$. There exists $\varepsilon_0 > 0$, depending only on γ, such that for all $j,k,l \in \mathbb{D}$,*

(i) *if $\delta \le \varepsilon_0(\alpha + (c\wedge l))l$ and either $k/\varepsilon_0 < j \in \mathbb{D}_X$ or $k/\varepsilon_0 < l \in \mathbb{D}_X$, then we have $\langle n_j^C E_k^C \mid F_l^C\rangle_{t,x} = 0 = \langle n_j^C E_k^C \mid I^C F_l^C\rangle_{t,x}$.*

(ii) *if $\delta \le \varepsilon_0(c+j+k+l)c$, then we have $\langle n_j^C E_k^{*C} \mid F_l^C\rangle_{t,x} = 0 = \langle n_j^C E_k^{*C} \mid I^C F_l^C\rangle_{t,x}$.*

For the proof we refer to [19, Lemma 5.1]. We use the $X^{s,b}$ norms for the parts far from the characteristics to gain δ^{-b}. For I^F, we have

$$\|I^F\|_{L^1(\mathbb{R})} \lesssim |I| \wedge \delta^{-1}. \tag{3.24}$$

For the proof, see [19, (6.12)].

3.3. **Main estimates.** First we estimate the $X^{s,b}$ norm $\mathcal{X}$, using mainly the Hölder inequality, and also the bilinear estimate for some interaction of frequency $\gg c$.

Lemma 3.5 ($X^{s,b}$ bound)**.** *Let $(\mathbb{E}^\sharp, N^\sharp) = \Phi(\mathbb{E}, N)$, given by* (3.16)*. Then*

$$\|(\mathbb{E}^\sharp, N^\sharp)\|_{\mathcal{X}} \lesssim \|(\mathbb{E}, N)\|_{\mathcal{H}}(1 + \|\mathbb{E}\|_{\mathcal{M}} + c^{-1/2}\|(\mathbb{E}, N)\|_{\mathcal{X}}). \tag{3.25}$$

Proof. Since the free part is trivial, it suffices to consider the nonlinear part $(\mathbb{E}^1, N^1)$. By (2.22) with $\theta = 0$ and the duality (2.18), we have

$$\|I_c^\nu N^1\|_{\alpha Y^{0,1}} \lesssim \sup\{\langle I_c^\nu|\nabla|\langle \mathbb{E}, \mathbb{E}+\mathbb{E}^*\rangle \mid u\rangle_{t,x} \mid \|u\|_{L_t^2 L_x^2} \le 1\}. \tag{3.26}$$

Decomposed into the frequency as in (2.13), the above integral is bounded by

$$\int dt \sum_{\substack{(j,k,l)\in\mathcal{T}\\ k\ge l}} \frac{\langle l/c\rangle^\nu j}{\langle j/c\rangle^\nu k}\|\mathbb{E}_k\|_{H_x^1}\|I_c^\nu \mathbb{E}_l\|_{L_x^\infty}\|u_j\|_{L_x^2} \lesssim \|\mathbb{E}\|_{L^\infty H^1}\|I_c^\nu \mathbb{E}\|_{L^2 B^0_{\infty,1}}\|u\|_{L^2L^2} \tag{3.27}$$

where we first took the summation over $\mathcal{T}$ using the ℓ_k^2, ℓ_l^1 and ℓ_j^2 of the spatial norms, and then integrated in time. Using (3.15), we obtain

$$\|N^1\|_{I_c^{-\nu}\alpha Y^{0,1}} \lesssim \|\mathbb{E}\|_{L^\infty H^1}\|\mathbb{E}\|_{\mathcal{M}}. \tag{3.28}$$

Similarly by (2.22) and (2.18), we have

$$\|\mathbb{E}^1\|_{I_c^{1-\nu}X^{0,1}+I_c c^{-2\nu}X^{0,1-\nu}} \lesssim \sup\{\langle n(\mathbb{E}+\mathbb{E}^*) \mid u\rangle_{t,x} \mid \|u\|_{I_c^\nu L_t^2L_x^2\cap c^{2\nu}X^{0,\nu}} \le 1\}, \tag{3.29}$$

where the (t,x) integral is decomposed by using (2.13),

$$\langle n(\mathbb{E}+\mathbb{E}^*) \mid u\rangle_{t,x} = \int dt \sum_{(j,k,l)\in\mathcal{T}} \langle n_j(\mathbb{E}_k+\mathbb{E}_k^*) \mid u_l\rangle_x. \tag{3.30}$$

Let $m := \min(j,k,l)$ and $h := \max(j,k,l)$. The x integral can be bounded by

$$\langle l/c\rangle^{-\nu}\langle k/c\rangle^\nu (m/k)^{\theta/2}\|N_j\|_{L_x^2}\|\mathbb{E}_k\|_{I_c^{-\nu}B^{\theta/2}_{6/\theta,2}}\|u_l\|_{I_c^\nu L_x^2}. \tag{3.31}$$

Next we consider the summation over $\mathcal{T}$. In view of the spaces $\mathcal{H}$ and $\mathcal{M}$, we may take ℓ_j^2, ℓ_k^1 and ℓ_l^2 on the space norms of N_j, $\mathbb{E}_k$ and u_l, respectively. If $m = k \le j \sim l$, then the coefficient is bounded, and summability for $m = k$ is provided by the norm of $\mathbb{E}_k$, and for $h \sim l \sim j$ by the norms of N_j and u_l. If $m = j$ or $m = l \le k \sim j \lesssim c$, then the coefficient is bounded by $(m/h)^{\theta/2}$, which gives summability for m, while that for h comes from two of the norms of n_j, $\mathbb{E}_k$ and u_l which we put in ℓ^2. Hence, in these three cases, (3.30) is bounded by

$$\|N_j\|_{L_t^\infty \ell_j^2 L_x^2}\|\mathbb{E}_k\|_{L_t^2\ell_k^1 I_c^{-\nu}B^{\theta/2}_{6/\theta,2}}\|u_l\|_{L_t^2\ell_l^2 I_c^\nu L_x^2} \lesssim \|N\|_{L^\infty L^2}\|\mathbb{E}\|_{\mathcal{M}}, \tag{3.32}$$

where we used the Hölder in t.

In the remaining case, namely $l + c \ll j \sim k$, the coefficient is not bounded, since $\theta/2 < \nu$ by (3.13) and the right inequality in (3.9). If we allow to lose some I_c, then the integral in (3.30) can be bounded by

$$\langle l/c\rangle^{\mu-5/6}\langle k/c\rangle^{5/6-\mu}(l/k)^{1/2}\|N_j\|_{L_x^2}\|\mathbb{E}_k\|_{I_c^{\mu-5/6}B^{1/2}_{6,2}}\|u_l\|_{I_c^{\mu-5/6}L_x^2}, \tag{3.33}$$

where the coefficient is summable for l because $\mu > 1/3$. In particular, we have

$$\|I_c^{5/6-\mu}N\mathbb{E}\|_{L^2L^2} \lesssim \|N\|_{L^\infty L^2}\|\mathbb{E}\|_{\mathcal{M}}. \tag{3.34}$$

This is useful in the later nonlinear energy estimate, but not sufficient to close all the estimates in $\mathcal{X}$, $\mathcal{M}$ and $\mathcal{H}$. To recover summability, we utilize its non-resonant property with distance $\delta \sim ch = ck$ from the characteristics. By Lemma 3.4, we can decompose the above (t,x) integral as

$$\langle n_j^F E_k \mid u_l \rangle_{t,x} + \langle n_j^C E_k^F \mid u_l^C \rangle_{t,x} + \langle n_j^C E_k \mid u_l^F \rangle_{t,x} =: A_1 + A_2 + A_3, \tag{3.35}$$

where we denote both $\mathbb{E}_k$ and $\mathbb{E}_k^*$ by E_k, since they have the same non-resonance property in this case. Hence, each integral is bounded by

$$\begin{aligned} A_1 &\lesssim (ch)^{-1}\alpha(j/c)^\nu k^{-1} l^{3/2} \|N_j\|_{I_c^{-\nu}\alpha Y^{0,1}} \|\mathbb{E}_k\|_{L^\infty H^1} \|u_l\|_{I_c^\nu L^2 L^2}, \\ A_2 &\lesssim (ch)^{-1}(k/c)^{-1+\nu} l^{3/2} \|N_j\|_{L^\infty L^2} \|\mathbb{E}_k\|_{I_c^{1-\nu}X^{0,1}+c^{-2\nu}I_c X^{0,1-\nu}} \|u_l\|_{I_c^\nu L^2 L^2}, \\ A_3 &\lesssim (ch)^{-\nu}(k/c)^\nu (l/k)^{\theta/2} c^{2\nu} \|N_j\|_{L^\infty L^2} \|\mathbb{E}_k\|_{I_c^{-\nu} L^2 B^{\theta/2}_{6/\theta,2}} \|u_l\|_{c^{2\nu}X^{0,\nu}}, \end{aligned} \tag{3.36}$$

where the powers of ch come from δ, and those of m from the Sobolev embedding. For $l + c \ll k \sim j$, the coefficient on each line is bounded respectively by

$$c^{-\nu}\frac{\alpha}{c}h^{-2+\nu}l^{3/2}, \quad c^{-\nu}h^{-2+\nu}l^{3/2}, \quad (l/h)^{\theta/2}. \tag{3.37}$$

The first two are summable for (l,h) in the region $l + c \ll h \sim j \sim k$, and the sum is bounded by $c^{-1/2}$. The last one is summable only for l and the sum is bounded by 1, hence we use the summability for k of the norm on $\mathbb{E}_k$. Thus we obtain

$$\|\mathbb{E}^1\|_{\mathcal{X}'} \lesssim \|(\mathbb{E},N)\|_{\mathcal{H}}(\|\mathbb{E}\|_{\mathcal{M}} + c^{-1/2}\|(\mathbb{E},N)\|_{\mathcal{X}}). \tag{3.38}$$

Gathering all the estimates, we obtain (3.25). □

Next we estimate $\mathbb{E}$ in $\mathcal{M}$, using the interpolation Lemma 2.4 and also the non-resonant bilinear estimate for the frequency $\mathbb{D}_X \ni k \lesssim c$.

Lemma 3.6 (Strichartz bound)**.** *Let* $(\mathbb{E}^\sharp, N^\sharp) = \Phi(\mathbb{E}, N)$ *and* $\mathbb{E}^\sharp = \mathbb{E}^0 + \mathbb{E}^1$, *given by* (3.16)*. Then*

$$\begin{aligned} &\|\mathbb{E}^0\|_{\mathcal{M}} \lesssim \|\mathbb{E}(0)\|_{H^1}, \quad \|\mathbb{E}^0\|_{\mathcal{M}(0,T)} \lesssim (T^{1/4} + c^{-1/2})\|\mathbb{E}(0)\|_{H^1}, \\ &\|\mathbb{E}^1\|_{\mathcal{M}} \lesssim (T^{1/4} + c^{-1/2})\|\mathbb{E}^\sharp\|_{L^\infty H^1 \cap \mathcal{X}'} \\ &\qquad + T^{1/4}\|(\mathbb{E},N)\|_{\mathcal{H}}\big[\|(\mathbb{E},N)\|_{\mathcal{H}\cap\mathcal{X}} + \|\mathbb{E}\|_{\mathcal{M}}\big]. \end{aligned} \tag{3.39}$$

Proof. Using the real interpolation, we have for any $a \in (0,1]$ and $I \subset \mathbb{R}$,

$$\begin{aligned} \|\mathbb{E}^0\|_{I_c^\mu \mathrm{St}^{1/2,2}_{a,1}(I)} &\lesssim \|\mathbb{E}^0_{\le 2c}\|_{(\mathrm{St}^{1,2}_1, \mathrm{St}^{0,2}_1)_{1/2,1}(I)} + \sum_{c \le k \in \mathbb{D}} (k/c)^\mu k^{-1/2} \|\mathbb{E}^0_k\|_{\mathrm{St}^{1,2}_a(I)} \\ &\lesssim \|\mathbb{E}^0\|^{1/2}_{\mathrm{St}^{1,2}_1} \|\mathbb{E}^0\|^{1/2}_{L^2_t L^6_x(I)} + c^{-1/2}\|\mathbb{E}^0_{>c}\|_{\mathrm{St}^{1,2}_{a,\infty}(I)} \end{aligned} \tag{3.40}$$

where we have used that in the frequency $\le c$, St_1 dominates the other St_θ by the Sobolev embedding. Then we get the desired bounds on $\mathbb{E}^0$ for $I = (0,T)$ and $I = \mathbb{R}$ by the Strichartz estimate and the Sobolev embedding $H^1_x \subset L^6_x$.

Next we estimate $\sum_{l>c}\|\mathbb{E}^1_l\|_{\mathcal{M}}$ and $\sum_{l<c}\|\mathbb{E}^1_l\|^2_{\mathcal{M}}$ by interpolation between $\mathcal{X}$ and $\mathcal{H}$ as follows. First there is a canonical splitting for the sum space in $\mathcal{X}$, namely

$$\mathbb{E}^1=\mathbb{E}^2+\mathbb{E}^3,\quad \widetilde{\mathbb{E}}^2(\tau,\xi)=\widetilde{\mathbb{E}}^1(\tau,\xi)\{|\tau-\omega_c(\xi)|<c|\xi|\}. \tag{3.41}$$

Then we have

$$\begin{aligned}&\|\mathbb{E}^2\|_{L^2H^1}+\|\mathbb{E}^3\|_{L^2H^1}\lesssim\|\mathbb{E}^1\|_{L^2H^1}\le T^{1/2}\|\mathbb{E}^1\|_{L^\infty H^1},\\&\|I_c^{-1+\nu}\mathbb{E}^2\|_{X^{0,1}}+\|I_c^{-1+\nu}\mathbb{E}^3\|_{c^{-\nu}X^{\nu,1-\nu}}\lesssim\|\mathbb{E}^1\|_{\mathcal{X}'},\end{aligned} \tag{3.42}$$

where we used the embedding $c^{-\nu}I_c^{\nu}X^{0,1-\nu}\subset X^{\nu,1-\nu}$. Since $1-\nu>1/2$, we can use Lemma 2.4 for $\mathbb{E}^2$ and $\mathbb{E}^3$, deriving

$$\begin{aligned}\|I_c^{(-1+\nu)/2}\mathbb{E}^1\|_{\mathrm{St}^{1/2,2}_{a,2}}&\lesssim\|\mathbb{E}^2\|^{1/2}_{L^2H^1}\|I_c^{-1+\nu}\mathbb{E}^2\|^{1/2}_{X^{0,1}}+\|\mathbb{E}^3\|^{1-b}_{L^2H^1}\|I_c^{-1+\nu}\mathbb{E}^3\|^{b}_{X^{\nu,1-\nu}}\\&\lesssim(T^{1/4}+T^{(1-b)/2}c^{-b\nu})\|\mathbb{E}^1\|_{L^\infty H^1\cap\mathcal{X}'},\end{aligned} \tag{3.43}$$

for any $a\in(0,1]$, where we set $b=1/(2-2\nu)\in(1/2,1)$. Hence, we have

$$\|\mathbb{E}^1_{\le c}\|_{\mathrm{St}^{1/2,2}_{1,2}}+\|\mathbb{E}^1_{>c}\|_{\mathcal{M}}\lesssim(T^{1/4}+c^{-1/2})\|\mathbb{E}^1\|_{L^\infty H^1\cap\mathcal{X}'}, \tag{3.44}$$

where we used the condition $\mu<(1-\nu)/2$ in (3.9) for the summability in the frequencies $>c$.

Thus it remain to bound $\mathbb{E}^1_l$ for $\mathbb{D}_X\ni l\le c$[†]. Indeed, the resonant frequencies $l\notin\mathbb{D}_X$ have a finite number and hence the above ℓ^2 bound controls the ℓ^1 norm. By the Strichartz estimate, we have

$$\|\mathbb{E}^1_l\|_{\mathrm{St}^{1/2,2}_1}\lesssim\|(0,T)(nE)_l\|_{L^1B^{1/2}_{2,1}+L^{4/3}B^{1/2}_{3/2,1}+X^{1/2,-1/2+\varepsilon}}. \tag{3.45}$$

Hence, by duality and (2.13), it is enough to estimate

$$\sum_{\substack{(j,k,l)\in\mathcal{T}\\ \mathbb{D}_X\ni l\le c}}\langle n_jE_k\,|\,(0,T)u_l\rangle_{t,x}, \tag{3.46}$$

for $E=\mathbb{E}$ and $\mathbb{E}^*$, and for all $u\in C_0^\infty(\mathbb{R}^4)$ satisfying

$$\sup_l\|u_l\|_{L^\infty H^{-1/2}\cap L^4B^{-1/2}_{3,\infty}}+\|u\|_{X^{-1/2-\varepsilon,1/2-\varepsilon}}\le 1. \tag{3.47}$$

For the summation on $l\lesssim j\sim k$, we bound the (t,x) integral in (3.46) by

$$(l/k)^{1/2}\langle k/c\rangle^{5/6-\mu}\|N_j\|_{L^\infty_tL^2_x}\|k^{1/2}\mathbb{E}_k\|_{I_c^{\mu-5/6}L^2_tL^6_x}T^{1/4}\|l^{-1/2}u_l\|_{L^4_tL^3_x}. \tag{3.48}$$

Since $\mu>1/3$, we can bound the coefficient as

$$\sup_k\sum_{l\lesssim c\wedge k}(l/k)^{1/2}\langle k/c\rangle^{5/6-\mu}\lesssim\sup_k\langle k/c\rangle^{-1/2+5/6-\mu}\le 1, \tag{3.49}$$

[†]As the following argument shows, we have actually better bound for the non-resonant frequency. For example, we can derive the same bound for the weighted norm $\|w(k)k^{1/2}\mathbb{E}_{\le c}\|_{L^2L^6}$, where $w(k)^4=\min(\max(k,\alpha/k),\max(k/\alpha,c/k))$.

so the summation for $l \lesssim j \sim k$ in (3.46) is bounded by using the Hölder in (j,k,l)

$$\lesssim T^{1/4}\|N_j\|_{\ell_j^\infty L_t^\infty L_x^2}\|\mathbb{E}_k\|_{\ell_k^1 I_c^\mu \mathrm{St}_1^{1/2,2}}\|u_l\|_{\ell_l^\infty L_t^4 B_{3,\infty}^{-1/2}} \lesssim T^{1/4}\|N\|_{L^\infty L^2}\|\mathbb{E}\|_{\mathcal{M}}. \tag{3.50}$$

The case $j < k \sim l (\le c)$ is treated similarly, but we sum for j first, and then integrate in space and time. Hence, (3.46) for this part is bounded by

$$\sum_{k\sim l\le c} T^{1/4}\|N_{<k}\|_{L^\infty L^2}\|k^{1/2}\mathbb{E}_k\|_{L^2L^6}\|l^{-1/2}u_l\|_{L_t^4L^3}, \tag{3.51}$$

Then, we take the summation in k and deduce that the contribution in (3.46) enjoys the same bound as (3.50).

In the remaining case, namely $k < j \sim l \le c$, we use the bilinear estimate with the non-resonant distance $\delta \sim (\alpha + j)j$, decomposing the space-time integral as

$$\begin{aligned}\langle n_j E_k \mid I u_l\rangle_{t,x} =& \langle n_j^F E_k \mid (0,T) I_c u_l\rangle_{t,x} + \langle n_j^C E_k^F \mid (0,T) I_c u_l\rangle_{t,x} \\ &+ \langle n_j^C E_k^C \mid (0,T) I_c u_l^F\rangle_{t,x} + \langle n_j^C E_k^C \mid (0,T)^F I_c u_l^C\rangle_{t,x} \\ =:& B_1 + B_2 + B_3 + B_4,\end{aligned} \tag{3.52}$$

by Lemma 3.4. We estimate the first three terms by

$$\begin{aligned} B_1 &\lesssim (\alpha j)^{-1}\alpha l^{1-\varepsilon}\|N_j^F\|_{\alpha Y^{0,1}}\|\mathbb{E}_k\|_{L^\infty H^1}\|u_l\|_{L^2B_{3,2}^{-1+\varepsilon}}, \\ B_2 &\lesssim (j^2)^{-1} k l^{1-\varepsilon}\|N_j^C\|_{L^\infty L^2}\|\mathbb{E}_k^F\|_{X^{0,1}+c^{-2\nu}X^{0,1-\nu}}\|u_l\|_{L^2B_{3,2}^{-1+\varepsilon}} \\ B_3 &\lesssim (j^2)^{-1/2+\varepsilon} k^{1-3/4} l^{3/4-3\varepsilon} T^{1/4}\|N_j^C\|_{L^\infty L^2}\|\mathbb{E}_k^C\|_{L^4B_{3,2}^{3/4}}\|u_l^F\|_{X^{-3/4+3\varepsilon,1/2-\varepsilon}},\end{aligned} \tag{3.53}$$

where we choose $\varepsilon \in (0, 1/12)$, and the first factor on each line is coming from δ. Since the coefficient is summable on $k \lesssim j \sim l$, it suffices to bound the norms on N_j, $\mathbb{E}_k$ and u_l. The norms on u_l have regularity room at least $1/4 - 4\varepsilon$. By using the above interpolation argument together with another interpolation $[L^\infty H^1, L^2 B_{6,2}^{1/2}]_{1/2} = L^4 B_{3,2}^{3/4}$, we have for $k \lesssim c$,

$$\|\mathbb{E}_k^C\|_{L^4B_{3,2}^{3/4}} \lesssim \|\mathbb{E}_k^C\|_{L^\infty H^1}^{1/2}\|\mathbb{E}_k^C\|_{L^2B_{6,2}^{1/2}}^{1/2} \lesssim \|\mathbb{E}_k^C\|_{L^\infty H^1}^{3/4}\|\mathbb{E}_k^C\|_{\mathcal{X}'}^{1/4}. \tag{3.54}$$

For B_4, we use (3.24), the Hölder and the Sobolev inequalities

$$B_4 \lesssim (T \wedge j^{-2}) k^{1/2} l \|n_j^C\|_{L^\infty L^2}\|\mathbb{E}_k^C\|_{L^\infty H^1}\|u_l^C\|_{L^\infty H^{-1}}, \tag{3.55}$$

where we have $1/2$ regularity room for u_l, and the sum of the coefficient for $k \ll l \sim j$ is bounded by $T^{1/4}$.

Thus we can bound the summation of (3.53). Gathering all the terms, we obtain

$$\sum_{\mathbb{D}_X \ni l \le c}\|\mathbb{E}_l^1\|_{\mathrm{St}_1^{1/2,2}} \lesssim T^{1/4}\|(\mathbb{E},N)\|_{\mathcal{H}}\big[\|(\mathbb{E},N)\|_{\mathcal{H}\cap\mathcal{X}} + \|\mathbb{E}\|_{\mathcal{M}}\big]. \tag{3.56}$$

Adding the previously estimated parts, we get (3.39). □

Finally we estimate the energy norm by using a modified nonlinear energy.

Lemma 3.7 (Energy bound). *Assume that* $(\mathbb{E}^\flat, N^\flat)$ *solves* (3.8) *on* $(0,T)$. *Let* $(\mathbb{E}, N) = \Phi(\mathbb{E}^\flat, N^\flat)$ *given by* (3.16), *and* $H_S := \sup_{0\le t\le S} \|(\mathbb{E}(t), N(t))\|_{H^1\times L^2}$. *Then*

$$\begin{aligned} H_T \lesssim{} & H_0 + H_0^2 + T^{1/3} H_T^{7/3} \\ & + \|\mathbb{E}\|_{\mathcal{M}(0,T)}^2 H_T + \|\mathbb{E}\|_{\mathcal{M}(0,T)} \|(\mathbb{E}, N)\|_{\mathcal{X}} + c^{-1/2} \|(\mathbb{E}, N)\|_{\mathcal{H}\cap\mathcal{X}}^2. \end{aligned} \tag{3.57}$$

Remark 3.8. This estimate concerns the true nonlinear solutions, in contrast with the preceeding lemmas 3.5 and 3.6, which are essentially iterative. The difference is reflected by the notation $(\mathbb{E}, N) = \Phi(\mathbb{E}^\flat, N^\flat)$ in the above and $(\mathbb{E}^\sharp, N^\sharp) = \Phi(\mathbb{E}, N)$ in the previous lemmas 3.5 and 3.6. In practice, we start with a local solution $(\mathbb{E}^\flat, N^\flat)$ and define the extended solutions iteratively by

$$(\mathbb{E}, N) := \Phi(\mathbb{E}^\flat, N^\flat), \quad (\mathbb{E}^\sharp, N^\sharp) := \Phi(\mathbb{E}, N). \tag{3.58}$$

Then $(\mathbb{E}^\sharp, N^\sharp) = (\mathbb{E}, N)$ by Lemma 3.3. We distinguished them in the previous lemmas in order to emphasize their iterative nature with minimal assumptions.

Proof. First we recall that $\|(\mathbb{E}, N)\|_{\mathcal{H}} \sim H_T$ and $(\mathbb{E}, N) = (\mathbb{E}^\flat, N^\flat)$ on $(0,T)$ by Lemma 3.3. Inner multiplying the first equation of (3.8) with $2I_c^{-1}(\dot{\mathbb{E}} + i\mathbb{E})$, and the second one with $|\alpha\nabla|^{-1}\dot N$ and adding the real parts, we get

$$\begin{aligned} \partial_t \left[\mathcal{E}(t) + \langle n\mathbb{E} \mid \mathbb{E}\rangle_x\right] &= -\langle n\mathbb{E}^* \mid 2\dot{\mathbb{E}}\rangle_x - \langle \dot n\mathbb{E}^* \mid \mathbb{E}\rangle_x - \langle n\mathbb{E}^* \mid 2i\mathbb{E}\rangle_x \\ &= \langle in\mathbb{E}^* \mid (2-\Delta_c)\mathbb{E} + I_c n\mathbb{E})\rangle_x - \langle (\operatorname{Re} i|\alpha\nabla|N)\mathbb{E}^* \mid \mathbb{E}\rangle_x, \end{aligned} \tag{3.59}$$

where we denote the linear part of energy by

$$\mathcal{E}(t) := \langle I_c^{-1}(2-\Delta_c)\mathbb{E} \mid \mathbb{E}\rangle_x + \frac{1}{2}\|N\|_{L_x^2}^2 \sim \|\mathbb{E}\|_{H_x^1}^2 + \|N\|_{L_x^2}^2, \tag{3.60}$$

The last trilinear term in (3.59) has a stronger Fourier multiplier than the other one when N has frequency $\ll \alpha$, for which we need to integrate by parts. Denoting

$$P_\alpha f := \frac{|\alpha\nabla|}{2c^2} \operatorname{Re} f_{<\alpha}, \tag{3.61}$$

we have

$$\begin{aligned} & \langle (\operatorname{Re} i|\alpha\nabla|N_{<\alpha})\mathbb{E}^* \mid \mathbb{E}\rangle_x \\ &= \partial_t \langle (P_\alpha iN)\mathbb{E}^* \mid i\mathbb{E}\rangle_x - \langle (P_\alpha i\dot N)\mathbb{E}^* \mid i\mathbb{E}\rangle_x - \langle (P_\alpha iN)\mathbb{E}^* \mid 2i\dot{\mathbb{E}}\rangle_x, \end{aligned} \tag{3.62}$$

and the last two terms are equal to

$$\langle P_\alpha|\alpha\nabla|(n + \langle \mathbb{E}, \mathbb{E}+\mathbb{E}^*\rangle) \mid \langle i\mathbb{E}, \mathbb{E}^*\rangle\rangle_x - \langle (P_\alpha iN)\mathbb{E}^* \mid \Delta_c\mathbb{E} - I_c n\mathbb{E}\rangle_x. \tag{3.63}$$

Thus we obtain

$$\begin{aligned} & \partial_t \left[\mathcal{E}(t) + \langle n\mathbb{E} \mid \mathbb{E}\rangle_x - \langle (P_\alpha iN)\mathbb{E}^* \mid i\mathbb{E}\rangle_x\right] \\ &= \langle in\mathbb{E}^* \mid (2-\Delta_c)\mathbb{E} + I_c n\mathbb{E}\rangle_x - \langle (\operatorname{Re} i|\alpha\nabla|N_{\ge\alpha})\mathbb{E}^* \mid \mathbb{E}\rangle_x \\ &\quad + \langle P_\alpha|\alpha\nabla|(n + \langle \mathbb{E}, \mathbb{E}+\mathbb{E}^*\rangle) \mid \langle i\mathbb{E}, \mathbb{E}^*\rangle\rangle_x - \langle (P_\alpha iN)\mathbb{E}^* \mid \Delta_c\mathbb{E} - I_c n\mathbb{E}\rangle_x. \end{aligned} \tag{3.64}$$

In order to derive the uniform bound on the energy norm, it suffices to dominate those error terms as well as the trilinear part on the left, which is bounded by

$$\|N\|_{L^2_x}\|(\mathbb{E})^2\|_{L^2_x} \lesssim \|N\|_{L^2_x}\|\mathbb{E}\|_{L^4_x}^2 \lesssim H_T\|\mathbb{E}(0)\|_{H^{3/4}}^2 + H_T^{8/3}\|\mathbb{E}^1\|_{H_x^{-1/2}}^{1/3}, \tag{3.65}$$

where we used $[H^{-1/2}, H^1]_{5/6} = H^{3/4} \subset L^4$. The $H^{-1/2}$ norm is estimated by using the equation

$$\|\mathbb{E}^1\|_{L^\infty H^{-1/2}} \lesssim \|n\mathbb{E}\|_{L^1 H^{-1/2}(0,T)} \lesssim T\|n\|_{L^\infty L^2}\|\mathbb{E}\|_{L^\infty H^1}. \tag{3.66}$$

Thus we obtain

$$|\langle n\mathbb{E} \,|\, \mathbb{E}\rangle_x| + |\langle (P_\alpha iN)\mathbb{E}^* \,|\, i\mathbb{E}\rangle_x| \lesssim H_T H_0^2 + T^{1/3}H_T^{10/3}. \tag{3.67}$$

Next we estimate the time integral of the error terms on $I := [0, T_1]$, for any $T_1 \in (0,T)$. Thanks to (3.34), we can dominate the quartic terms including N by

$$\|I_c^{1/2}N\mathbb{E}\|_{L^2_t L^2_x(0,T)}^2 \lesssim \|N\|_{L^\infty L^2(0,T)}^2\|\mathbb{E}\|_{\mathcal{M}(0,T)}^2, \tag{3.68}$$

since $5/6 - \mu < 1/2$. Those without N are bounded by

$$\|I_c^\nu(\mathbb{E})^2\|_{L^2_t H^1_x(0,T)}^2 \lesssim \|\mathbb{E}\|_{L^\infty H^1}^2\|I_c^\nu\mathbb{E}\|_{L^2 B^0_{\infty,1}(0,T)}^2 \lesssim H_T^2\|\mathbb{E}\|_{\mathcal{M}(0,T)}^2, \tag{3.69}$$

since $P_\alpha|\alpha\nabla| \lesssim I_c^{2\nu}|\nabla|^2$ on L^2_x.

The trilinear terms are of the following form:

$$\sum_{(j,k,l)\in\mathcal{T}} \langle g_j F_k \,|\, E_l I\rangle_{t,x}, \tag{3.70}$$

where $I = (0,T_1) \subset (0,T)$, $g = M_1 N$ or $g = M_1\overline{N}$, $F = \mathbb{E}^*$ and $E = M_2\mathbb{E}$ with some Fourier multipliers M_a, $a = 1,2$, which are bounded on any L^p with the norm

$$\prod_{a=1}^{2}\|M_a\|_{\mathcal{L}(L^p)} \lesssim h\min(c,h), \quad h := \max(j,k,l). \tag{3.71}$$

We further decompose it by the distance $\delta = \varepsilon c(c+h)$ from the characteristics:

$$\begin{aligned}\sum_{(j,k,l)\in\mathcal{T}} \Big[&\langle g_j^F F_k \,|\, E_l I\rangle_{t,x} + \langle g_j^C F_k \,|\, E_l^F I\rangle_{t,x} + \langle g_j^C F_k^F \,|\, E_l I\rangle_{t,x} \\ &- \langle g_j^C F_k^{F*} \,|\, E_l^F I\rangle_{t,x} + \langle g_j^C F_k^C \,|\, E_l^C I^F\rangle_{t,x}\Big], \\ =:& K_1 + K_2 + K_3 + K_4 + K_5,\end{aligned} \tag{3.72}$$

by Lemma 3.4. First, we estimate K_1. We assume that $l \le k$, otherwise we change the role of l and k. Hence, $k \sim h$ and

$$\begin{aligned}|K_1| &\lesssim \int_0^T dt \sum_{\substack{(j,k,l)\in\mathcal{T}\\ k\sim h}} w_1(j,l,h)\|\delta N_j^F\|_{I_c^{-\nu}\alpha L^2_x}\|l^{1/2}\mathbb{E}_l\|_{I_c^{\mu-6/5}L^6_x}\|\mathbb{E}_k\|_{H^1_x} \\ &\lesssim \sum_{\substack{(j,k,l)\in\mathcal{T}\\ k\sim h}} w_1(j,l,h)\|N_j\|_{I_c^{-\nu}\alpha Y^{0,1}}\|\mathbb{E}_l\|_{\mathcal{M}(0,T)}\|\mathbb{E}_k\|_{L^\infty H^1},\end{aligned} \tag{3.73}$$

where we put

$$w_1(j,l,h) := h(c \wedge h)\delta^{-1}\alpha\langle j/c\rangle^{\nu}(m/l)^{1/2}\langle l/c\rangle^{5/6-\mu}h^{-1}. \tag{3.74}$$

For $m = j \le l \sim k$, we have

$$\begin{aligned}\sum_h \sum_{j\le h} w_1(j,l,h) &\lesssim \sum_h \sum_{j\le h} \frac{\langle j/c\rangle^{\nu}(j/h)^{1/2}}{\langle c/h\rangle\langle h/c\rangle^{1/6+\mu}} \\ &\lesssim \sum_h \langle c/h\rangle^{-1}\langle h/c\rangle^{-1/6-\mu+\nu} \lesssim 1,\end{aligned} \tag{3.75}$$

and we can bound the norms of $N_j, \mathbb{E}_l, \mathbb{E}_k$ by those of N and $\mathbb{E}$. For $m = l \le j \sim k$, we have

$$\sum_h \sup_{l\le h} w_1(j,l,h) \lesssim \sum_h \langle c/h\rangle^{-1}\langle h/c\rangle^{-1/6-\mu+\nu} \lesssim 1 \tag{3.76}$$

since $\mu + 1/6 - \nu > 0$. To get summability, we use ℓ_l^1 for $\|\mathbb{E}_l\|_{\mathcal{M}(0,T)}$. Hence, we obtain

$$|K_1| \lesssim H_T \|\mathbb{E}\|_{\mathcal{M}(0,T)}\|(\mathbb{E},N)\|_{\mathcal{X}}. \tag{3.77}$$

Similarly, $|K_2|$ is bounded by

$$\begin{aligned}&\int_0^T dt \sum_{(j,k,l)\in\mathcal{T}} w_2(j,k,l)\|N_j\|_{L_x^2}\|k^{1/2}\mathbb{E}_k\|_{I_c^{\mu-6/5}L_x^6}\|I_c^{-1}\delta\mathbb{E}_l^F\|_{I_c^{-\nu}L_x^2+\delta^{\nu}c^{-2\nu}L_x^2} \\ &\lesssim \sum_{(j,k,l)\in\mathcal{T}} w_2(j,k,l)\|N_j\|_{L^\infty L^2}\|\mathbb{E}_k\|_{I_c^{\mu}\mathrm{St}_1^{1/2,2}(0,T)}\|\mathbb{E}_l\|_{I_c^{1-\nu}X^{0,1}+c^{-2\nu}I_cX^{0,1-\nu}}, \\ &w_2(j,k,l) := h(c\wedge h)\delta^{-1}\langle h/c\rangle^{\nu}(m/k)^{1/2}\langle k/c\rangle^{5/6-\mu}\langle l/c\rangle^{-1},\end{aligned} \tag{3.78}$$

where the contribution from the second member of the sum space $I_c^{-\nu}L_x^2 + \delta^{\nu}c^{-2\nu}L_x^2$ is bigger than the first one. If $m = j \le l \sim k$, then we have

$$\begin{aligned}\sum_h \sum_{j\le h} w_2 &\lesssim \sum_h \langle c/h\rangle^{-2}\sum_{j\le h}(j/h)^{1/2}\langle h/c\rangle^{-1/6-\mu+\nu} \\ &\lesssim \sum_h \langle c/h\rangle^{-2}\langle h/c\rangle^{-1/6-\mu+\nu} \lesssim 1,\end{aligned} \tag{3.79}$$

and we can bound the norms of $N_j, \mathbb{E}_l, \mathbb{E}_k$ by those of N and $\mathbb{E}$. If $m = k$, then we have

$$\begin{aligned}\sum_h \sup_{k\le h} w_2 &\lesssim \sum_h \langle c/h\rangle^{-2}\sup_{k\le h}\langle k/c\rangle^{5/6-\mu}\langle h/c\rangle^{-1+\nu} \\ &\lesssim \sum_h \langle c/h\rangle^{-2}\langle h/c\rangle^{-1/6-\mu+\nu} \lesssim 1\end{aligned} \tag{3.80}$$

and we get the summability using ℓ^1_k for $\|\mathbb{E}_k\|_{I^\mu_c \mathrm{St}^{1/2,2}_1}$. If $m = l$, then we have

$$\begin{aligned}\sum_h \sum_{l\le h} w_2 &\lesssim \sum_h \langle c/h\rangle^{-2} \sum_{l\le h} (l/h)^{1/2} \langle h/c\rangle^{5/6-\mu+\nu} \langle l/c\rangle^{-1} \\ &\lesssim \sum_h \langle c/h\rangle^{-2} \langle h/c\rangle^{1/3+\nu-\mu} \lesssim 1,\end{aligned} \tag{3.81}$$

where the first summand attains the maximum around $l \sim c \wedge h$. Thus we obtain

$$|K_2| \lesssim H_T \|\mathbb{E}\|_{\mathcal{M}(0,T)} \|(\mathbb{E}, N)\|_{\mathcal{X}}. \tag{3.82}$$

K_3 satisfies the same estimate thanks to the symmetry. $|K_4|$ is bounded by

$$\begin{aligned}&\int dt \sum_{(j,k,l)\in\mathcal{T}} w_4(j,k,l) \|N_j\|_{L^2_x} \|\delta\langle h/c\rangle^{-\nu} \mathbb{E}^F_k\|_{I_c L^2_x} \|\delta\langle h/c\rangle^{-\nu} \mathbb{E}^F_l\|_{I_c L^2_x} \\ &\lesssim \sum_{(j,k,l)\in\mathcal{T}} w_4(j,k,l) \|N\|_{L^\infty L^2} \|\mathbb{E}\|^2_{I^{1-\nu}_c X^{0,1} + c^{-2\nu} I_c X^{0,1-\nu}},\end{aligned} \tag{3.83}$$

where we put $w_4(j,k,l) := h(c\wedge h)\delta^{-2}\langle h/c\rangle^{2\nu} \langle k/c\rangle^{-1} \langle l/c\rangle^{-1} m^{3/2}$. Hence

$$\sum_h \sum_{m\le h} w_4(j,k,l) \lesssim \sum_h h(c\wedge h)\delta^{-2}\langle h/c\rangle^{2\nu-2} h^{3/2} \lesssim c^{-1/2}, \tag{3.84}$$

where the sum over h converges since $\nu < 3/4$. Thus we obtain

$$|K_4| \lesssim c^{-1/2} H_T \|(\mathbb{E}, N)\|^2_{\mathcal{X}}. \tag{3.85}$$

Finally, $|K_5|$ is bounded by

$$\sum_{(j,k,l)\in\mathcal{T}} w_5(j,k,l) \|N_j\|_{L^\infty L^2_x} \|\mathbb{E}_k\|_{L^\infty H^1_x} \|\mathbb{E}_l\|_{L^\infty H^1_x} \lesssim \sum_{(j,k,l)\in\mathcal{T}} w_5(j,k,l) \mathcal{H}^3, \tag{3.86}$$

where $w_5(j,k,l) := h(c\wedge h)k^{-1}l^{-1}m^{3/2}\|I^F\|_{L^1}$. Using (3.24), we have

$$\sum_h \sum_{m\le h} w_5(j,k,l) \lesssim \sum_h (c\wedge h) h^{1/2} \delta^{-1} \lesssim c^{-1/2}. \tag{3.87}$$

Thus we obtain

$$|K_5| \lesssim c^{-1/2} H_T^3. \tag{3.88}$$

Gathering all the terms, we get

$$\begin{aligned}H_T^2 &\lesssim H_0^2 + H_0^2 H_T + T^{1/3} H_T^{10/3} + H_T^2 \|\mathbb{E}\|^2_{\mathcal{M}(0,T)} \\ &\quad + H_T \|\mathbb{E}\|_{\mathcal{M}(0,T)} \|(\mathbb{E}, N)\|_{\mathcal{X}} + c^{-1/2} H_T \|(\mathbb{E}, N)\|^2_{\mathcal{H}\cap\mathcal{X}},\end{aligned} \tag{3.89}$$

which implies (3.57). □

3.4. **Concluding uniform estimate.** Let $(\mathbb{E}^\flat, N^\flat)$ be a solution of (3.8) on $(0,\widetilde{T})$ for some $\widetilde{T} > 0$, given by [21]. $\widetilde{T}$ may depend on (c,α) and the solution itself; indeed we do not care about the size of $\widetilde{T}$ here. We will prove a uniform bound on the energy norm in terms of the initial norm only, which is *a priori* valid as long as $\widetilde{T}$ and $1/c$ are below some bounds which are also determined by the initial norm only. Actually they depend also on $\sup(\alpha/c)$, but we are assuming that it is fixed once for all.

Let $\kappa := \widetilde{T}^{1/4} \vee c^{-1/2} \le 1$ and

$$H_T := \|(\mathbb{E}, N)\|_{L^\infty(0,T;H^1\times L^2)}, \tag{3.90}$$

which is continuous for $0 < T < \widetilde{T}$. We will prove that if κ is sufficiently small, then H_T is uniformly bounded, depending only on H_0.

For any $T \in (0,\widetilde{T})$, let $(\mathbb{E}_T, N_T) = \Phi(\mathbb{E}^\flat, N^\flat)$ given by (3.16), and let

$$X_T := \|(\mathbb{E}_T, N_T)\|_{\mathcal{X}}, \quad M_T := \|\mathbb{E}_T\|_{\mathcal{M}}, \quad M'_T := \|\mathbb{E}_T\|_{\mathcal{M}(0,T)}. \tag{3.91}$$

We recall that $(\mathbb{E}_T, N_T) = (\mathbb{E}, N)$ on $(0,T)$ and $\|(\mathbb{E}_T, N_T)\|_{\mathcal{H}} \sim H_T$ by Lemma 3.3. By (3.25), (3.39), (3.57) and (3.17), we have

$$\begin{aligned} &X_T \le CH_T(1 + M_T + \kappa X_T),\\ &M'_T \vee (M_T - CH_0) \le C\kappa(H_T + X_T + H_T(H_T + X_T + M_T)),\\ &H_T \le C(H_0 + H_0^2) + C\kappa(H_T^{7/3} + H_T^2 + X_T^2) + CM'_T(M'_T H_T + X_T), \end{aligned} \tag{3.92}$$

for some universal constant $C \ge 1$. Now we assume that $\widetilde{T}$ is sufficiently small and c is large enough in the sense that

$$999C^9\kappa(1 + H_0)^6 \le 1/2. \tag{3.93}$$

There exists, by continuity, the maximal $T \in (0,\widetilde{T}]$ for which we have

$$H_T \le 1 + 2C(H_0 + H_0^2). \tag{3.94}$$

Once we show that this inequality is strict, then $T = \widetilde{T}$ by continuity, and so we have the uniform bound (3.94) as long as $(\widetilde{T}, c)$ satisfies (3.93). ¿From (3.93) and (3.94), we have

$$100C^6\kappa(1 + H_T)^3 < 1/2. \tag{3.95}$$

In particular, $C\kappa H_T < 1/2$. Hence from the first inequality of (3.92), we have

$$X_T \le 2CH_T(1 + M_T), \tag{3.96}$$

and plugging this into the second inequality,

$$M'_T \vee (M_T - CH_0) \le 2C^2\kappa(1 + M_T)(1 + H_T)^2 + C\kappa(H_T + H_T^2). \tag{3.97}$$

Since $2C^2\kappa(1+H_T)^2 < 1/2$, we get

$$M_T/2 - CH_0 \le 2C^2\kappa(1+H_T)^2 + C\kappa(H_T + H_T^2) \le 3C^2\kappa(1+H_T)^2 < 1/2. \quad (3.98)$$

In particular, we have $M_T + 2 \le 2C(1+H_0)$, and plugging it into (3.96) and (3.97),

$$X_T + H_T \le 4C^2(1+H_T)^2, \quad M'_T \le 5C^3\kappa(1+H_T)^3 < 1. \quad (3.99)$$

By using them we can estimate the terms in the third inequality of (3.92)

$$\begin{aligned} &C\kappa(H_T^{7/3} + H_T^2) + C\kappa X_T^2 + CM'_T(M'_T H_T + X_T) \\ &\le 2C\kappa(H_T+1)H_T^2 + 16C^5\kappa(1+H_T)^4 + 20C^6\kappa(1+H_T)^5 \\ &\le 38C^6\kappa(1+H_T)^2 < H_T/2. \end{aligned} \quad (3.100)$$

Thus we obtain from the third inequality of (3.92),

$$H_T \le 2C(H_0 + H_0^2), \quad (3.101)$$

which implies the strict inequality in (3.94). Thus we conclude that $T = \widetilde{T}$, which means that we have (3.94) as long as the solution exists and (3.93) holds.

In addition, the local wellposedness result in [21] implies the following:

(i) The local solution can be extended until $\|\mathbb{E}(t)\|_{H^1} + \|N(t)\|_{L^2}$ blows up.

(ii) For any bounded set of (c,α) with $\sup(\alpha/c) < 1$ and bounded set of initial data, we have a uniform bound on the energy norm on a common time interval.

Combining these with the above uniform bound for large c, we obtain

Proposition 3.9. *For any $\gamma \in (0,1)$ and b, there exists $\widetilde{T}(\gamma,b) > 0$ and $B(\gamma,b) > 0$ with the following property: Let $1 \le \alpha \le \gamma c$, $0 < T \le \widetilde{T}(\gamma,b)$ and $(\mathbb{E},N) = \Phi(\mathbb{E}^\flat, N^\flat)$ given by (3.16). Assume that $(\mathbb{E}^\flat, N^\flat)$ solves (3.8) on $(0,T)$, $(\mathbb{E},N) \in \mathcal{H}\cap\mathcal{X}$, $\mathbb{E} \in \mathcal{M}$ and $\|(\mathbb{E}(0),N(0))\|_{H^1\times L^2} \le b$. Then we have*

$$\|(\mathbb{E},N)\|_{\mathcal{H}\cap\mathcal{X}} + \|\mathbb{E}\|_{\mathcal{M}} \le B(\gamma,b). \quad (3.102)$$

The above assumptions $(\mathbb{E},N) \in \mathcal{H}\cap\mathcal{X}$ and $\mathbb{E} \in \mathcal{M}$ are fulfilled for example if one of the following conditions holds:

(i) $(\mathbb{E}^\flat, N^\flat) \in \mathcal{H}(0,T)$ and $\mathbb{E}^\flat \in \mathrm{St}_0^{1,p}(0,T)$ for all $p > 2$.

(ii) $(\mathbb{E}^\flat, N^\flat) \in X^{1,b} \times Y^{0,b}$ for some $b > 1/2$.

(iii) $(\mathbb{E}^\flat, N^\flat) \in \mathcal{H}\cap\mathcal{X}$ and $\mathbb{E}^\flat \in \mathcal{M}$.

The sufficiency of (i) was proved in Lemma 3.3. (ii) implies (i) by (2.25), and the sufficiency of (iii) is clear from (3.17), (3.25) and (3.39). We can use (i) for the solutions with finite (Klein-Gordon) Strichartz norm of the H^1 level, (ii) for those constructed by the $X^{s,b}$ argument as in [21], and (iii) for those obtained by iterating Φ on a shorter time interval. The distinction between those solution classes would become irrelevant if we can prove the uniqueness of finite energy solution, but we do not pursue it here.

In particular, the above uniform bound in $\mathcal{H}$ implies that the unique local solution constructed in [21] exists on some time interval, determined by the upper bounds on α/c and the initial energy norm only. Notice that the assumption of uniform decay for high frequency (3.2) is not needed for the above uniform bounds. That is used only for the convergence proved in the following section.

4. Convergence from the Klein-Gordon-Zakharov to the NLS

In this section we prove the strong convergence in Theorem 3.1 by using the uniform energy bound in Proposition 3.9. It suffices to prove the convergence on the small time interval $[0,T]$, since we can repeat the same argument for later time as long as the limit solution is bounded in H^1, i.e., up to the maximal existence time T^*.

The proof proceeds in the following three steps. First we extract a subsequence which converges weakly. Then the uniqueness of the weak solution to the limit system implies that the whole sequence converges. Finally by using asymptotic conservation of the modified energy, we deduce the convergence is indeed strong.

We consider the uniformly bounded solution $(\mathbb{E}, N)$ in Proposition 3.9, and assume in addition that $\mathbb{E}(0) \to \mathbb{E}^\infty(0)$ in H^1 as $(c,\alpha) \to \infty$, and

$$\lim_{R\to\infty} \sup_{(c,\alpha)} \|N_{>R}(0)\|_{L^2} = 0, \tag{4.1}$$

under the condition $\alpha \le \gamma c$. We are going to prove the strong convergence of $\mathbb{E}$.

4.1. **Weak convergence.**

First we consider the weak limit of $\mathbb{E}$. By the equation (3.16) and the energy bound, we have

$$\|\dot{\mathbb{E}}\|_{L^\infty(H^{-1}\cap cL^2)} \lesssim \|\Delta\mathbb{E}\|_{L^\infty H^{-1}} + \|n\mathbb{E}\|_{L^\infty H^{-1}} \lesssim \|\mathbb{E}\|_{L^\infty H^1}(1 + \|n\|_{L^\infty L^2}). \tag{4.2}$$

Combined with the energy bound, this implies that $\{\mathbb{E}\}_{(c,\alpha)}$ is equicontinuous for $t \in \mathbb{R}$ in the weak topology of H^1. Hence by the standard compactness argument, there is a subsequence of $(c,\alpha) \to \infty$, along which

$$\mathbb{E} \to \mathbb{E}^\infty \text{ in } C(\mathbb{R}; (w\text{-}H^1) \cap L^p_{loc}), \tag{4.3}$$

for some $\mathbb{E}^\infty$ and for any $p < 6$.

Next, for any test function $u \in C_0^\infty((0,T)\times\mathbb{R}^3)$, we have from the equation of N and partial integration in t,

$$\begin{aligned} &\langle N + |\mathbb{E}|^2 \mid u\rangle_{t,x} = -\langle i|\alpha\nabla|^{-1}\dot{N} + \langle \mathbb{E}, \mathbb{E}^*\rangle \mid u\rangle_{t,x} \\ &= \langle i|\alpha\nabla|^{-1}N \mid \dot{u}\rangle_{t,x} + (2c^2)^{-1}\langle \mathbb{E}^* \mid i(\operatorname{Re}\dot{u})\mathbb{E} + 2i(\operatorname{Re} u)\dot{\mathbb{E}}\rangle_{t,x} \to 0, \end{aligned} \tag{4.4}$$

by the uniform bounds on $\|N\|_{L^2_x}$, $\|\mathbb{E}\|_{H^1_x}$ and $\|\dot{\mathbb{E}}\|_{H^{-1}_x}$. Combined with the L^2_x bound and the convergence (4.3), it implies that

$$N + |\mathbb{E}^\infty|^2 \to 0 \text{ in } w\text{-}L^p_t L^2_x(0,T), \tag{4.5}$$

for any $p \in (1,\infty)$. Similarly we have, denoting $\mathbb{E}^\dagger := e^{-2ic^2t}(\overline{\mathbb{E}_2}, \overline{\mathbb{E}_1})$,

$$\begin{aligned}\langle n\mathbb{E}^* \,|\, u\rangle_{t,x} &= -(2c^2)^{-1}\langle i\mathbb{E}^* \,|\, \dot{n}u + n\dot{u}\rangle_{t,x} - (2c^2)^{-1}\langle ie^{-2ic^2t}\partial_t\mathbb{E}^\dagger, \,|\, nu\rangle_{t,x} \\ &\to 0,\end{aligned} \tag{4.6}$$

by the uniform bounds on $\|\dot{n}/\alpha\|_{H^{-1}_x}$, $\|\dot{\mathbb{E}}/c\|_{L^2_x}$ and the energy norm. From this, (4.5) and (4.3) as well as the energy bound, we deduce that

$$I_c n(\mathbb{E} + \mathbb{E}^*) \to -|\mathbb{E}^\infty|^2\mathbb{E}^\infty \text{ in } w\text{-}L^p_t L^q_x(0,T), \tag{4.7}$$

for any $p \in (1,\infty)$ and $q \in (1, 3/2)$. Thus we conclude that the limit function $\mathbb{E}^\infty$ is a weak solution of (1.3) on $[0,T]$ in the class $C([0,T]; w\text{-}H^1)$, and the uniqueness of such a solution implies that the whole sequence along $(c,\alpha) \to \infty$ is converging to this unique limit on $(0,T)$. It is convergent actually on $\mathbb{R}$, due to (3.18). Thus $\mathbb{E}^\infty \in C(\mathbb{R}; H^1 \times L^2)$ is the unique solution of

$$\mathbb{E}^\infty = e^{-it\Delta/2}\left[\chi(t)\mathbb{E}^\infty(0) - \frac{i}{2}\mathcal{I}_T e^{it\Delta/2}|\mathbb{E}^\infty|^2\mathbb{E}^\infty\right]. \tag{4.8}$$

4.2. **Estimates on the limit solution.** Next we derive a few bounds on the above limit solution $\mathbb{E}^\infty$. By the Strichartz estimate for $e^{-it\Delta/2}$, we have

$$\begin{aligned}\|\mathbb{E}^\infty - e^{-it\Delta/2}\mathbb{E}^\infty(0)\|_{L^\infty H^1 \cap L^2 B^1_{6,2}(0,S)} &\lesssim \||\mathbb{E}^\infty|^2\mathbb{E}^\infty\|_{L^1H^1(0,S)} \\ &\lesssim S^{1/2}\|\mathbb{E}^\infty\|_{L^\infty H^1}\|\mathbb{E}^\infty\|^2_{L^4L^\infty},\end{aligned} \tag{4.9}$$

for any $S \in (0,T)$. By the real interpolation we have

$$\begin{aligned}\|\mathbb{E}^\infty\|_{L^4L^\infty} \lesssim \|\mathbb{E}^\infty\|_{L^4B^{1/2}_{6,1}} &\lesssim \|\|\mathbb{E}^\infty(t)\|^{1/2}_{L^6_x}\|\mathbb{E}^\infty(t)\|^{1/2}_{B^1_{6,2}}\|_{L^4_t} \\ &\lesssim \|\mathbb{E}^\infty\|^{1/2}_{L^\infty H^1}\|\mathbb{E}^\infty\|^{1/2}_{L^2B^1_{6,2}}\end{aligned} \tag{4.10}$$

Hence if $S \ll \|\mathbb{E}^\infty\|^{-4}_{\mathcal{H}}$, then we have

$$\|\mathbb{E}^\infty\|_{L^\infty H^1 \cap L^2B^1_{6,2} \cap L^4L^\infty(0,S)} \lesssim \|\mathbb{E}^\infty\|_{\mathcal{H}}. \tag{4.11}$$

We can repeat this argument for finite times $\sim T/S$ to cover $(0,T)$, deducing that $\mathbb{E}^\infty \in L^2B^1_{6,2} \subset \mathrm{St}^{1,2}_a$ for all $a \in [0,1]$.

4.3. **Strong convergence.** First we prove L^2 convergence. Inner multiplying the first equation of (3.8) with $I_c^{-1}i\mathbb{E}$, we obtain for any $T_1 \in (0,T)$,

$$\left[\langle I_c^{-1}\mathbb{E} \,|\, \mathbb{E}\rangle_x\right]_0^{T_1} = \langle in\mathbb{E}^* \,|\, \mathbb{E}(0,T_1)\rangle_{t,x}. \tag{4.12}$$

The trilinear estimates in the previous section (3.77), (3.82), (3.85), (3.88) imply that the right hand side is of order $O(c^{-2})$. Since $I_c^{-1/2}\mathbb{E}(0) \to \mathbb{E}^\infty(0)$ in L^2_x and $\langle I_c^{-1/2}\mathbb{E} - \mathbb{E}^\infty \,|\, \mathbb{E}^\infty\rangle_x \to 0$ in L^∞_t by the weak convergence, we have

$$\|I_c^{-1/2}\mathbb{E} - \mathbb{E}^\infty\|_{L^2_x}^2 = \|I_c^{-1/2}\mathbb{E}\|_{L^2_x}^2 - \|\mathbb{E}^\infty\|_{L^2_x}^2 - 2\langle I_c^{-1/2}\mathbb{E} - \mathbb{E}^\infty \,|\, \mathbb{E}^\infty\rangle_x \to 0, \quad (4.13)$$

uniformly in $t \in \mathbb{R}$.

Next we consider the H^1 convergence. Let

$$\mathbb{E} = \mathbb{E}^\infty + \mathbb{E}', \quad N + |\mathbb{E}|^2 = N^I + N', \quad N^I = e^{i|\alpha\nabla|t}(N(0) + |\mathbb{E}(0)|^2). \quad (4.14)$$

The modified energy can be rewritten as

$$\begin{aligned}\mathcal{E}(t) + \langle n\mathbb{E} \,|\, \mathbb{E}\rangle_x =& \langle I_c^{-1}(2-\Delta_c)\mathbb{E} \,|\, \mathbb{E}\rangle_x + \frac{1}{2}\|N + |\mathbb{E}|^2\|_{L^2_x}^2 - \frac{1}{2}\|\mathbb{E}\|_{L^4_x}^4 \\ =& \langle I_c^{-1}(2-\Delta_c)\mathbb{E}' \,|\, \mathbb{E}'\rangle_x + \frac{1}{2}\|N'\|_{L^2}^2 \\ &+ 2\|\mathbb{E}^\infty\|_{L^2}^2 + \|\nabla\mathbb{E}^\infty\|_{L^2}^2 + \frac{1}{2}\|N^I\|_{L^2}^2 - \frac{1}{2}\|\mathbb{E}^\infty\|_{L^4}^4 \\ &+ 2\langle I_c^{-1}(2-\Delta_c)\mathbb{E}^\infty \,|\, \mathbb{E}\rangle_x + \langle N' \,|\, N^I\rangle_x \\ &+ \langle(\Delta - I_c^{-1}\Delta_c)\mathbb{E}^\infty \,|\, \mathbb{E}^\infty\rangle_x + \frac{1}{2}(\|\mathbb{E}^\infty\|_{L^4}^4 - \|\mathbb{E}\|_{L^4}^4),\end{aligned} \quad (4.15)$$

where we have dropped the term $\langle (P_\alpha iN)\mathbb{E}^* \,|\, i\mathbb{E}\rangle_x$ because it is vanishing by the $L^\infty_t L^4_x$ convergence of $\mathbb{E}$. The second line is 0 at $t = 0$. The third line is a conserved quantity. As $(c,\alpha) \to \infty$, the last line tends to 0 in L^∞_t, by the strong continuity of $\mathbb{E}^\infty$, the L^2 strong convergence, and the H^1 uniform bound of $\mathbb{E}$. On the forth line, the first term tends to 0 by the weak convergence. For the second term, we split N^I in the frequency at $R \to \infty$ to exploit its uniform decay for high frequency (4.1). For any $\varepsilon > 0$, we can choose $R > 1$ such that $|\langle N' \,|\, N^I_{>R}\rangle_x| \le \varepsilon$ uniformly on $t \in (0,T)$ and for (c,α). For the low frequency part, we have

$$\langle N' \,|\, N^I_{\le R}\rangle_x = \int_0^t \langle e^{-i|\alpha\nabla|s}(|\mathbb{E}|_t^2 + i|\alpha\nabla|\langle\mathbb{E},\mathbb{E}^*\rangle)(s) \,|\, N^I_{\le R}(0)\rangle_x ds, \quad (4.16)$$

where the contribution from $\mathbb{E}^*$ tends to 0 by partial integration in the same way as (4.4). The contribution of $|\mathbb{E}|^2$ is also vanishing by the Strichartz estimate

$$\left\|\int_0^t e^{i|\alpha\nabla|s} f(s)ds\right\|_{L^\infty H^{-3/2}} \lesssim \alpha^{-1/4}\|f\|_{L^{4/3}B^{-1}_{4/3,2}}. \quad (4.17)$$

Hence we deduce $\langle N' \,|\, N^I\rangle_x \to 0$ in $L^\infty_t(0,T)$, and therefore

$$\|\mathbb{E}'(T_1)\|_{H^1}^2 + \|N'(T_1)\|_{L^2}^2 \lesssim o(1) + \int_0^{T_1} dt(\text{RHS of (3.64)}), \quad (4.18)$$

for any $T_1 \in (0,T)$. Here and after, $o(1)$ denotes arbitrary positive quantity tending to 0 as $(c,\alpha) \to \infty$.

Next we expand the right hand side of (4.18), decomposing each function by

$$\mathbb{E} = \mathbb{E}_{\le R} + \mathbb{E}_{>R}, \quad N = N_{\le R} + N_{>R}, \tag{4.19}$$

for $(c, \alpha) \gg R \gg 1$. We denote

$$\begin{aligned} &H := \|(\mathbb{E}, N)\|_{\mathcal{H}}, \quad M := \|\mathbb{E}\|_{\mathcal{M}}, \quad X := \|(\mathbb{E}, N)\|_{\mathcal{X}}, \\ &H' := \|(\mathbb{E}', N')\|_{\mathcal{H}}, \quad M' := \|\mathbb{E}'\|_{\mathcal{M}}, \\ &H_R := \|(\mathbb{E}, N)_{>R}\|_{\mathcal{H}}, \quad M_R := \|\mathbb{E}_{>R}\|_{\mathcal{M}}, \quad X_R := \|(\mathbb{E}, N)_{>R}\|_{\mathcal{X}}. \end{aligned} \tag{4.20}$$

For the trilinear terms, we apply the same estimates as in the proof of Lemma 3.7. If at least two of the three functions have the lower frequency, then we have $h \lesssim R$ in those estimates, so that the factor $\langle c/h \rangle^{-1}$ or $c^{-1/2}$ kills those terms in the limit. Thus the trilinear terms are bounded by

$$H_R M_R X + H_R M X_R + H M_R X_R + o_R(1), \tag{4.21}$$

where $o_R(1)$ denotes arbitrary positive quantity tending to 0 as $(c, \alpha) \to \infty$, depending on R. As for the quartic terms with N, if at most one of the four functions is the higher frequency part, then we partially integrate $e^{2ic^2 t}$. By the equations, (3.34), and the Sobolev embedding $H^2 \subset B^{1/2}_{6,1}$, we have

$$\|\dot{N}_{\le R}\|_{L^2_x} \lesssim \alpha R(H + H^2), \quad \|\dot{\mathbb{E}}_{\le R}\|_{\mathrm{St}^{1/2,2}_{1,1}} \lesssim R^2(M + HM), \tag{4.22}$$

so that we gain at least α/c^2 for fixed R. For the quartic terms without N, we gain R^2/c^2 from $P_\alpha |\alpha \nabla|$ if more than two functions are the lower frequency. Thus the quartic term is bounded by

$$H_R^2 M^2 + H_R M_R H M + M_R^2 H^2 + o_R(1), \tag{4.23}$$

and therefore

$$\begin{aligned} \|\mathbb{E}'\|^2_{\mathcal{H}} \lesssim\ & H_R M_R X + H_R M X_R + H M_R X_R \\ & + H_R^2 M^2 + H_R M_R H M + M_R^2 H^2 + o_R(1). \end{aligned} \tag{4.24}$$

By the uniform decay assumption (3.2) and (3.18), we have

$$H_R \lesssim \|\mathbb{E}'\|_{\mathcal{H}} + o(1; R \to \infty), \tag{4.25}$$

here and after $o(1; R \to \infty)$ denotes arbitrary positive quantity tending to 0 as $R \to \infty$, uniformly in (c, α). Since $\mathbb{E}^\infty \in L^2 B^1_{6,2}$, we have

$$M_R \lesssim M' + o(1; R \to \infty), \tag{4.26}$$

and by the L^2 strong convergence, we have also

$$M' \le \|\mathbb{E}'_{>R}\| + \|\mathbb{E}'_{\le R}\| \lesssim M_R + o(1; R \to \infty) + o_R(1). \tag{4.27}$$

Next we apply the arguments for (3.25) to $(\mathbb{E}_{>R}, N_{>R})$. Then one of the functions in bilinear terms must have frequency $\gtrsim R$. If it is estimated in $\mathcal{X}$, then we get $c^{-1/2}$ decay, otherwise we apply (4.25) or (4.26). Thus we obtain

$$X_R \lesssim (H' + M')H + o(1; R \to \infty) + o(1). \tag{4.28}$$

Similarly, by the argument for (3.39), we obtain

$$M_R \lesssim T^{1/4}(H' + M')(1 + H^2) + o(1; R \to \infty) + o(1), \tag{4.29}$$

by using (4.25), (4.26) and (4.28), where the contribution of $\mathbb{E}^0_{>R}$ is contained in $o(1; R \to \infty)$. Plugging them into (4.24) and (4.27), we obtain for any $0 < T' < T$,

$$\begin{aligned} (H')^2 &\lesssim T^{1/4}(H' + M')^2(1 + H^6) + o(1; R \to \infty) + o_R(1), \\ M' &\lesssim T^{1/4}(H' + M')(1 + H^2) + o(1; R \to \infty) + o_R(1) \end{aligned} \tag{4.30}$$

Choosing T sufficiently small compared with $1 + H^6$, and letting $(c, \alpha) \to \infty$ and then $R \to \infty$, we deduce that

$$\lim_{(c,\alpha)\to\infty} H' + M' = 0. \tag{4.31}$$

Remark 4.1. As can be seen from the proof, the convergence stated in theorem 3.1 holds for any family of solutions such that $(\mathbb{E}, N) \in \mathcal{H} \cap \mathcal{X}$ and $\mathbb{E} \in \mathcal{M}$. This class is larger than the uniqueness class of [21].

5. Klein-Gordon-Zakharov to Zakharov

In this section we prove convergence of solutions from the Klein-Gordon-Zakharov system to the Zakharov system. There is no resonance blow-up in this case, so the iterative argument works uniformly with respect to c, relying on the bilinear estimate, without using the energy conservation.

However, this problem cannot be solved by a simple "interpolation" between the available estimates on fixed c [21] and in the limit [6, 12]. The trouble comes from the term $n_l \mathbb{E}_h$, $l + c \ll h$. If the parameter c is fixed, then it can be treated just by the Strichartz estimate, of the Schrödinger type for $c = \infty$ as in [12], or of the wave type for finite c as in [21], but neither argument works uniformly in c. More precisely, the former loses $(h/c)^{1/4}$, and the latter $c^{1/2}$, at least. Thus we are forced to apply the bilinear estimate to this term, even though there is resonance interaction $(n^C \mathbb{E}^C)^C$ in this case. We exploit the smallness of the set of resonant frequency. The idea is similar to that in [6], which used an improved Sobolev inequality, but we need a sharper estimate (Lemma 5.4 below), which is an improvement of the interpolated Strichartz estimate (2.39).

Theorem 5.1. *Consider the limit $c \to \infty$ with $\alpha = \alpha(c) > 0$ bounded from above and below. For each c, let (E^c, n^c) be a solution of* (1.1) *given by* [21], $(\widehat{\mathbb{E}}^c, \widehat{N}^c)$ *be a solution of* (1.7) *given by* [6, 12], *and let*

$$\mathbb{E}^c := \frac{e^{-ic^2t}}{2}(1 - ic^{-2}I_c\partial_t)(E^c, \overline{E^c}), \quad (N^c, \widehat{N}^c) := (1 - i|\alpha\nabla|^{-1}\partial_t)(n^c, \widehat{n}^c). \tag{5.1}$$

Assume that $(\mathbb{E}^c(0), N^c(0))$ is bounded in $H^1 \times L^2$,

$$(\mathbb{E}^c(0), N^c(0)) - (\widehat{\mathbb{E}}^c(0), \widehat{N}^c(0)) \to 0 \text{ in } H^1 \times L^2, \tag{5.2}$$

and uniform decay for high frequency:

$$\lim_{R\to\infty} \limsup_{c\to\infty} \|(\mathbb{E}^c(0), N^c(0))_{>R}\|_{H^1\times L^2} = 0. \tag{5.3}$$

Let T^c and $\widehat{T}^c$ be the maximal bounded time, namely

$$\begin{aligned} T^\infty &:= \sup\{T > 0 \mid \limsup_{c\to\infty} \|(\mathbb{E}^c, N^c)\|_{\mathcal{H}(0,T)} < \infty\}, \\ \widehat{T}^\infty &:= \sup\{T > 0 \mid \limsup_{c\to\infty} \|(\widehat{\mathbb{E}}^c, \widehat{N}^c)\|_{\mathcal{H}(0,T)} < \infty\}. \end{aligned} \tag{5.4}$$

Then we have $T^\infty = \widehat{T}^\infty > 0$, and for all $T \in (0, T^\infty)$,

$$(\mathbb{E}^c - \widehat{\mathbb{E}}^c, N^c - \widehat{N}^c) \to 0 \text{ in } C([0,T]; H^1 \times L^2). \tag{5.5}$$

Remark 5.2. $\widehat{T}^\infty > 0$ is bounded from below in terms of the initial bound

$$\limsup_{c\to\infty} \|(\widehat{\mathbb{E}}^c(0), \widehat{N}^c(0))\|_{H^1\times L^2}, \tag{5.6}$$

by the local wellposedness of (1.7). If $\alpha(c)$ and the initial data $(\widehat{\mathbb{E}}^c(0), \widehat{n}^c(0), \partial_t\widehat{n}^c(0))$ are converging, then $\widehat{T}^\infty$ is the maximal existence time of the limit solution of $\widehat{\mathbb{E}}^c$.

Remark 5.3. One can easily observe that the necessary and sufficient condition for the same convergence for the free equations is given by replacing (5.3) with

$$\sup_{R>0} \limsup_{c\to\infty} \|(\mathbb{E}^c(0), N^c(0))_{>Rc}\|_{H^1\times L^2} = 0. \tag{5.7}$$

In other words, some part of the norm is allowed to escape to the infinite frequency, if it is slower than c. In this case, the $X^{1,b}$ norm of the asymptotic profile $\widehat{\mathbb{E}}$ can become unbounded as $c \to \infty$ for any $b > 0$. If the high frequency leak is slower than $\sqrt{c}$, then the $X^{1,b}$ norm remains bounded and we can probably prove the same convergence result.

It suffices to prove the convergence on a uniform small interval $[0,T]$, since we can continue it to $t > T$ by the same argument until the solutions become unbounded. Fix $\varepsilon \in (0, 0.01)$ and let

$$\begin{aligned} \mathcal{X}_2 :=& X^{1-4\varepsilon,1/2+\varepsilon} \times Y^{0,1-10\varepsilon}, \\ \mathcal{X}_3 :=& \left[X^{1-4\varepsilon,1/2+\varepsilon} \cap X^{1,1/2-\varepsilon} \cap L^\infty(H^1)\right] \times Y^{0,1-10\varepsilon}. \end{aligned} \tag{5.8}$$

We will prove that for $(\mathbb{E}^\sharp, N^\sharp) = \Phi(\mathbb{E}^\flat, N^\flat)$ in (3.16),

$$\|(\mathbb{E}^\sharp, N^\sharp)\|_{\mathcal{X}_3} \lesssim \|(\mathbb{E}(0), N(0))\|_{H^1\times L^2} + T^\varepsilon \|(\mathbb{E}^\flat, N^\flat)\|_{\mathcal{X}_3}^2, \tag{5.9}$$

without assuming that it is a solution of (3.8). In addition, we will get $c^{-\varepsilon}$ decay in $\mathcal{X}_2$ for those nonlinear terms involving $\mathbb{E}^*$.

Now let $(\mathbb{E}^\flat, N^\flat)$ be a solution of (3.8) on $(0,T)$ given by [21]. Then $(\mathbb{E}^\flat, N^\flat) \in X^{1,b} \times Y^{0,b}$ for some $b > 1/2$, so (3.20) implies that $(\mathbb{E}^\flat, N^\sharp) \in \mathcal{X}_3$. Hence the above estimate implies that $(\mathbb{E}^\sharp, N^\sharp) = \Phi(\mathbb{E}^\flat, N^\sharp) \in \mathcal{X}_3$. Since $(\mathbb{E}^\sharp, N^\sharp) = \Phi(\mathbb{E}^\sharp, N^\sharp)$, the above estimate (5.9) implies that for small $T > 0$

$$\|(\mathbb{E}^\sharp, N^\sharp)\|_{\mathcal{X}_3} \lesssim \|(\mathbb{E}(0), N(0))\|_{H^1\times L^2}. \tag{5.10}$$

We prove (5.1) in the following, and then the convergence in the last subsection.

It seems impossible to get uniform $X^{1,b}$ bound for $b > 1/2$ due to the term $(n_{\mathrm{High}}\mathbb{E}_{\mathrm{Low}})^{\mathrm{Far}}$. The reason is that the regularity gap between n and $\mathbb{E}$ equals what we can recover from the nonresonance distance. This criticality is more evident in the limit system, see [12].

We estimate the nonlinearity in duality coupling with frequency decomposition:

$$\begin{aligned}
\langle I_c nE \mid u\rangle_{t,x} &= \sum_{(j,k,l)\in\mathcal{T}} \langle n_j E_k \mid I_c u_l\rangle_{t,x},\\
\langle |\alpha\nabla|\langle E, F\rangle \mid N\rangle_{t,x} &= \sum_{(j,k,l)\in\mathcal{T}} \langle (|\alpha\nabla| n_j) E_k \mid F_l\rangle_{t,x},
\end{aligned} \tag{5.11}$$

where $n = \operatorname{Re} N$ and $E, \overline{F} \in \{\mathbb{E}, \mathbb{E}^*\}$. Let $h = \max(j,k,l)$, $\ell = \min(k,l)$ and $m = \min(j,k,l)$.

5.1. Estimates on $\langle n_{high}\mathbb{E}_{low} \mid \mathbb{E}_{high}\rangle_{t,x}$ and $\langle n\mathbb{E}^* \mid \mathbb{E}\rangle_{t,x}$.

First we consider non-resonant interactions. If $E = \overline{F} = \mathbb{E}$ and $j \sim h \gg m$, then we have non-resonance distance $\delta \sim (c\wedge h)h$ by Lemma 3.4. If $E = \mathbb{E}^*$ and $\overline{F} = \mathbb{E}$, then we have $\delta \sim (c \vee h)c$. In both cases we have

$$\delta\langle h/c\rangle \gtrsim h^2, \quad \delta\langle m/c\rangle h/m \gtrsim h^2. \tag{5.12}$$

We decompose (5.11) by the distance from the characteristics. For $N^\sharp$, we consider

$$\begin{aligned}
&\langle (|\alpha\nabla| n_j) E_k \mid F_l\rangle_{t,x}\\
&= \langle (|\alpha\nabla| n_j^F) E_k \mid F_l\rangle_{t,x} + \langle (|\alpha\nabla| n_j^C) E_k^F \mid F_l\rangle_{t,x} + \langle (|\alpha\nabla| n_j^C) E_k^C \mid F_l^F\rangle_{t,x}\\
&=: A_1 + A_2 + A_3,
\end{aligned} \tag{5.13}$$

where $n = \operatorname{Re} N$ is a test function in $Y^{0,10\varepsilon}$, and $E, F \in \{\mathbb{E}, \mathbb{E}^*\}$. For A_1, we have

$$\begin{aligned}
A_1 &\lesssim j\|N_j^F\|_{L^2L^2}\|\mathbb{E}_\ell\|_{L^{2/(1-\varepsilon)}L^{2/\varepsilon}}\|\mathbb{E}_h\|_{L^{2/\varepsilon}L^{2/(1-\varepsilon)}}\\
&\lesssim j\delta^{-9\varepsilon}\ell^{3\varepsilon}h^{-1+5\varepsilon}\|N_j^F\|_{Y^{0,9\varepsilon}}\|\mathbb{E}_\ell\|_{X^{1-4\varepsilon,1/2+\varepsilon}}\|\mathbb{E}_h\|_{X^{1-4\varepsilon,1/2+\varepsilon}},
\end{aligned} \tag{5.14}$$

where we used the Strichartz estimate (2.25) of the wave type for $X^{1-4\varepsilon,1/2+\varepsilon}$. Since the coefficient is bounded by $m^{3\varepsilon}h^{5\varepsilon}\delta^{-9\varepsilon}$, the right hand side is summable for the dyadic frequency in any case by $\delta \gtrsim h$, with a decay factor $c^{-9\varepsilon}$ if the term involves $\mathbb{E}^*$ by $\delta \gtrsim ch$. The resulting contribution to $N^\sharp$ is bounded in $T^\varepsilon Y^{0,1-10\varepsilon}$ by (2.22). The estimates on A_2 and A_3 are essentially the same. For A_2, we have

$$\begin{aligned} A_2 &\lesssim T^\varepsilon j m^{3/2-27\varepsilon}\|N_j^C\|_{L^{2/(1-10\varepsilon)}_{t,x}}\|E_k^F\|_{L^2_{t,x}}\|F_l\|_{L^{1/4\varepsilon}L^{2/(1-8\varepsilon)}} \\ &\lesssim T^\varepsilon j^{1+10\varepsilon}m^{3/2-27\varepsilon}\delta^{-1/2-\varepsilon}k^{-1+4\varepsilon}l^{-1+12\varepsilon} \\ &\quad\times\|N_j\|_{Y^{0,10\varepsilon}}\|\mathbb{E}_k^F\|_{X^{1-4\varepsilon,1/2+\varepsilon}}\|\mathbb{E}_l\|_{X^{1-4\varepsilon,1/2+\varepsilon}}, \end{aligned} \tag{5.15}$$

where we used the Strichartz estimate for F_l, and the interpolation for N_j:

$$Y^{0,b} = (Y^{0,0}, Y^{0,1/2,1})_{2b,2} \subset (L^2L^2, L^4B^{-1/2}_{4,2})_{2b,2} \subset L^{2/(1-b)}B^{-b}_{2/(1-b),2}, \tag{5.16}$$

with $b = 10\varepsilon < 1/2$. The coefficient for A_2 is bounded by $T^\varepsilon m^\varepsilon\delta^{-\varepsilon}h^{-2\varepsilon}$, and so its contribution to $N^\sharp$ is bounded in $T^\varepsilon Y^{0,1-10\varepsilon}$ with a decay factor $c^{-\varepsilon}$ if $\mathbb{E}^*$ is involved. For A_3, we just switch the roles of E_k and F_l in the above argument.

For the equation of $\mathbb{E}^\sharp$, we consider the same decomposition as in (3.52). Then B_1 is the most regular term, which we estimate

$$\begin{aligned} B_1 &\lesssim \langle l/c\rangle^{-1}m^{3/2}\|N_j^F\|_{L^2L^2}\|E_k\|_{L^\infty L^2}\|u_l\|_{L^2L^2} \\ &\lesssim \langle l/c\rangle^{-1}m^{3/2}\delta^{-1+10\varepsilon}k^{-1+4\varepsilon}l\|N_j^F\|_{Y^{0,1-10\varepsilon}}\|\mathbb{E}_k\|_{X^{1-4\varepsilon,1/2+\varepsilon}}\|u_l\|_{X^{-1,0}}. \end{aligned} \tag{5.17}$$

The coefficient is bounded by

$$\begin{cases} \delta^{-\varepsilon}(\delta\langle l/c\rangle k/l)^{-1+11\varepsilon}m^{3/2}l^{4\varepsilon} & (l=m) \\ \delta^{-\varepsilon}(\delta\langle l/c\rangle)^{-1+11\varepsilon}m^{1/2+4\varepsilon}l & (l=h) \end{cases} \tag{5.18}$$

and they are both dominated by $\delta^{-\varepsilon}m^{1/2}h^{-1+26\varepsilon}$, due to (5.12). Hence B_1's contribution to $\mathbb{E}^\sharp$ is bounded in $T^\varepsilon X^{1,1-\varepsilon}$, with additional $c^{-\varepsilon}$ for those terms with $\mathbb{E}^*$. For B_2, we have

$$\begin{aligned} B_2 &\lesssim \langle l/c\rangle^{-1}m^{19/20}\|N_j^C\|_{L^\infty L^2}\|E_k^F\|_{L^2L^2}\|u_l\|_{L^2L^{60/19}} \\ &\lesssim \langle l/c\rangle^{-1}m^{19/20}k^{-1+4\varepsilon}\delta^{-1/2-\varepsilon}l\langle l/c\rangle^{11/24} \\ &\quad\times\|N_j\|_{Y^{0,1/2+\varepsilon}}\|\mathbb{E}_k^F\|_{X^{1-4\varepsilon,1/2+\varepsilon}}\|u_l\|_{X^{-1,11/40}}, \end{aligned} \tag{5.19}$$

where we applied (2.39) to u_l with $V = I_c^{-5/6}B^{-1}_{6,2}$:

$$X^{s,\theta/2} \subset I_c^{-5\theta/6}L^2B^s_{q,2}, \quad 1/q = 1/2 - \theta/3 \quad (0 \le \theta < 1) \tag{5.20}$$

The coefficient is bounded by

$$\begin{cases} \delta^{-\varepsilon}(\delta\langle l/c\rangle k/l)^{-1/2}k^{-1/2+4\varepsilon}m^{1/2+19/20} & (l=m) \\ \delta^{-\varepsilon}(\delta\langle l/c\rangle)^{-1/2}hk^{-1+4\varepsilon}m^{19/20} & (l=h) \end{cases} \tag{5.21}$$

which is bounded by $\delta^{-\varepsilon}m^{-1/20+4\varepsilon}$. Thus B_2's contribution to $\mathbb{E}^\sharp$ is bounded in $T^\varepsilon X^{1,29/40-\varepsilon}$ with $c^{-\varepsilon}$ for $\mathbb{E}^*$. The term B_3 is the only place where we have to distinguish the three spaces in $\mathcal{X}_3$. We start with the Hölder inequality as above:

$$B_3 \lesssim \langle l/c\rangle^{-1} T^\varepsilon m^{19/20} \|N_j^C\|_{L^\infty L^2} \|\mathbb{E}_k^C\|_{L^{2/(1-2\varepsilon)} L^{60/19}} \|u_l^F\|_{L^2 L^2}, \tag{5.22}$$

and then estimate $\mathbb{E}_k$ by interpolation between (2.25) and (2.39):

$$\begin{aligned} X^{1,\theta+\varepsilon} = (X^{1,1/2,1}, X^{1,\theta})_{\alpha,2} &\subset ([L^\infty H^1, L^2 S]_{2\theta}, L^2((H^1,S)_{2\theta,2}))_{\alpha,2} \\ &\subset L^p([H^1,S]_{2\theta}), \end{aligned} \tag{5.23}$$

where $S := I_c^{-5/6} B^1_{6,2}$, $\alpha \in (0,1)$ satisfies $(1-\alpha)/2 + \alpha\theta = \theta + \varepsilon$ and $1/p := (1-\alpha)\theta + \alpha/2 = 1/2 - \varepsilon$. Choosing $\theta = 11/40$, we obtain

$$\|\mathbb{E}_k^C\|_{L^{2/(1-2\varepsilon)} L^{60/19}} \lesssim k^{-1} \langle k/c\rangle^{11/24} \|\mathbb{E}_k\|_{X^{1,11/40+\varepsilon}}. \tag{5.24}$$

For $0 \le a \le \varepsilon$, we have

$$\|u_l^F\|_{L^2L^2} \lesssim \delta^{-1/2+a} l^{1-4a} \|u_l\|_{X^{-1+4a,1/2-a,\infty}}, \tag{5.25}$$

Plugging these bounds into (5.22), we get

$$\begin{aligned} B_3 \lesssim\, & \langle l/c\rangle^{-1} T^\varepsilon m^{19/20} k^{-1} \langle k/c\rangle^{11/24} \delta^{-1/2+a} l^{1-4a} \\ & \times \|N_j\|_{Y^{0,1/2+\varepsilon}} \|\mathbb{E}_k\|_{X^{1,11/40+\varepsilon}} \|u_l\|_{X^{-1+4a,1/2-a,\infty}}. \end{aligned} \tag{5.26}$$

Using (5.12) together with $\langle h/c\rangle/\langle m/c\rangle \lesssim h/m$, we can bound the coefficient on the first line by $T^\varepsilon m^{-1/20}\delta^{-a}$. Hence the contribution of $\sum_{j,k} B_3$ to $\mathbb{E}_l^\sharp$ is estimated in $T^\varepsilon \ell_l^2(X_1)$ by putting $a = \varepsilon$ and using (2.22), and also in $T^\varepsilon \ell_l^2(X^{1,1/2,\infty} \cap L^\infty(H^1))$ by putting $a = 0$ and using (2.23). By using $X^{1,1/2,\infty} \subset X^{1,1/2-\varepsilon}$ and $\ell_l^2 L_t^\infty \subset L^\infty \ell_l^2$, we can sum it for l in $T^\varepsilon \mathcal{X}_3$, as desired. In addition, if the term contains $\mathbb{E}^*$, it is bounded also in $T^\varepsilon c^{-\varepsilon}\mathcal{X}_2$. Finally we estimate B_4 by using (5.24) for u_l^C,

$$\begin{aligned} B_4 \lesssim\, & \|(0,T)^F\|_{L^{2/(1+2\varepsilon)}} \langle l/c\rangle^{-1} k^{-1} m^{19/20} l \langle l/c\rangle^{11/24} \\ & \times \|N_j^C\|_{L^\infty L^2} \|\mathbb{E}_k^C\|_{L^\infty H^1} \|u_l^C\|_{X^{-1,11/40+\varepsilon}}, \end{aligned} \tag{5.27}$$

where the coefficient is bounded by using (3.24) and (5.12),

$$(\delta\langle l/c\rangle)^{-1/2-\varepsilon} k^{-1} l m^{19/20} \lesssim \delta^{-\varepsilon} m^{-1/20}. \tag{5.28}$$

Thus B_4's contribution is bounded in $T^\varepsilon X^{1,29/40-2\varepsilon}$.

5.2. **Estimates on $\left\langle n_{low}\mathbb{E}_{high} \,\middle|\, \mathbb{E}_{high}\right\rangle_{t,x}$.** Next we consider the remaining case $E = \overline{F} = \mathbb{E}$ and $j \lesssim m$, where the nonlinearity may be resonant. Here we do not look for $c^{-\varepsilon}$ decay. For $N^\sharp$, we use an argument similar to (5.23):

$$\begin{aligned} X^{1,\theta+\alpha} = (X^{1,0}, X^{1,1/2,1})_{2\theta+2\alpha,2} &\subset (L^2H^1, [L^\infty H^1, L^2 S]_{\theta/(\theta+\alpha)})_{2\theta+2\alpha,2} \\ &\subset L^{2/(1-2\alpha)}([H^1,S]_{2\theta}), \end{aligned} \tag{5.29}$$

with $S := B^0_{\infty,2}$ and $(\theta,\alpha) := (1/2-2\varepsilon, \varepsilon)$, $(2\varepsilon, 1/2-3\varepsilon)$. Then we get

$$\begin{aligned} \left\langle (|\alpha\nabla| n_j) E_k \,\middle|\, F_l \right\rangle_{t,x} &\lesssim j \|n_j\|_{L^{2/(1-4\varepsilon)} L^2} \|E_k\|_{L^{2/(1-2\varepsilon)} L^{1/(2\varepsilon)}} \|F_l\|_{L^{1/(3\varepsilon)} L^{2/(1-4\varepsilon)}} \\ &\lesssim j k^{-4\varepsilon} l^{-1+4\varepsilon} \|n_j\|_{Y^{0,2\varepsilon}} \|\mathbb{E}_k\|_{X^{1,1/2-\varepsilon}} \|\mathbb{E}_l\|_{X^{1,1/2-\varepsilon}}, \end{aligned} \tag{5.30}$$

hence its contribution to $N^\sharp$ is bounded in $T^\varepsilon Y^{0,1-3\varepsilon}$.

For $\mathbb{E}^\sharp$, we split $l \le c$ and $l > c$ and consider the former case first. We have

$$\begin{aligned}\sum_{j \lesssim k \sim l \le c} \langle n_j \mathbb{E}_k \mid I_c u_l \rangle_{t,x} &\lesssim \sum_{k \sim l \le c} \|\sum_{j \lesssim k} N_j\|_{L^\infty L^2} \|\mathbb{E}_k\|_{L^2L^4} \|u_l\|_{L^2L^4} \\ &\lesssim \|N\|_{L^\infty L^2} \|\mathbb{E}\|_{X^{1,3/8}} \|u\|_{X^{-1,3/8}},\end{aligned} \tag{5.31}$$

where we used (5.20) for $\mathbb{E}_k$ and u_l. Hence the contribution of the above interactions to $\mathbb{E}^\sharp$ is bounded in $T^\varepsilon X^{1,5/8-\varepsilon}$.

In the case $l > c$, the Strichartz estimate as above does not give uniform bound, so we use bilinear estimates. Setting the distance threshold from characteristics as $\delta = m$, we decompose the duality coupling

$$\begin{aligned}&\langle n_j \mathbb{E}_k \mid I_c u_l \rangle_{t,x} \\ &= \langle n_j^F \mathbb{E}_k \mid I_c u_l \rangle_{t,x} + \langle n_j^C \mathbb{E}_k^F \mid I_c u_l \rangle_{t,x} + \langle n_j^C \mathbb{E}_k^C \mid I_c u_l^F \rangle_{t,x} + \langle n_j^C \mathbb{E}_k^C \mid I_c u_l^C \rangle_{t,x} \\ &=: C_1 + C_2 + C_3 + C_4,\end{aligned} \tag{5.32}$$

where the resonant interaction C_4 does not vanish. The other three terms are nonresonant, and estimated by using the $X^{s,b}$ spaces[‡]. For C_1, we use (5.23) with $(\theta, \varepsilon) \to (1/2 - 3\varepsilon, 2\varepsilon)$ for $\mathbb{E}_k$.

$$\begin{aligned}C_1 &\lesssim \langle l/c \rangle^{-1} m^{1/2+6\varepsilon} \|N_j^F\|_{L^2L^2} \|\mathbb{E}_k\|_{L^{2/(1-4\varepsilon)} L^{6/(1+12\varepsilon)}} \|u_l\|_{L^{1/(2\varepsilon)} L^2} \\ &\lesssim \langle h/c \rangle^{-1+5/6-5\varepsilon} m^{1/2+6\varepsilon} \delta^{-1+10\varepsilon} \|N_j^F\|_{Y^{0,1-10\varepsilon}} \|\mathbb{E}_k\|_{X^{1,1/2-\varepsilon}} \|u_l\|_{X^{-1,1/2-2\varepsilon}},\end{aligned} \tag{5.33}$$

whose contribution to $\mathbb{E}^\sharp$ is bounded in $T^\varepsilon X^{1,1/2+\varepsilon}$. The estimate for C_2 and C_3 are essentially the same. For C_2 we use (5.20) for $\mathbb{E}_k$ and u_l, getting

$$\begin{aligned}C_2 &\lesssim \langle l/c \rangle^{-1} m^{3/10} \|N_j^C\|_{L^\infty L^2} \|\mathbb{E}_k^F\|_{L^2 L^{5/2}} \|u_l\|_{L^2L^5} \\ &\lesssim m^{3/10} \delta^{-7/20+2\varepsilon} \|N_j\|_{Y^{0,1/2+\varepsilon}} \|\mathbb{E}_k^F\|_{X^{1,1/2-2\varepsilon}} \|u_l\|_{X^{-1,9/20}},\end{aligned} \tag{5.34}$$

so its contribution to $\mathbb{E}^\sharp$ is bounded in $T^\varepsilon X^{1,11/20-\varepsilon}$. For C_3, we just switch the roles of $\mathbb{E}_k$ and u_l.

For the resonant interaction C_4, we use the following improvement of the Strichartz estimate on a radially thin Fourier support.

Lemma 5.4. *Assume that $u(t,x) \in X^{0,1/4,1}$ is supported in the Fourier space on*

$$R < |\xi| < R + w, \tag{5.35}$$

for some R and w satisfying $c \lesssim R \gtrsim w > 0$. Then we have

$$\|u\|_{L_t^2 L_x^4} \lesssim c^{-1/4} w^{1/4} R^{1/4} \|u\|_{X^{0,1/4,1}}. \tag{5.36}$$

The same estimate holds for the wave equation $e^{itc|\nabla|}$ without the restriction $R \gtrsim c$. We gain $(w/R)^{1/4}$ compared with the Strichartz estimate without the support

[‡]For the estimates on the nonresonant terms, δ can be slightly smaller, such as $m^{2/3+}$.

condition (2.39). The choice of exponents in the above estimate is an amazing just-fit both for the proof and for the requirement of our problem, except for the second exponent of $X^{0,1/4,1}$, for which we have $1/4-$ room to increase for our use.

Proof. We start with the idea in [6] to use the Fourier restriction on the sphere, which they used for interactions of the same type in the Zakharov system[§].

Let F_r be the Fourier restriction on the sphere of radius $r > 0$, defined by

$$F_r\varphi = \mathcal{F}^{-1}\delta(|\xi| - r)\varphi(\xi). \tag{5.37}$$

By scaling and the Fourier restriction theorem on the sphere, we have for any $\varphi(x)$,

$$\|F_r\varphi\|_{L^4_x} = \|r^{9/4}\mathcal{F}^{-1}r^{-1}\delta(|\xi|-1)\varphi(r\xi)\|_{L^4_x} \lesssim r^{5/4}\|\varphi(r\theta)\|_{L^2_\theta(S^2)}. \tag{5.38}$$

Now assume that $\operatorname{supp}\mathcal{F}\varphi \subset \{R < |\xi| < R + w\}$. Applying this to the identity

$$\varphi = \int_0^\infty F_r\mathcal{F}\varphi dr, \tag{5.39}$$

and using the Schwarz inequality in r, we obtain

$$\|\varphi\|_{L^4_x} \lesssim \int_0^\infty r^{5/4}\|\mathcal{F}\varphi(r\theta)\|_{L^2_\theta} dr \lesssim w^{1/2}R^{1/4}\|\varphi\|_{L^2_x}. \tag{5.40}$$

Just by integrating in t, we get

$$\|u\|_{L^2_tL^4_x} \lesssim w^{1/2}R^{1/4}\|u\|_{L^2_tL^2_x}. \tag{5.41}$$

Next we decompose the space-time Fourier transform $\widetilde{u}$ for $|\tau - \omega_c(\xi)| \sim \delta \in \mathbb{D}$. It suffices to prove the desired estimate on each piece, because the third exponent in (5.36) is 1. Hence we assume that $|\tau - \omega_c(\xi)| \sim \delta$ on $\operatorname{supp}\widetilde{u}$. If $\delta \gtrsim wc$, then we have

$$\|u\|_{L^2_tL^4_x} \lesssim w^{1/2}R^{1/4}\|u\|_{L^2_tL^2_x} \lesssim w^{1/2}R^{1/4}\delta^{-1/4}\|u\|_{X^{0,1/4}}, \tag{5.42}$$

which implies the desired estimate in this case. If $\delta \ll wc$, we further decompose the Fourier support into squares in $(\tau, |\xi|)$ of size δ by

$$\widetilde{u} = \sum_{k\in\mathbb{Z}} \psi_k(\tau/\delta)\widetilde{u}(\tau,\xi), \tag{5.43}$$

where $\psi_k(s) = \psi(s-k) - \psi(s-k+1) \in C_0^\infty(\mathbb{R})$ and $\psi \in C^\infty(\mathbb{R})$ is chosen such that

$$\psi(s) = \begin{cases} 1 & (s < 1/3) \\ 0 & (s > 2/3). \end{cases} \tag{5.44}$$

Hence $\psi_k(\tau/\delta)$ localizes the τ frequency onto $|\tau - \delta k| < 2\delta/3$. Denote the summand by $\widetilde{u}_k$ and define an operator $R : (v_k(t,x))_{k\in\mathbb{Z}} \mapsto (Rv)(t,x)$ by

$$\mathcal{F}_tRv = \sum_{k\in\mathbb{Z}} \sum_{j=-1,0,1} \psi_{k+j}(\tau/\delta)\mathcal{F}_tv_k, \tag{5.45}$$

[§]That estimate could be avoided in their case by the argument in [12], or the above argument for $h \lesssim c$. But we need even sharper estimates to recover uniformity.

where $\mathcal{F}_t$ is the time Fourier transform. By the Plancherel identity and trivial summation, we have

$$\|Rv\|_{L_t^2 L_x^2} \lesssim \|v_k\|_{\ell_k^2 L_t^2 L_x^2}, \quad \|Rv\|_{L_t^2 L_x^\infty} \lesssim \|v_k\|_{\ell_k^1 L_t^2 L_x^\infty}. \tag{5.46}$$

Hence the complex interpolation implies that

$$\|Rv\|_{L_t^2 L_x^4} \lesssim \|v_k\|_{\ell_k^{4/3} L_t^2 L_x^4}. \tag{5.47}$$

Since $u = R(u_k)$, we deduce that

$$\|u\|_{L_t^2 L_x^4} \lesssim \|u_k\|_{\ell_k^{4/3} L_t^2 L_x^4} \lesssim N^{1/4} \|u_k\|_{\ell_k^2 L_t^2 L_x^4}, \tag{5.48}$$

where N is the number of k's satisfying $u_k \neq 0$. The support conditions

$$R < |\xi| < R + w, \quad |\tau - \omega_c(\xi)| \sim \delta, \quad k - 1 < \tau/\delta < k + 1, \tag{5.49}$$

together with $\omega_c'(r) \sim c$ for $r \gtrsim c$, imply that the radial width for ξ is $O(\delta/c)$ for each k and so $N \lesssim wc/\delta$. The width bound also implies via (5.41) that

$$\|u_k\|_{L_t^2 L_x^4} \lesssim (\delta/c)^{1/2} R^{1/4} \|u_k\|_{L_t^2 L_x^2}. \tag{5.50}$$

Plugging this into the above estimate together with the bound on N, we arrive at

$$\|u\|_{L_t^2 L_x^4} \lesssim w^{1/4} c^{-1/4} R^{1/4} \delta^{1/4} \|u\|_{L_t^2 L_x^2}, \tag{5.51}$$

which implies the desired estimate in this case. $\square$

The resonance condition for C_4 implies

$$\pm\alpha|\xi_0| + \omega_c(\xi_1) - \omega_c(\xi) = O(j), \tag{5.52}$$

where ξ_0, ξ_1 and ξ are the Fourier variable on $\mathbb{R}^3$ for N_j^C, $\mathbb{E}_k^C$ and u_l^C, respectively. Since $k \sim l > c$, we have $\omega_c(\xi_1) - \omega_c(\xi) \sim c(|\xi_1| - |\xi|)$, and so

$$||\xi_1| - |\xi|| \lesssim j/c. \tag{5.53}$$

To exploit this, we further decompose $\mathbb{E}$ and u into shells of width j/c:

$$C_4 = \sum_{\substack{|a-b| \lesssim 1,\ a,b \in \mathbb{N} \\ aj/c \sim k,\ bj/c \sim l}} \left\langle N_j^C \mathbb{E}_{a,j/c}^C \,\middle|\, I_c u_{b,j/c}^C \right\rangle_{t,x}, \tag{5.54}$$

where $\varphi_{a,\lambda}$ with $a \in \mathbb{N}$ and $\lambda > 0$ is the Fourier restriction onto the shell of radius $a\lambda$ and width λ defined by

$$\mathcal{F}\varphi_{a,\lambda} := [\psi(|\xi|/\lambda - a) - \psi(|\xi|/\lambda - a + 1)]\mathcal{F}\varphi, \tag{5.55}$$

where ψ is the cut-off function defined in (5.44). Applying the above lemma, we obtain

$$\begin{aligned} &\|\mathbb{E}_{a,j/c}^C\|_{L^2 L^4} \lesssim c^{-1/4} (jk/c)^{1/4} k^{-1} \|\mathbb{E}_{a,j/c}\|_{X^{1,1/4,1}}, \\ &\|u_{b,j/c}^C\|_{L^2 L^4} \lesssim c^{-1/4} (jl/c)^{1/4} l^1 \|u_{b,j/c}\|_{X^{-1,1/4,1}}, \end{aligned} \tag{5.56}$$

Hence the summand in (5.54) is dominated by

$$\langle l/c\rangle^{-1} j^{1/2} c^{-1} k^{-3/4} l^{5/4} \|N_j\|_{L^\infty L^2} \|\mathbb{E}_{a,j/c}\|_{X^{1,1/4,1}} \|u_{b,j/c}\|_{X^{-1,1/4,1}}. \tag{5.57}$$

Applying the Schwarz inequality for $a = b + O(1)$, we obtain

$$C_4 \lesssim \sum_{j \lesssim k \sim l} (j/l)^{1/2} \|N_j\|_{Y^{0,1/2+\varepsilon}} \|\mathbb{E}_k\|_{X^{1,1/4+\varepsilon}} \|u_l\|_{X^{-1,1/4+\varepsilon}}, \tag{5.58}$$

so its contribution to $\mathbb{E}^\sharp$ is bounded in $T^\varepsilon X^{1,3/4-2\varepsilon}$.

5.3. **Uniform bounds and convergence.** Let $V := (\mathbb{E}, N) = (\mathbb{E}^c, N^c)$ and $\widehat{V} := (\widehat{\mathbb{E}}, \widehat{N}) = (\widehat{\mathbb{E}}^c, \widehat{N}^c)$ be as in Theorem 5.1. Let $V^\sharp := \Phi(V)$, and similarly we define $\widehat{V}^\sharp = (\widehat{\mathbb{E}}^\sharp, \widehat{N}^\sharp) = \widehat{\Phi}(\widehat{V})$ by

$$\begin{aligned} \widehat{\mathbb{E}}^\sharp &:= e^{-it\Delta/2}\left[\chi(t)\widehat{\mathbb{E}}(0) + \frac{i}{2}\mathcal{I}_T e^{it\Delta/2}\widehat{n}\widehat{\mathbb{E}}\right], \\ \widehat{N}^\sharp &:= e^{it|\alpha\nabla|}\left[\chi(t)\widehat{N}(0) + i\mathcal{I}_T e^{-it|\alpha\nabla|}|\alpha\nabla||\widehat{\mathbb{E}}|^2\right]. \end{aligned} \tag{5.59}$$

Let V^0 and $\widehat{V}^0$ be the free parts of $V^\sharp$ and $\widehat{V}^\sharp$, respectively. The integral equations $V^\sharp = \Phi(V^\sharp)$, $\widehat{V}^\sharp = \widehat{\Phi}(\widehat{V}^\sharp)$ can be written schematically as

$$V^\sharp = V^0 + Q[V^\sharp] + Q^*[V^\sharp], \quad \widehat{V}^\sharp = \widehat{V}^0 + Q^\infty[\widehat{V}^\sharp], \tag{5.60}$$

where $Q[V] = Q[V, V]$ denotes the quadratic parts without $\mathbb{E}^*$, while Q^* consists of those with $\mathbb{E}^*$, and Q^∞ is the limit ones. The estimates in the previous subsections can be written as

$$\begin{aligned} &\|Q[V,W]\|_{\mathcal{X}_3} + \|Q^*[V,W]\|_{\mathcal{X}_3} \lesssim T^\varepsilon \|V\|_{\mathcal{X}_3}\|W\|_{\mathcal{X}_3}, \\ &\|Q^*[V,W]\|_{\mathcal{X}_2} \lesssim T^\varepsilon c^{-\varepsilon}\|V\|_{\mathcal{X}_3}\|W\|_{\mathcal{X}_3}. \end{aligned} \tag{5.61}$$

Hence for small T and large c, we obtain uniform bound in $\mathcal{X}_3$, for which we do not need the assumption of uniform decay for higher frequency (5.3). If we assume it, then it is inherited by V^0 and $V^\sharp$ as follows. We have

$$\|V^0_{>R}\|_{\mathcal{X}_3} \lesssim \|V(0)_{>R}\|_{H^1\times L^2}, \quad \|V^\sharp_{>R}\|_{\mathcal{X}_3} \lesssim \|V^0_{>R}\|_{\mathcal{X}_3} + T^\varepsilon \|V^\sharp_{>R/8}\|_{\mathcal{X}_3}\|V^\sharp\|_{\mathcal{X}_3}. \tag{5.62}$$

Hence we obtain

$$\lim_{R\to\infty} \limsup_{c\to\infty} \|V^\sharp_{>R}\|_{\mathcal{X}_3} = 0, \tag{5.63}$$

for small $T > 0$. One can observe from the arguments in the previous subsections that $\widehat{V}^\sharp$ has the same estimates if $\mathcal{X}_3$ is replaced with the limit space $\mathcal{X}_3^\infty$, i.e.,

$$\|Q^\infty[V,W]\|_{\mathcal{X}_3^\infty} \lesssim T^\varepsilon \|V\|_{\mathcal{X}_3^\infty}\|W\|_{\mathcal{X}_3^\infty}, \tag{5.64}$$

hence

$$\lim_{R\to\infty} \limsup_{c\to\infty} \|\widehat{V}^\sharp_{>R}\|_{\mathcal{X}_3^\infty} = 0. \tag{5.65}$$

Thus it suffices to show

$$\limsup_{c\to\infty} \|(V^\sharp - \widehat{V}^\sharp)_{\le R}\|_{\mathcal{X}_2+\mathcal{X}_3+\mathcal{X}_3^\infty} = 0 \tag{5.66}$$

for all fixed $R \ge 1$, since the $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_3^\infty$ norms are all equivalent in the frequency $\le R$. We have

$$\begin{aligned} V^\sharp - \widehat{V}^\sharp =& Q^*[V^\sharp] + (Q[V^\sharp] - Q[V^\sharp_{\le R}]) - (Q^\infty[\widehat{V}^\sharp] - Q^\infty[\widehat{V}^\sharp_{\le R}]) \\ &+ (Q - Q^\infty)[\widehat{V}^\sharp_{\le R}] + (Q[V^\sharp_{\le R}] - Q[\widehat{V}^\sharp_{\le R}]). \end{aligned} \tag{5.67}$$

On the right, the first term is vanishing as $c^{-\varepsilon}$ in $\mathcal{X}_2$. The second term is vanishing in $\mathcal{X}_3$ by (5.63). The third term is vanishing in $\mathcal{X}_3^\infty$ by (5.65). The fourth term is vanishing because $c^2(\langle \xi/c\rangle - 1) \to |\xi|^2/2$ and $\langle \xi/c\rangle^{-1} \to 1$ uniformly on $|\xi| \le R$. The fifth term is bounded by $T^\varepsilon \|(V^\sharp - \widehat{V}^\sharp)_{\le R}\|_{\mathcal{X}_3}$, and so absorbed by the left hand side. Thus we obtain the desired convergence.

6. Zakharov to NLS

For the convergence from the Zakharov system to the nonlinear Schrödinger equation, we have a very simple proof, relying on the time-local a priori bound by the nonlinear energy.

Theorem 6.1. *Consider the limit $\alpha \to \infty$. For each α, let (u^α, n^α) be a solution of* (1.7) *given by* [6]*, and denote its maximal existence time by T^α. Assume that $u^\alpha(0)$ converges in H^1, that $(n^\alpha(0), |\alpha\nabla|^{-1}\dot n^\alpha(0))$ is bounded in L^2, and that the latter has uniform decay for high frequency, namely,*

$$\lim_{R\to\infty} \limsup_{\alpha\to\infty} \|(n^\alpha(0), |\alpha\nabla|^{-1}\dot n^\alpha(0))_{>R}\|_{L^2} = 0. \tag{6.1}$$

Let u^∞ be the solution of (1.3) *with $u^\infty(0) = \lim_{\alpha\to\infty} u^\alpha(0)$, and T^∞ be the maximal existence time. Then we have $\liminf_{\alpha\to\infty} T^\alpha \ge T^\infty$, and for all $0 < T < T^\infty$,*

$$\begin{aligned} &u^\alpha - u^\infty \to 0 \text{ in } C([0,T]; H^1), \\ &n^\alpha + |u^\infty|^2 - n^\alpha_f \to 0 \text{ in } C([0,T]; L^2), \\ &|\alpha\nabla|^{-1}(\dot n^\alpha - \dot n^\alpha_f) \to 0 \text{ in } C([0,T]; L^2), \end{aligned} \tag{6.2}$$

where n^α_f is the free wave defined by

$$\begin{cases} \alpha^{-2}\ddot n^\alpha_f - \Delta n^\alpha_f = 0, \\ n^\alpha_f(0) = n^\alpha(0) + |u^\infty(0)|^2, \quad \dot n^\alpha_f(0) = \dot n^\alpha(0). \end{cases} \tag{6.3}$$

Proof. We omit the superscript α. First we derive a uniform bound from the conserved energy $\mathcal{E}_N = \mathcal{E}(t) + \mathcal{N}(t)$, where both

$$\begin{aligned} &\mathcal{E}(t) := \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \|n\|_{L^2}^2/2 + \||\alpha\nabla|^{-1}\dot n\|_{L^2}^2/2, \\ &\mathcal{N}(t) := \int n|u|^2 dx, \end{aligned} \tag{6.4}$$

are initially bounded by the assumption. We will derive a priori bound on the energy norm $H_T := \sup_{0\le t\le T} \mathcal{E}(t)^{1/2}$ for small $T > 0$ independent of α. Decomposing u into the linear and nonlinear parts

$$u(t) = u^0 + u^1, \quad u^0 := e^{-i\Delta t}u(0), \tag{6.5}$$

we estimate the nonlinear energy

$$\begin{aligned} |\mathcal{N}(t)| &\lesssim \|n(t)\|_{L^2_x} \left[\|u^0(t)\|_{L^4_x}^2 + \|u^1(t)\|_{L^4_x}^2\right] \\ &\lesssim H_T\|u(0)\|_{H_x^{3/4}}^2 + H_T^{8/3}\|u^1(t)\|_{H_x^{-1/2}}^{1/3}, \end{aligned} \tag{6.6}$$

where we used the Hölder inequality and the Sobolev embedding $[H^{-1/2}, H^1]_{5/6} = H^{3/4} \subset L^4$. By the equation for u, we have on $(0, T)$

$$\|u^1\|_{L_t^\infty H_x^{-1/2}} \lesssim T\|nu\|_{L_t^\infty L_x^{3/2}} \lesssim T\|n\|_{L_t^\infty L_x^2}\|u\|_{L_t^\infty H_x^1} \lesssim T H_T^2. \tag{6.7}$$

Hence, by the conservation of energy, we obtain

$$H_T^2 \le \mathcal{E}_N + C(H_T\|u(0)\|_{H^{3/4}}^2 + T^{1/3}H_T^{10/3}), \tag{6.8}$$

which implies via the Schwarz inequality,

$$H_T^2 \le 2\mathcal{E}_N + C^2\|u(0)\|_{H^{3/4}}^4 + 2CT^{1/3}H_T^{10/3}, \tag{6.9}$$

with an absolute constant $C > 0$. Then the continuity on T implies that

$$H_T^2 \le 2B, \quad B := 2\mathcal{E}_N + C^2\|u(0)\|_{H^{3/4}}^4, \tag{6.10}$$

provided that $T \le (2C)^{-3}2^{-5}B^{-2}$.

Next we derive the weak convergence. The energy bound together with the equation of u implies that $\dot{u}$ is bounded in $L^\infty H^{-1}$, and so u is equi-continuous with respect to α in the weak topology of H^1, hence it is convergent, along some subsequence of $\alpha \to \infty$, in $C([0, T]; w\text{-}H^1 \cap L^p_{loc})$ for any $p < 6$. By the equation of n and the energy bound, we have

$$\Delta(n + |u|^2) \to 0 \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^3), \tag{6.11}$$

and $n + |u|^2$ is bounded in $L_t^p L^2$, so weakly goes to 0 for any $p < \infty$. Hence u^∞ is a weak solution of (1.3) in $C([0, T]; w\text{-}H^1)$, and its uniqueness implies the convergence for the whole limit.

Finally we prove the strong convergence. By the L^2 conservation law for both the equations and by the weak convergence, we have

$$\|u(t) - u^\infty(t)\|_{L_x^2}^2 = 2\langle u^\infty(t) - u(t) \mid u^\infty(t)\rangle_x + \|u(0)\|_{L_x^2}^2 - \|u^\infty(0)\|_{L_x^2}^2 \to 0, \tag{6.12}$$

uniformly in $t \in [0,T]$. Interpolating with the weak H^1 convergence, we get L^4 strong convergence. Let $N := n - i|\alpha\nabla|^{-1}\dot{n}$ and $N^I := e^{i|\alpha\nabla|t}(n(0) + |u(0)|^2)$. The conserved energy $\mathcal{E}$ can be decomposed as

$$\begin{aligned}\mathcal{E} =&\|\nabla u\|_{L^2}^2 - \frac{\|u\|_{L^4}^4}{2} + \|N + |u|^2\|_{L^2}^2/2 \\ =&\|\nabla u^\infty\|_{L^2}^2 - \frac{\|u^\infty\|_{L^4}^4}{2} + \|N^I\|_{L^2}^2/2 \\ &+ \|\nabla(u - u^\infty)\|_{L^2}^2 + \|N + |u|^2 - N^I\|_{L^2}^2/2 \\ &+ 2\langle u^\infty - u \mid \Delta u^\infty\rangle_x - \frac{\|u\|_{L^4}^4 - \|u^\infty\|_{L^4}^4}{2} - \langle N + |u|^2 - N^I \mid N^I\rangle_x,\end{aligned} \tag{6.13}$$

where the second line is a conserved quantity, the third one at $t = 0$ goes to 0, and on the last line, the first and second terms tend to 0, uniformly in t, by the weak H^1 and strong L^4 convergence. Hence it suffices to show that the last term is also vanishing. For any $\varepsilon > 0$, there exists $R > 0$, independent of α, such that $|\langle N + |u|^2 - N^I \mid N^I_{>R}\rangle_x| < \varepsilon$, because of the assumption (6.1). We can rewrite the lower frequency part as

$$\langle N + |u|^2 - N^I \mid N^I_{\le R}\rangle_x = \int_0^t \langle e^{-i|\alpha\nabla|s}|u|_t^2(s) \mid N^I(0)_{\le R}\rangle_x ds. \tag{6.14}$$

Its absolute value is bounded by the Strichartz estimate (4.17)

$$\lesssim \alpha^{-1/4}T^{3/4}\||u|_t^2\|_{L^\infty B^{-1}_{4/3,2}}\|N^I(0)\|_{L^2}R^{3/2}, \tag{6.15}$$

where the norm for $|u|_t^2 = \nabla\cdot\langle\nabla u, iu\rangle$ is bounded by H_T^2. Thus we obtain $u - u^\infty \to 0$ in $L^\infty H^1$ and $N + |u|^2 - N^\infty \to 0$ in $L^\infty L^2$, as desired. □

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