On the Diffusion limit of a semiconductor Boltzmann-Poisson system without micro-reversible process

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Abstract. This paper deals with the diffusion approximation of a semiconductor Boltzmann-Poisson system. The statistics of collisions we are considering here, is the Fermi-Dirac operator with the Pauli exclusion term and without the detailed balance principle. Our study generalizes, the result of Goudon and Mellet [14], to the multi-dimensional case.

keywords: Semiconductor, Boltzmann-Poisson, diffusion approximation, Fermi-Dirac, detailed balance, Hybrid-Hilbert expansion, entropy dissipation.

1 Introduction

We are concerned with the study of the diffusion approximation of nonlinear semiconductor Boltzmann equations. In semi-classical kinetic description, electrons (for example) in semiconductor devices can be described by their distribution function $f(t,x,k)$. The time variable is $t \in \mathbf{R}^+$, x is the position which is in a bounded domain ω of $\mathbf{R}^d$ which we assume for simplicity to be periodic, namely $\omega = \mathbf{T}^d := (\mathbf{R}/2\pi\mathbb{Z})^d$. The pseudo wave-vector k lies in the first Brillouin zone $\mathcal{B}$. $\mathcal{B}$ is identified with the d dimensional torus $\mathbf{T}^d$.The Boltzmann equation reads as

$$\frac{\partial f}{\partial t} + v(k).\nabla_x f - \frac{q}{\hbar}\nabla_x \Phi \,.\, \nabla_k f = Q(f).$$

Here the velocity and the acceleration terms of the electrons solve the Newton's equations

$$\frac{dx}{dt} = v(k) := \nabla_k \varepsilon(k), \qquad \frac{dk}{dt} = -\frac{q}{\hbar}\nabla_x \Phi.$$

The diagram band, $\varepsilon(k)$ is cell-periodic, while the constants q and $\hbar$ are respectively, the elementary charge and the reduced Planck constant. We will assume that the energy band has a finite number of critical values. This will be detailed later on (see for example [4]). The electrostatic potential is self-consistent, it solves the Poisson equation

$$-\Delta_x \Phi(t,x) = \rho(t,x) - D$$

where

$$\rho(t,x) = \int_{\mathcal{B}} f(t,x,k)\, dk$$

and D is a doping profile. The collision dynamics, we are considering here is the statistics of Fermi-Dirac. It reads as

$$Q(f) = \int_{k' \in \mathcal{B}} \left[\sigma(k,k')(1-f(k))f(k') - \sigma(k',k)(1-f(k'))f(k)\right]\, dk'$$

when there is no ambiguity, we will denote by f, f', σ, σ' respectively $f(k)$, $f(k')$, $\sigma(k,k')$ and $\sigma(k',k)$. This operator describes binary collisions between impurities and phonons in degenerate semiconductor [16]. It takes into account the Pauli's exclusion principle which means that two electrons can not occupy the same state. This hypothesis implies (in particular) that during the evolution the density of particles takes bounded values $0 \le f \le 1$. We consider, here, a cross section which does not satisfy the so-called micro-reversibility principle. This means that $e^{\varepsilon(k)}\,\sigma(k,k')$ is not necessary symmetric. Usually when modelling collisions, it is assumed that the cross section satisfies the detailed balance principle:

$$\sigma(k,k')\, e^{\varepsilon(k)} = \sigma(k',k)\, e^{\varepsilon(k')} \tag{1}$$

This assumption implies that the equilibrium state ($Q(F) = 0$) is described by the Fermi-Dirac function :

$$F(\mu(t,x),k) = \frac{1}{1 + \exp(\mu(t,x) + \varepsilon(k))}.$$

The parameter $\mu \in \bar{\mathbf{R}}$, is the chemical potential. This situation has been recently, investigated in asymptotic analysis of kinetic equations [1, 2, 5, 12, 15, 17]. From physical point of view, in heterogeneous media, like biological systems, the detailed balance principle does not hold [21]. In [8], Degond, Goudon and Poupaud have studied the diffusion approximation for a linear transport for non-micro-reversible processes. The derivation of a non linear fluid model, has been obtained by Goudon and Mellet in [13]. In that work the authors studied, the present model for a given potential. In [14], a rigorous analysis of existence, uniqueness and regularity of solutions for the limit fluid model (coupled to Poisson equation) is presented. However, the diffusion approximation is only studied for the one-dimensional case. We point out that this analysis is based on the contraction property of the collision operator [20], and the use of a Hybrid-Hilbert expansion introduced in [5] for the resolution of such nonlinear asymptotics. We, also cite the work of Ben Abdallah, Escobedo and Mischler [3] where the authors, use the contraction property of the Pauli operator to obtain a decay of the solution for long time.

Our aim in the present work is to analyze the multi-dimensional case. In particular, we would like to prove a strong convergence and exhibit a rate of convergence without restriction on the dimension. The main idea here is to replace a Gronwall lemma by an Osgood lemma (see Chemin [7] for an application to fluid mechanics). This idea was used by Yudovich [32] to prove existence and uniqueness to the incompressible Euler system when the vorticity is bounded. The difficulty there was coming from the fact that the Riesz transform is not bounded in L^∞. In our case the problem is coming from L^1 and we have to replace L^1 by some L^p for p close to 1.

2 Setting of the problem and main result

We consider the diffusion approximation of the Boltzmann equation

$$\frac{\partial f^\alpha}{\partial t} + \frac{1}{\alpha}\left(v(k)\,.\,\nabla_x f^\alpha - \nabla_x \Phi^\alpha\,.\,\nabla_k f^\alpha \right) = \frac{Q(f^\alpha)}{\alpha^2} \tag{2}$$

The electrostatic potential solves the Poisson equation

$$-\Delta_x \Phi^\alpha(t,x) = \int_{\mathcal{B}} f^\alpha(t,x,k)\,dk - D \tag{3}$$

We assume that f^α is periodic in x and k and that Φ^α is periodic in x. We also impose the following initial data

$$f^\alpha(t=0,x,k) = f_{in}(x,k), \qquad \forall\,(x,k)\in\Omega. \tag{4}$$

Assumptions. We shall assume that

A1. The initial data f_{in} is well-prepared in the sense that $Q(f_{in})=0$. We also assume the natural bound (see [2, 11]):

$$0 \le f_{in} \le 1$$

A2. The cross section σ is smooth, bounded and does not necessary verify the detailed balance principle. It satisfies

$$\exists\,\sigma_1,\ \sigma_2 \quad / \quad 0<\sigma_1\le\sigma(k,k')\le\sigma_2, \quad \forall\,(k,k')\in\mathcal{B}^2.$$

A3. Non degeneracy condition :

$$\text{For all } \xi\in\mathbf{R}^d-\{0\}, \quad meas\,(\{k\in\mathcal{B},\ /\ \nabla_k\varepsilon(k)\,.\,\xi\neq 0\})>0$$

A4. Solvability condition : For all $\rho\in[0,1]$, we have

$$\int v(k)\frac{\partial F}{\partial \rho}\,dk = 0$$

where $F(\rho,k)$ is defined in Proposition 3.2.

Main result. Our main result is the following (we refer to the next two sections for some notations):

Theorem 2.1 *Under the assumptions A1, A2, A3 and A4, the solution* (f^α,Φ^α) *of the scaled Boltzmann-Poisson system (2-3-4)converges, when* α *goes to zero, towards* $(F(\rho,k),\Phi)$: $\forall\,T>0, \exists\,C>0\,/$

$$\|f^\alpha - F(\rho,k)\|_{L^\infty(0,T;\,L^p(\Omega))} \;\le\; (C\alpha)^{\frac{1}{p}e^{-C\,T}}\,e^{\frac{2C\,T}{p}}, \quad \forall\,p\in[1,\infty) \tag{5}$$

$$\|\Phi^\alpha - \Phi\|_{L^\infty(0,T;\,W^{2,p}(\omega))} \;\le\; \frac{Cp}{p-1}(C\alpha)^{\frac{1}{p}e^{-C\,T}}\,e^{\frac{2C\,T}{p}}, \quad \forall\,p\in(1,\infty) \tag{6}$$

where (ρ,Φ) *is the solution of the Drift-Diffusion Poisson system (10) and* C *depends only on* T. $\square$

3 Preliminaries

Before starting our analysis, we would like to precise some existence results. The existence result for the initial system can be found in Poupaud [24].

Proposition 3.1 *[Existence and uniqueness [24]]. Let α be a fixed nonnegative parameter and $f_{in} \in W^{1,1}(\Omega)$ satisfy $0 \leq f_{in} \leq 1$. Under the previous assumptions, there exists a unique solution (f^α, Φ^α) to the Boltzmann-Poisson system such that*

$$\begin{cases} f^\alpha \in W^{1,1} \cap W^{1,\infty}_{t,x,k}, \qquad 0 \leq f^\alpha \leq 1, \\ \Phi^\alpha \in W^{2,\infty}_{t,x}. \end{cases}$$

□

The existence and the uniqueness of an equilibrium state $F(\rho, k)$ for a given mass ρ can be proved by applying an implicit-function theorem. More precisely,

Proposition 3.2 *[14]. Let $\rho \in [0,1]$. Under the assumption A2, there exists a unique $F(\rho, k)$ solution of*

$$\begin{cases} 0 \leq F(\rho,k) \leq 1, \qquad \displaystyle\int_{\mathcal{B}} F(\rho,k)dk = \rho, \\ Q(F(\rho)) = 0. \end{cases}$$

Moreover, F is smooth with respect to the density ρ and for all $n \geq 0$, the derivatives $\frac{\partial^n F}{\partial \rho^n}$ belong to $L^\infty([0,1] \times \mathcal{B})$. □

The collision operator Q is nonlinear. To derive (formally) the limit fluid model from the scaled Boltzmann equation (when tending α to zero) we shall study the linearized operator DQ. Let us remark that: for all f and h,

$$Q(f+h) = Q(f) + L_f(h) + R(h,h)$$

where the linearized part L_f of Q is

$$L_f(g) = DQ(f)g = \int_{\mathcal{B}} (\, s_f(k,k')g(k') - s_f(k',k)g(k))\, dk' \tag{7}$$

and the cross section s is

$$s_f(k,k') = \sigma(k,k')(1-f(k)) + \sigma(k',k)f(k).$$

The quadratic part R reads as

$$R(g,h) = \frac{1}{2}\int_{\mathcal{B}} (\sigma(k,k') - \sigma(k',k))\ [g(k)\, h(k') + g(k')\, h(k)]\, dk'.$$

The operator L_f satisfies the following spectral properties

Proposition 3.3 *[19]. Let $f \in L^2(\mathcal{B})$, satisfy $0 \leq f \leq 1$. Then,*

1. *Equilibrium state: $\exists\, M_f \in L^2_+(\mathcal{B})\ /\ \int_{\mathcal{B}} M_f\, dk = 1$ and*

$$\mathcal{N}(L_f) = \mathbf{R}\, M_f$$

Moreover,

$$\frac{\sigma_1}{\sigma_2} \leq M_f \leq \frac{\sigma_2}{\sigma_1}$$

2. *Entropy dissipation: $\exists\, \delta > 0$ such that for all $g \in L^2(\mathcal{B})$*

$$\langle -L_f(g)\,,\, \frac{g}{M_f}\rangle = -\int_{\mathcal{B}} L_f(g)\,\frac{g}{M_f} dk \geq \delta \left\| g - \left(\int_{\mathcal{B}} g dk\right) M_f \right\|^2_{L^2(\mathcal{B})}$$

3. *Solvability condition: $\forall\, h \in L^2(\mathcal{B})$, $\exists\, g \in L^2(\mathcal{B})/L_f(g) = h$ if and only if $\int_{\mathcal{B}} h\, dk = 0$. This solution is unique under the condition $\int_{\mathcal{B}} g\, dk = 0$.* □

Actually, it is easy to see that if $f(k) = F(\rho, k)$ for some $\rho \in [0,1]$, then $M_f(k) = \frac{\partial F(\rho,k)}{\partial \rho}$.

4 Formal expansion

Let us introduce the following Hilbert expansion

$$f^{\alpha} := f_0 + \alpha\, f_1 + \alpha^2\, f_2 + \cdots \tag{8}$$

Inserting this expansion in the scaled Boltzmann equation, it becomes

$$\frac{\partial f_0}{\partial t} + \frac{1}{\alpha}\left(v(k).\nabla_x f_0 - \nabla_x\Phi\,.\,\nabla_k f_0\right) + \left(v(k).\nabla_x f_1 - \nabla_x\Phi\,.\,\nabla_k f_1\right) = \frac{1}{\alpha^2}Q(f_0)$$
$$+\frac{1}{\alpha}\,L_{f_0}(f_1) + L_{f_0}(f_2) + R(f_1,f_1) + \cdots$$

Identifying terms with the same powers in α, we get

$$\begin{aligned}
Q(f_0) &= 0,\\
L_{f_0}(f_1) &= v(k).\nabla_x f_0 - \nabla_x\Phi\,.\,\nabla_k f_0,\\
L_{f_0}(f_2) &= \frac{\partial f_0}{\partial t} + v(k).\nabla_x f_1 - \nabla_x\Phi\,.\,\nabla_k f_1 - R(f_1,f_1)
\end{aligned}$$

The first approximation of f^{α} is a Fermi-Dirac function:

$$f_0(t,x,k) = F(t,\rho(t,x),k)$$

where ρ is its associated density $\rho := \int_{\mathcal{B}} f_0\, dk$. To solve the second equation, we replace f_0 by the above expression. We obtain

$$L_{f_0}(f_1) = \frac{\partial F}{\partial \rho}(\rho,k)\, v(k).\nabla_x\rho - \nabla_x\Phi\,.\,\nabla_k F(\rho,k).$$

This requires the introduction of two auxiliary functions $\lambda(\rho,k), \nu(\rho,k)$ which are respectively the solutions in $[L^{\infty}(L^2)]^d$ of

$$\begin{cases} L_{F(\rho)}(\lambda) = v(k)\frac{\partial F}{\partial \rho}(\rho,k); \qquad L_{F(\rho)}(\nu) = \nabla_k\, F(\rho,k) \\ \displaystyle\int_{\mathcal{B}} \lambda(\rho,k)\, dk = \int_{\mathcal{B}} \nu(\rho,k)\, dk = 0. \end{cases} \tag{9}$$

Here, the existence of λ is a consequence of A4 and proposition 3.3. For ν, the existence is insured by the fact that $\int \nabla_k F\ dk = 0$. Then, the function f_1 has the form

$$f_1(t,x,k) = \lambda\,.\,\nabla_x\rho - \nu\,.\,\nabla_x\Phi + \theta(t,x)\frac{\partial\, F}{\partial \rho}$$

where $\theta\dfrac{\partial\, F}{\partial\, \rho}$ is an arbitrary function of the null space $\mathcal{N}(L_{F(\rho)})$. Integrating the third equation we obtain a condition on the density ρ and this is formally the limit fluid model. We remark, here that the integral with respect to k of the remainder $R(g,g)$ vanishes for all g and the solvability condition stands as

$$\frac{\partial \rho}{\partial t} + \nabla_x.\int_{\mathcal{B}} v(k) f_1\, dk = 0.$$

Replacing f_1 by its expression in this equation, we get that $F(t,\rho(t,x),k)$ satisfies

$$\begin{cases} \dfrac{\partial \rho}{\partial t} - \nabla_x[\,\Pi(\rho,x)\nabla_x\rho - \Theta(\rho,x)\nabla_x\Phi\,] = 0, \\ -\Delta_x\Phi = \displaystyle\int_{\mathcal{B}} F(\rho,k)dk - D, \\ \rho(t=0,x) = \rho_{in}(x) \end{cases} \tag{10}$$

where the matrix Π and Θ are defined by

$$\begin{aligned} \Pi(\rho) &= -\int_{\mathcal{B}} v(k) \otimes \lambda(\rho, k) dk, \\ \Theta(\rho) &= -\int_{\mathcal{B}} v(k) \otimes \nu(\rho, k) dk \end{aligned} \tag{11}$$

and λ and ν are defined in (9).

The following propositions and remark summarize the properties of this above fluid system. We point out that the regularity of the limit model is very useful for the study of the diffusion limit by Hilbert or Chapman-Enskog like expansions.

Proposition 4.1 *[19]. Under the assumptions A2, A3 and A4:*

1. *The coefficients Π and Θ are continuous Lipschitz functions of the density $\rho \in \mathbf{R}$ and smooth with respect to $\rho \in]0,1[$.*
2. *$\Pi_{i,j}$ and $\theta_{i,j}$ belong to $L^\infty([0,1])$.*
3. *The matrix $\Pi(\rho)$ is positive definite and there exists a constant $\beta > 0$ such that for all $\xi \in \mathbf{R}^d$,*

$$\Pi(\rho)\, \xi \,.\, \xi \geq \beta\, |\xi|^2, \qquad \forall\, \rho \in [0,1].$$

Proposition 4.2 *[14]. Let $T > 0$. Then, the system (10-11) has a unique solution (ρ, Φ) satisfying $\rho \in L^2((0,T;\ W^{1,2}(\omega)) \cap W^{1,2}(0,T;\ W^{-1,2}(\omega))$ and $\Phi \in L^2(0,T;\ W^{1,2}(\omega))$. Moreover, $0 \leq \rho \leq 1$.*

Remark 4.3 *[14]. The solution (ρ, Φ) of the system (10-11) is $\mathcal{C}^\infty$ with respect to t and x for $t > 0$.*

5 Proof of the convergence result

To explain the difficulty of the asymptotic analysis, we recall that when we consider the Boltzmann equation with a cross section satisfying the detailed balance principle ; the analysis of the convergence uses the entropy dissipation. This idea allows to control the distance between f^α and its local equilibrium $F(\rho^\alpha(t,x), k)$ (see [23]). The function F is Lipschitz continuous with respect to the variable ρ. By applying the velocity averaging lemma [12, 17] we can prove the compactness of ρ^α in $L^2_{t,x}$. This property is enough to pass to the limit (as α tends to zero) in the non linear terms of the scaled equation and then we obtain a strong convergence of f^α towards a Fermi-Dirac function $F(\rho(t,x), k)$. We note that this method gives a L^p-convergence. However, in the self consistent case and a cross section which dos not satisfy the detailed balance principle ; the averaging lemma does not give immediately the convergence of f^α. In this case, the contraction property of the operator Q is used. In [14], Goudon and Mellet have studied the one dimensional case. The convergence proof uses the Gronwall lemma to approximate $f^\alpha - F(\rho, k)$ in L^1. In the present work we deal with the multi-dimensional case.

The contraction property of the collision operator (see [3, 20]) reads as

$$\int_{\mathcal{B}} (Q(f) - Q(g)) sgn(f-g) dk \leq 0 \tag{12}$$

where sgn is the sign function. Denoting

$$r^\alpha = f^\alpha - (F(\rho) + \alpha\, f_1 + \alpha^2\, f_2), \tag{13}$$

we can remark that

$$Q(f^\alpha) = Q(F(\rho) + \alpha\, f_1^\alpha + \alpha^2\, f_2) + N_{1-f^\alpha}(r^\alpha) - N_{r^\alpha}(f^\alpha - r^\alpha)$$

where

$$N_f(g) = \int_{\mathcal{B}} (\sigma(k,k') f g' - \sigma(k',k) f' g)\, dk'.$$

As a consequence, r^α satisfies the following scaled Boltzmann equation

$$\left\{\begin{array}{lcl}\dfrac{\partial r^\alpha}{\partial t}+\dfrac{1}{\alpha}\Big(v(k)\,.\,\nabla_x r^\alpha-\frac{q}{\hbar}\nabla_x\Phi^\alpha\,.\,\nabla_k r^\alpha\Big) & = & \dfrac{1}{\alpha^2}N_{1-f^\alpha}(r^\alpha)-\dfrac{1}{\alpha^2}N_{r^\alpha}(F(\rho)+\alpha\,f_1+\alpha^2\,f_2)\\ & - & \dfrac{1}{\alpha}\nabla_x(\Phi^\alpha-\Phi).\nabla_k(F(\rho)+\alpha f_1)+\alpha\,S^\alpha\\ r^\alpha(t=0)=\alpha(f_1+\alpha\,f_2) & & \end{array}\right.$$

where

$$S^\alpha:=R(f_1,f_2)+R(f_2,f_1)+\alpha\,R(f_2,f_2)-\frac{\partial f_1}{\partial t}-\alpha\,\frac{\partial f_2}{\partial t}-v(k)\,.\,\nabla_x f_2-\nabla_x\Phi^\alpha\,.\,\nabla_k f_2.$$

The term $\dfrac{1}{\alpha}\nabla_x(\Phi^\alpha-\Phi)$ is singular (at this stage). This is why we replace (13) by the Hybrid Hilbert expansion (see [5, 13]):

$$\tilde{r^\alpha}=f^\alpha-(f_0+\alpha\,f_1^\alpha+\alpha^2\,f_2). \tag{14}$$

The function f_1^α is the solution in $\mathcal{N}(L_{f_0})$ of

$$L_{f_0}(f_1^\alpha)=\frac{\partial F}{\partial\rho}(\rho,k)\,v(k).\nabla_x\rho-\nabla_x\Phi^\alpha\,.\,\nabla_k F(\rho,k)$$

It has the form

$$f_1^\alpha(t,x,k)=\lambda(\rho)\,.\,\nabla_x\rho-\nu\,.\,\nabla_x\Phi^\alpha.$$

Using the Hybrid-Hilbert expansion (14), the remainder $\tilde{r^\alpha}$ satisfies

$$\left\{\begin{array}{lcl}\dfrac{\partial\tilde{r^\alpha}}{\partial t}+\dfrac{1}{\alpha}\Big(v(k)\,.\,\nabla_x\tilde{r^\alpha}-\frac{q}{\hbar}\nabla_x\Phi^\alpha\,.\,\nabla_v\tilde{r^\alpha}\Big) & = & \dfrac{1}{\alpha^2}N_{1-f^\alpha}(\tilde{r^\alpha})-\dfrac{1}{\alpha^2}N_{\tilde{r^\alpha}}(f^\alpha-\tilde{r^\alpha})+\alpha\,S_1^\alpha+S_2^\alpha\\ \tilde{r^\alpha}(t=0)=\alpha(f_1+\alpha\,f_2)(t=0) & & \end{array}\right.$$

where S_1^α and S_2^α are given by

$$S_1^\alpha=R(f_1^\alpha,f_2)+R(f_2,f_1^\alpha)+\alpha\,R(f_2,f_2)-\alpha\frac{\partial f_2}{\partial t}-v(k)\,.\,\nabla_x f_2-\nabla_x\Phi^\alpha\,.\,\nabla_k f_2$$

and

$$\begin{array}{rcl}S_2^\alpha & = & v\,.\,\nabla_x(f_1-f_1^\alpha)+\nabla_x(\Phi-\Phi^\alpha).\nabla_k f_1+\nabla_x\Phi^\alpha.\nabla_k(f_1-f_1^\alpha)\\ & + & R(f_1^\alpha+f_1,f_1^\alpha-f_1)-\alpha\,\frac{\partial f_1^\alpha}{\partial t}.\end{array}$$

We refer to [13] for the regularity of the source terms S_1^α and S_2^α. We point out that these terms do not contain any singular term. By carefully analyzing S_1^α and S_2^α using the regularity of ρ, Φ and the uniform bounds on the potential Φ^α in $W^{2,p}$ for all $p\in[1,\infty[$, we can establish that

$$\|S_1^\alpha\|_{L^\infty(0,T;\,L^1(\Omega))}\lesssim 1$$

and

$$\|S_2^\alpha\|_{L^\infty(0,T;\,L^1(\Omega))}\lesssim\alpha+\|\alpha\,\partial_t f_1^\alpha\|_{L^1}+\|\tilde{r^\alpha}\|_{L^1}.$$

The contraction property of the collision operator gives the differential inequality

$$\left\{\begin{array}{l}\dfrac{d}{dt}\|\tilde{r^\alpha}\|_{L^1}\lesssim\alpha\|S_1^\alpha\|_{L^1}+\|S_2^\alpha\|_{L^1}\\ \tilde{r^\alpha}(t=0)=\alpha(f_1+\alpha\,f_2)(t=0)\end{array}\right.$$

Indeed,

$$\int_{\mathcal{B}} \left[Q(f^\alpha) - Q(F(\rho) + \alpha\, f_1^\alpha + \alpha^2\, f_2) \right]\; sgn(\tilde{r^\alpha})\, dk \le 0. \tag{15}$$

Hence,

$$\frac{d}{dt}\|\tilde{r^\alpha}\|_{L^1} \lesssim \alpha + \left\|\alpha\, \nabla_x \frac{\partial \Phi^\alpha}{\partial t}\right\|_{L^1(\omega)} + \|\tilde{r^\alpha}\|_{L^1} \tag{16}$$

Lemma 5.1

$$\left\|\alpha\; \nabla_x \frac{\partial \Phi^\alpha}{\partial t}\right\|_{L^1(\omega)} \lesssim \alpha + \|\tilde{r^\alpha}\|_{L^1(\Omega)} \left(1 + |\log\left(\|\tilde{r^\alpha}\|_{L^1(\Omega)}\right)|\right) \tag{17}$$

Proof of Lemma 5.1. The potential Φ^α solves the Poisson equation

$$-\Delta_x \Phi^\alpha = \rho^\alpha - D$$

with periodic boundary condition. In addition, the density ρ^α satisfies the local mass conservation

$$\frac{\partial \rho^\alpha}{\partial t} = -\nabla_x \,.j^\alpha$$

These equations imply that

$$\alpha\, \frac{\partial}{\partial t} \nabla_x \Phi^\alpha = \nabla_x \Delta_x^{-1} \nabla_x \,.\, \alpha\, j^\alpha$$

Now, let us write $\alpha\, j^\alpha$ in terms of $\tilde{r^\alpha}$:

$$j^\alpha \;\; = \;\; \frac{1}{\alpha} \int_{\mathcal{B}} v(k)\, \tilde{r^\alpha} + \int_{\mathcal{B}} v(k)\, f_1^\alpha dk + \alpha \int_{\mathcal{B}} v(k)\, f_2.$$

Using the fact that f_0 is even, the smoothness of (ρ, Φ) and

$$f_1^\alpha - f_1 = -\nu(\rho, k)\,.\,\nabla_x(\Phi^\alpha - \Phi)$$

we infer that

$$\begin{aligned}
\alpha\, j^\alpha \;\; &= \;\; \int_{\mathcal{B}} v(k)\, \tilde{r^\alpha} + \alpha\, \Theta(\rho)\,.\,\nabla_x(\Phi^\alpha - \Phi) + \alpha \int_{\mathcal{B}} v(k)\, f_1 + \alpha^2 \int_{\mathcal{B}} v(k) f_2 \\
&= \;\; \int_{\mathcal{B}} v(k)\, \tilde{r^\alpha} + \alpha\, \Theta(\rho)\,.\,\nabla_x \Delta^{-1}\left(\int_{\mathcal{B}} \tilde{r^\alpha}\, dk\right) + \alpha\, \Theta(\rho)\,.\,\nabla_x \Delta^{-1}\left[\int_{\mathcal{B}} f_1 + \alpha \int_{\mathcal{B}} f_2\right] \\
&+ \;\; \alpha \int_{\mathcal{B}} v(k)\, f_1 + \alpha^2 \int_{\mathcal{B}} v(k)\, f_2.
\end{aligned}$$

where $\Theta(\rho)$ is given by (11). This implies that

$$\nabla_x \Delta_x^{-1} \nabla_x \,.\, \alpha\, j^\alpha \;\; = \;\; \nabla_x \Delta_x^{-1} \nabla_x \,.\, \left[\int_{\mathcal{B}} v(k)\, \tilde{r^\alpha}\right] + \alpha\, \nabla_x \Delta_x^{-1} \nabla_x \,.\, \left[\Theta(\rho)\,.\,\nabla_x \Delta_x^{-1}\left(\int_{\mathcal{B}} \tilde{r^\alpha}\, dk\right)\right] + O(\alpha)_{L^\infty}.$$

Using the fact that $\tilde{r^\alpha} \in L^\infty$ and $v(k) \in L^\infty$, we get

$$\|\nabla_x \Delta_x^{-1} \nabla_x \,.\, \alpha\, j^\alpha\|_{L^1} \le \left\|\nabla_x \Delta_x^{-1} \nabla_x \,.\, \int_{\mathcal{B}} v(k)\, \tilde{r^\alpha} dk\right\|_{L^1_x} + O(\alpha)_{L^\infty}$$

Now, one can control the L^1 norm using interpolation argument

$$\begin{aligned}
\left\|\nabla_x \Delta_x^{-1} \nabla_x \,.\, \left[\int_{\mathcal{B}} v(k)\, \tilde{r^\alpha} dk\right]\right\|_{L^1_x} \;\; &\lesssim \;\; \left\|\nabla_x \Delta_x^{-1} \nabla_x \,.\, \left[\int_{\mathcal{B}} v(k)\, \tilde{r^\alpha}\right]\right\|_{L^p_x} \\
&\lesssim \;\; \frac{p}{p-1}\, \|\tilde{r^\alpha}\|_{L^p_{x,k}} \\
&\lesssim \;\; \frac{p}{p-1}\, \|\tilde{r^\alpha}\|_{L^1}^{1/p}\, \|\tilde{r^\alpha}\|_{L^\infty}^{1-1/p}
\end{aligned}$$

In the first inequality, we used that the L^p norm controls the L^1 norm; in the second inequality we used that $\nabla\Delta^{-1}\nabla$ is bounded from L^p to L^p for $1 < p < \infty$ with a norm controlled by $C\frac{p}{p-1}$ where C does not depend on p.

By remarking that $\tilde{r^\alpha}$ is uniformly bounded in L^∞ and denoting $\theta = (p-1)$, we get

$$\left\| \nabla_x \Delta_x^{-1} \nabla_x \cdot \left[\int_{\mathcal{B}} v(k)\, \tilde{r^\alpha} dk \right] \right\|_{L^1_x} \lesssim \frac{1}{\theta} \|\tilde{r^\alpha}\|_{L^1}^{1/(1+\theta)}$$

If we differentiate, the function $h(\theta) = \frac{a^{1/1+\theta}}{\theta}$ where $a > 0$ is a parameter, we obtain

$$h'(\theta) = -\frac{a^{1/\theta+1}}{\theta} \left(\frac{1}{\theta} + \frac{\log a}{(1+\theta)^2} \right).$$

This suggests that the function $h(\theta)$ attains its minimum when θ is close to $1/|\log a|$. We choose $\theta = 1/(1 + |\log a|)$, hence

$$\left\| \nabla_x \Delta^{-1} \nabla_x \cdot \left[\int_{\mathcal{B}} v(k)\, \tilde{r^\alpha} dk \right] \right\|_{L^1_x} \lesssim \left(1 + |\log \|\tilde{r^\alpha}\|_{L^1}|\right)\ \|\tilde{r^\alpha}\|_{L^1}\ \|\tilde{r^\alpha}\|_{L^1}^{\frac{-1}{2+|\log\ \|\tilde{r^\alpha}\|_{L^1}|}}$$

This ends the proof of the lemma 5.1 since $a^{\frac{-1}{2+|\log a|}}$ is uniformly bounded on $\mathbf{R}_+$. □

Now, we return to the proof of Theorem 2.1. Using the fact that the function $a \to a(1+|\log a|)$ is increasing for $a \geq 0$, we deduce from the inequality (16) that

$$\frac{d}{dt}\|\tilde{r^\alpha}\|_{L^1} \leq \alpha + 2(\|\tilde{r^\alpha}\|_{L^1} + \alpha)(1 + |\log(\|\tilde{r^\alpha}\|_{L^1} + \alpha)|)$$

Adding α and taking the log, we get

$$\frac{d}{dt}\log(\alpha + \|\tilde{r^\alpha}\|_{L^1}) \leq C(2 + |\log(\|\tilde{r^\alpha}\|_{L^1} + \alpha)|)$$

Since $\|\tilde{r^\alpha}\|_{L^1} + \alpha$ is small, we can replace $|\log(\|\tilde{r^\alpha}\|_{L^1} + \alpha)|$ by $-\log(\|\tilde{r^\alpha}\|_{L^1} + \alpha)$, hence,

$$\frac{d}{dt}\left[e^{C\,t} \log(\alpha + \|\tilde{r^\alpha}\|_{L^1}) \right] \leq 2\,C\,e^{C\,t}$$

Hence, by Gronwall lemma, we deduce that

$$\log(\alpha + \|\tilde{r^\alpha}\|_{L^1}) \leq \log(C\alpha)e^{-Ct} + 2Ct \tag{18}$$

and hence $\|\tilde{r^\alpha}\|_{L^1} \leq (C\,\alpha)^{e^{-Ct}} e^{2\,C\,t}$. This ends the proof of (5) when $p = 1$. The case $p \in [1, \infty)$ can be deduced by interpolation. Using elliptic regularity, we also deduce that (6) holds. This ends the proof of the main theorem. □

6 Acknowledgement

N. M was partially supported by an NSF grant DMS-0703145.

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