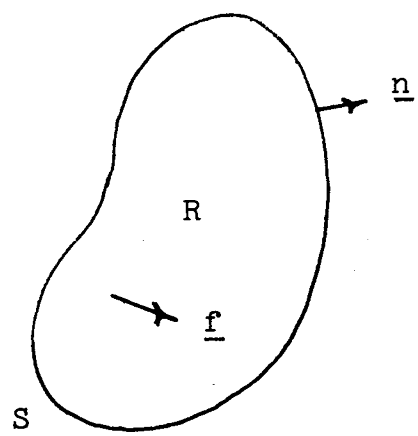
II. Cardiac Fluid Dynamics

Results from Vector Analysis

(1) Given a region R with surface S and an arbitrary vector field $\underline{f}$

$$\int_R dv\ \nabla\cdot\underline{f} = \int_S (\underline{f}\cdot\underline{n})da$$

where $\underline{n}$ is the unit normal to the surface



If $\underline{f}$ is the velocity field of some fluid and the region R moves with the fluid, then the rate of change of volume is given by

$$\frac{dV}{dt} = \int_S da\ \underline{f}\cdot\underline{n} = \int_R dv\ \nabla\cdot\underline{f}$$

Incompressible flows are characterized by volume conservation for <u>every</u> material region R. This is equivalent to $\nabla\cdot\underline{f} = 0$ everywhere.

(2) Using the relation $\nabla\cdot(\underline{f}\phi) = (\nabla\cdot\underline{f})\phi + \underline{f}\cdot\nabla\phi$ it follows that

$$\int_R dv\underline{f}\cdot\nabla\phi = \int_R dv\ \nabla\cdot(\underline{f}\phi) - \int_R dv(\nabla\cdot\underline{f})\phi$$

$$= \int_S da\ \underline{n}\cdot f\ \phi - \int_R dv(\nabla\cdot\underline{f})\phi$$

In particular if

$$\nabla\cdot\underline{f} = 0 \quad \text{in} \quad R$$

$$\underline{n}\cdot\underline{f} = 0 \quad \text{on} \quad S$$

$$\int_R \underline{f}\ \nabla\phi\ dv = 0$$

We shall refer to such a flow $\underline{f}$ satisfying the above conditions as an incompressible flow confined to the region R. Then the foregoing result can be stated as follows:

Incompressible flows confined to a region R are orthogonal to the gradient of an arbitrary scalar over the same region.

(3) Given an arbitrary vector field $\underline{f}$, one can decompose $\underline{f}$ as follows

$$\underline{f} = -\nabla\phi + f^D$$

where

$$\nabla\cdot\underline{f}^D = 0 \quad \text{in} \quad R$$

$$\underline{n}\cdot\underline{f}^D = 0 \quad \text{on} \quad S$$

and the fields $\nabla\phi$ and $\underline{f}^D$ are uniquely determined. The construction of $\nabla\phi$ is accomplished by solving the following problem

$$\left.\begin{aligned} \nabla^2\phi &= -\nabla\cdot\underline{f} \quad \text{in} \quad R \\ \underline{n}\cdot\nabla\phi &= -\underline{n}\cdot\underline{f} \quad \text{on} \quad S \end{aligned}\right\}$$

Since $\underline{f}$ is given, the right hand sides are known. The uniqueness of the decomposition follows from the orthogonality of $\nabla\phi$ and $\underline{f}^D$. To show this explicitly, let

$$\underline{f} = -\nabla\phi_1 + \underline{f}_1^D$$

$$\underline{f} = -\nabla\phi_2 + \underline{f}_2^D$$

$$0 = -\nabla(\phi_1-\phi_2) + (\underline{f}_1^D-\underline{f}_2^D)$$

$$0 = \int dv|\nabla(\phi_1-\phi_2)|^2 + \int dv|\underline{f}_1^D-\underline{f}_2^D|^2$$

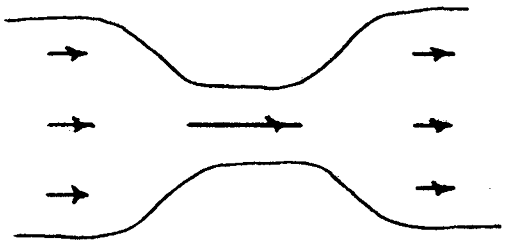
where the cross terms disappear because of the orthogonality. It follows that

$$\nabla\phi_1 = \nabla\phi_2$$

$$\underline{f}_1^D = \underline{f}_2^D \quad .$$

Time Derivatives in Fluid Dynamics

Consider the steady flow of an incompressible fluid through a narrowing in a pipe. By steady flow, we mean that the velocity at each point in space is independent of time. That is, $\frac{\partial \underline{u}}{\partial t} = 0$. On the other hand the flow is faster in the narrow part of the pipe, and a material particle in the flow moves from one region to the other and therefore has a non-zero acceleration. To compute this acceleration we have to define the <u>material</u> time derivative $\frac{D}{Dt}$. For a function $f(\mathbf{x}, \mathbf{t})$ the definition is



$$\frac{D}{Dt} f(\underline{x},t) = \lim_{\delta t \to 0} \frac{f(\underline{x}+\delta\underline{x}, t+\delta t) - f(\underline{x},t)}{\delta t}$$

where

$$\delta\underline{x} = \underline{u}\, \delta t \quad .$$

Therefore

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underline{u}\cdot\nabla f \quad .$$

One can also write

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$$

and this operator can be applied to vectors as well. In particular the material acceleration is

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} \quad .$$

Using the material time derivative $\frac{D}{Dt}$ we can derive some of the previous results in a more interesting way. Consider an incompressible flow $\underline{u}$ confined to a fixed region R so that $\underline{u}\cdot\underline{n} = 0$ on the boundary of R. Suppose that an arbitrary scalar field ϕ is also given with $\frac{D\phi}{Dt} = 0$. That is, the numerical values of the function ϕ are simply transported around at the velocity $\underline{u}$. In that case, since $\underline{u}$ conserves every element of volume dv, we have

$$0 = \frac{d}{dt}\int_R \phi \, dv = \int_R \frac{\partial\phi}{\partial t}\, dv$$

$$= -\int_R \underline{u}\cdot\nabla\phi \, dv \quad .$$

In the foregoing, the first equality depends on the volume-conserving character of the flow, the second on the fact that the region R is not changing with time, and the third on

$$0 = \frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \underline{u}\cdot\nabla\phi \quad .$$

We have proved again that

$$0 = \int_R \underline{u}\cdot\nabla\phi \, dv$$

for arbitrary ϕ (The equation $D\phi/Dt = 0$ places no restriction on the form of ϕ at a single instant) and for volume-conserving flow $\underline{u}$ confined to a region R. Here the fact that $\underline{u}$ conserves volume was not used in the form $\nabla\cdot\underline{u} = 0$, but directly. In fact, using the same identities as in (2), above, we can write

$$0 = \int dv\ \underline{u}\cdot\nabla\phi = -\int dv(\nabla\cdot\underline{u})\phi$$

which proves that $\nabla\cdot\underline{u} = 0$, since ϕ is arbitrary.

Incompressible Flows Bounded by Moving Walls

The flow $\underline{u}$ has the following constraints:

$$\nabla\cdot\underline{u} = 0 \qquad \text{in } R$$

$$\underline{u}\cdot\underline{n} = \text{given function} \qquad \text{on } S$$

Using (3) above write

$$\underline{u} = -\nabla\phi + \underline{w}$$

where

$$\nabla\cdot\underline{w} = 0 \quad \nabla^2\phi = 0 \quad \text{in } R$$

$$\underline{w}\cdot\underline{n} = 0 \quad \underline{n}\cdot\nabla\phi = -\underline{n}\cdot\underline{u} \text{ on } S$$

The total class of flows $\underline{u}$ satisfying the constraints can be found by picking ϕ defined by the above, and then picking an arbitrary $\underline{w}$ satisfying $\nabla\cdot\underline{w} = 0$, $\underline{n}\cdot\underline{w} = 0$.

Define

$-\nabla\phi$ = the forced part of the motion

$\underline{w}$ = the free part of the motion

Note that $-\nabla\phi$ takes care of the given motion of the walls, while $\underline{w}$ is an incompressible flow consistent with the instantaneous geometry for non-moving walls.

Using the result (2) that $\int \underline{w}\cdot\nabla\phi \, dv = 0$ we note that the total kinetic energy can be written

$$U = \frac{1}{2}\rho \int_R |u|^2 dv = U_1 + U_2$$

where

$$U_1 = \int |-\nabla\phi|^2 \, dv$$

$$U_2 = \int |w|^2 \, dv$$

thus the energies of the forced and free parts of the motion simply add, and

THEOREM: The lowest energy incompressible flow bounded by moving walls is the forced motion $-\nabla\phi$, with no free motion at all, i.e., with $\underline{w} = 0$.

Equations of Motion

Temporarily assume that the only forces acting are those due to the constraint of incompressibility. Call these $\underline{f}_c$ (per unit volume). Then Newton's law yields

$$\rho \frac{d^2\underline{x}}{dt^2} = \underline{f}_c$$

where

ρ = density

$\underline{x}(t)$ = trajectory of a fluid particle.

Constraint forces are not arbitrary but obey the following law:

The system of constraint forces is orthogonal to the free part of an arbitrary motion satisfying the constraint. (Example: consider a bead sliding about on an undulating surface. The forced motion is the motion of the bead which is required by the motion of the surface normal to itself. The free part is the motion of the bead parallel to the instantaneous configuration of the surface. The constraint force, the forces of the surface on the bead, is orthogonal to all possible free motions. That is, it is normal to the surface.)

Applying this principle to the present problem we have

$$\int dv\, \underline{f}_c \cdot \underline{w} = 0$$

for all $\underline{w}$ such that

$$\nabla \cdot \underline{w} = 0 \quad \text{in} \quad R$$

$$\underline{n} \cdot \underline{w} = 0 \quad \text{on} \quad S$$

Using (3) set $\underline{f}_c = -\nabla p + \underline{f}_c^D$. From (2) $\int dv(-\nabla p \cdot \underline{w}) = 0$. Therefore, setting $\underline{w} = \underline{f}_c^D$ (which is permissible since $\underline{w}$ satisfies the same conditions as $\underline{f}_c^D$) we get

$$\int dv |\underline{f}_c^D|^2 = 0 \rightarrow \underline{f}_c^D = 0 \quad .$$

Thus

$$\underline{f}_c = -\nabla p \quad .$$

We have proved that the constraint forces take the form of the gradient of a scalar. This scalar is called the pressure!

Thus the equation of motion (no viscosity) reads

$$\rho \frac{d^2\underline{x}}{dt^2} = -\nabla p \quad .$$

Equations of Motion from a Lagrangian

Minimize the Lagrangian subject to the constraint of volume conservation

$$L = \int \frac{1}{2}\rho \left|\frac{d\underline{x}}{dt}\right|^2 dv$$

$$\delta \int L \, dt = \int dt \int dv \, \rho \, \frac{d\underline{x}}{dt} \cdot \frac{d\delta \underline{x}}{dt}$$

$$= - \int dt \int dv \, \rho \, \frac{d^2 \underline{x}}{dt^2} \cdot \delta \underline{x} \quad .$$

But

$$\nabla \cdot \delta \underline{x} = 0 \quad \text{in} \quad R$$

$$\underline{n} \cdot \delta \underline{x} = 0 \quad \text{on} \quad S$$

Otherwise, $\delta \underline{x}$ is arbitrary. As before, it follows that there is a scalar p such that $\rho \, \frac{d^2 \underline{x}}{dt^2} = -\nabla p$.

Conservation of the Circulation

Definition: The circulation around a closed path C is given by

$$K = \oint_C \underline{u} \cdot d\underline{x}$$

where

$$\oint = \text{integration around a closed path.}$$

Let the path move with the fluid

$$\frac{dK}{dt} = \oint \frac{D\underline{u}}{Dt} \cdot d\underline{x} + \oint \underline{u} \cdot d \, \frac{D\underline{x}}{Dt}$$

$$= - \oint \frac{1}{\rho} \nabla p \cdot d\underline{x} + \frac{1}{2} \oint d \, |\underline{u}|^2 \quad *$$

$$= 0$$

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* In this step use equation of motion without viscosity.

(In particular, if $K = 0$ for a path moving with the fluid at some instant, $K = 0$ for all subsequent times.)

Reversing the above argument, we see that if the circulation is conserved for all closed material paths, then

$$\oint \frac{D\underline{u}}{Dt} \cdot d\underline{x} = 0$$

for all such paths. It follows that there exists some scalar ϕ such that

$$\frac{D\underline{u}}{Dt} = \nabla\phi$$

and this is all that is asserted by our (inviscid) equation of motion. Therefore, conservation of the circulation for all closed material paths is equivalent to the equation of motion.

Moreover, knowledge of the circulation (all paths) at $t = 0$ is equivalent to an initial condition. Suppose two velocity fields $\underline{u}_1$ and $\underline{u}_2$ have the same circulation and satisfy

$$\nabla\cdot\underline{u}_1 = \nabla\cdot\underline{u}_2 = 0 \quad \text{in} \quad R$$

$$\underline{n}\cdot\underline{u}_1 = \underline{n}\cdot\underline{u}_2 = \text{given function on} \quad S \quad .$$

Let

$$\underline{w} = \underline{u}_1 - \underline{u}_2 \quad .$$

Then

$$\nabla\cdot\underline{w} = 0 \quad \text{in} \quad R$$

$$\underline{n}\cdot\underline{w} = 0 \quad \text{on} \quad S$$

$$\oint \underline{w}\cdot d\underline{x} = 0 \quad \text{all paths}$$

so that

$$\underline{w} = \nabla\phi \quad .$$

It follows from these statements that

$$\phi = \text{constant}$$

$$\underline{w} = 0$$

$$\underline{u}_1 = \underline{u}_2 \quad .$$

Therefore, there is at most one velocity field with some given circulation on each closed path.

Irrotational Flow

Suppose K = 0, all paths.

Then $\underline{u} = -\nabla\phi$

$$\nabla\cdot\underline{u} = 0 \rightarrow \nabla^2\phi = 0 \quad .$$

The boundary condition becomes

$\underline{n}\cdot\nabla\phi$ = given function on the walls.

Properties of irrotational flow.

(1) Lowest energy flow consistent with the normal boundary conditions. Therefore it is stable against any process which lowers the energy.

(2) No matter how ϕ varies with time, $\underline{u} = -\nabla\phi$ satisfies the inviscid equations of motion! (the pressure will be different in each case). Thus the flow at each instant is completely

determined by the boundary conditions, independent of the history.

(3) Even when viscosity is added, $\underline{u} = -\nabla\phi$ remains a solution of the equations of motion in the interior, because the viscous force per unit volume vanishes identically for such flows. From (1) we see that this is not an accident but follows from the fact that viscosity tends to lower the energy.

(4) Despite the foregoing, flows of real fluids are not irrotational because...

Boundary Slip in Irrotational Flow

Consider a flow bounded by moving walls with the tangential component equal to zero everywhere. Along any curve Γ on the boundary (not necessarily closed)

$$\int_{\Gamma} \underline{u} \cdot d\underline{x} = 0$$

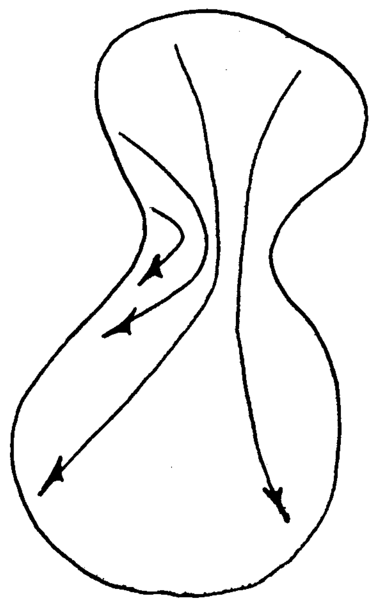
simply because $\underline{u} \cdot d\underline{x} = 0$ at each point. Now if the flow is also irrotational

$$0 = -\int_{\Gamma} \nabla\phi \cdot d\underline{x} = \phi_2 - \phi_1 \quad .$$

Since this holds for all boundary curves Γ, ϕ = constant on the boundary. The solution of $\nabla^2\phi = 0$ then yields ϕ = constant in the interior as well and the flow is zero. But the walls were given as moving, so we have a contradiction.

We have proved that irrotational incompressible flows bounded by moving walls must have a non-zero tangential component of velocity somewhere at the wall. In fact, the

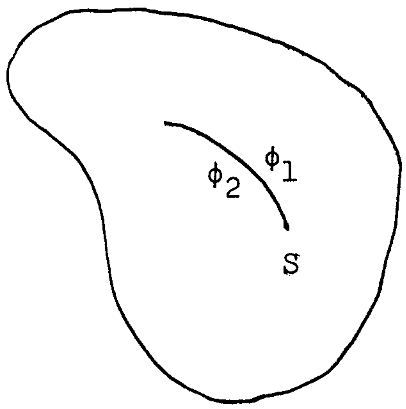
ϕ max



ϕ min

only points where there is no tangential component are ϕ_{max} and ϕ_{min}.

Tangential slip generated by a moving <u>immersed</u> boundary in potential flow:



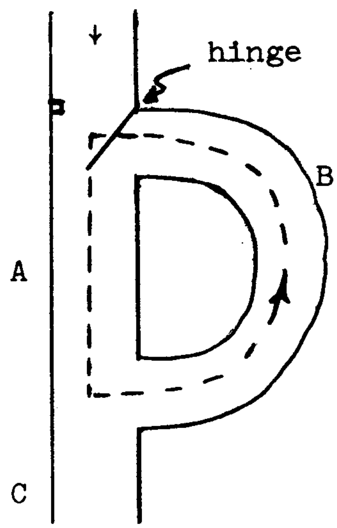
If $\phi_1 \equiv \phi_2$ on S, then tangential derivatives to all orders are equal. Moreover the normal first derivatives are also equal because no fluid is penetrating the boundary. Since $\nabla^2\phi = 0$ it follows that second derivatives in the normal direction are also equal. Under these conditions the surface S has no effect on the flow, which is determined by the external boundary conditions. Conversely, if the motion of S has any effect on the flow, then $\phi_1 \neq \phi_2$. There are two cases

(i) $\phi_1 - \phi_2$ = constant. Then there is no tangential slip, but there is infinite flow around the edges of S.

(ii) $\phi_1 - \phi_2 \neq$ constant. Then relative tangential slip is generated by the motion of S.

The Dynamics of Heart Valve Closure

This subject can be considered from the standpoint of non-viscous flow (and the conservation of the circulation). In doing this, one has to take as given certain features of the forward flow pattern. These features become clearer when viscosity is included.



An apparatus like this was used in 1912* to demonstrate efficient heart valve closure. During forward flow there is non-zero circulation around the D-shaped path shown, since there is forward flow in A, but the side tube B is occluded by the open valve. The circulation is given by

$$K = u_A L_A + 0 \cdot L_B$$

where $u_{A,B}$ = velocity in tube A, B

$L_{A,B}$ = length of tube A, B

If the forward flow is suddenly cut off**, the circulation is conserved, so that

$$K = u_A' L_A + u_B' L_B$$

But if the tubes are of equal cross sectional area

$$u_A' = u_B'$$

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* Henderson and Johnson, "Two Modes of Closure of the Heart Valves" Heart 4:69-82, 1912

** e.g. by shutting the tube at C.

and we have

$$u_A' = u_A \frac{L_A}{L_A+L_B} \quad .$$

This reduced flow, circulating around the D-shaped segment can shut the valve from the side without backflow.

An essential feature of the foregoing is that the flow region is not simply connected. The curve shown cannot be continuously deformed to a point without leaving the fluid. We can replace such a cut out region by a point vortex. This will be considered next.

Point Vortex

A point vortex is a singularity in a two dimensional flow characterized as follows. If there is only one point vortex in an otherwise irrotational flow, then $K = 0$ for all paths which do not enclose the vortex and $K = K_o$ for all paths which do. K_o is called the strength of the vortex. If there are more than one vortex then

$$K = \oint_\Gamma \underline{u} \cdot d\underline{x} = \sum K_i$$

where the sum extends over all the vortices enclosed by Γ.

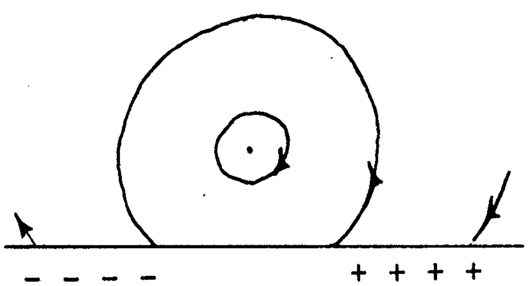
The prototype of a point vortex is a flow with circular streamlines with speed $u = K/2\pi r$. Since this $\to \infty$ as $r \to 0$, one can add on any finite flow and the streamlines still have to tend to circles (with the speed indicated) at points sufficiently close to the point vortex. Nevertheless the finite part of the flow is important because it determines the motion of the vortex itself, the infinite part being symmetric and contributing nothing.

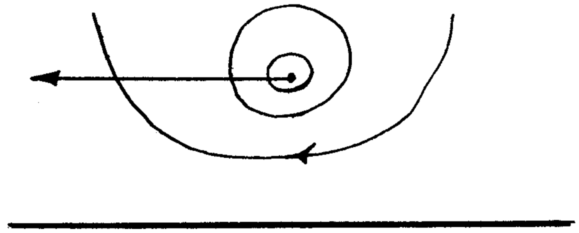
In fact, the problem of determining the motion of a point vortex reduces to writing the flow in the form

$$\underline{u} = \underline{u}_1 + \underline{u}_2$$

where $\underline{u}_1$ follows the circular streamlines with $\underline{u}_1 = K/2\pi r$ and $\underline{u}_2$ is finite at the center of the vortex. Then the vortex moves at velocity $\underline{u}_2$.

For example if a point vortex is placed near a line boundary, the circular streamlines of the flow $\underline{u}_1$ do not satisfy the boundary condition. To compensate for this one has to add a flow which arises from injecting fluid at the region marked ++++ and extracting fluid at the region marked ----. Clearly this second flow $\underline{u}_2$ will move the vortex to the left. And the final streamlines will look like this:

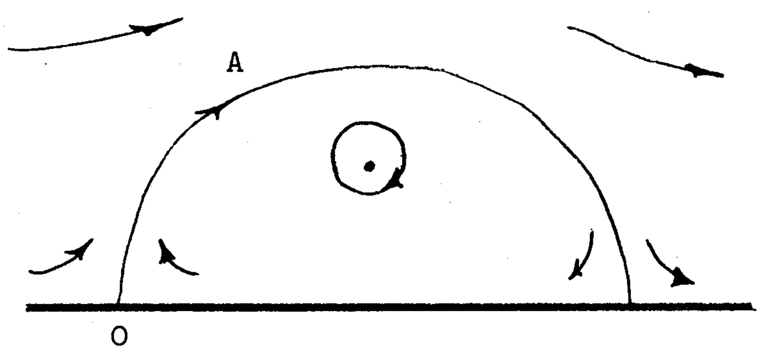




Note that the vortex is no longer symmetrically placed with respect to its own streamlines and this can be thought of as the source of the motion of the vortex.

This flow $\underline{u}_2$ which moves the vortex is easily computed in this case (straight boundary) by noticing that the required flow $\underline{u}_2$ is generated by placing an image vortex with opposite rotation below the plane boundary. This shows that the speed of the vortex is $K/2\pi(2d)$ where d = distance from the vortex to the boundary.

We can also hold the vortex at rest by adding on a main stream flow of precisely this magnitude from left to right. Then the streamlines look like this:



A valve leaflet anchored at 0 could remain steady* in such a flow along the streamline OA. If the main stream flow is suddenly shut off, the situation reverts to that discussed above.

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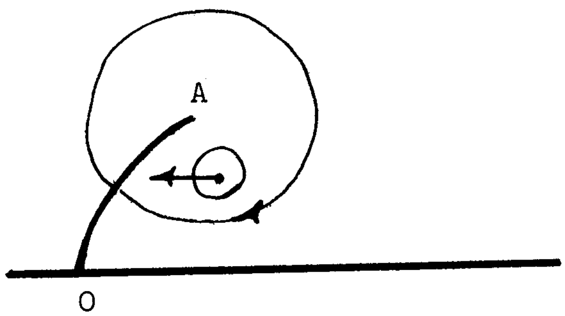
* This situation is clearly unstable. See below, however.



The vortex streamlines move the leaflet to the left, and the vortex itself moves left "chasing" the leaflet. The velocity of motion is equal and opposite to that of the stream which was just shut off.

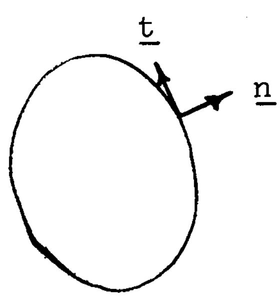
Complex Variables and Incompressible Potential Flow In Two Dimensions

The aim here is to map the solution outlined above onto a geometry more like that of a heart valve.

For incompressible, irrotational flow it two dimensions we have

$$\oint \underline{u}\cdot\underline{t}\; ds = 0 \rightarrow \phi,\; u_x = \frac{\partial\phi}{\partial x},\; u_y = \frac{\partial\phi}{\partial y}$$

$$\oint \underline{u}\cdot\underline{n}\; ds = 0 \rightarrow \psi,\; u_x = \frac{\partial\psi}{\partial y},\; u_y = -\frac{\partial\psi}{\partial x}$$



ψ is called the stream function. The relation between ψ and ϕ can be seen most clearly by defining $\underline{u}^{\perp}$ = the vector $\underline{u}$ rotated through 90°. Then $\underline{u}\cdot\underline{n} = \underline{u}^{\perp}\cdot\underline{t}$ and $\nabla\psi = \underline{u}^{\perp}$. Thus ψ is the potential function for a flow which has the same speed as u̲ but is everywhere perpendicular to u̲.

Note also that

$$(*)\qquad \left.\begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y} \\ \frac{\partial\phi}{\partial y} &= -\frac{\partial\psi}{\partial x} \end{aligned}\right] \qquad \text{Cauchy-Riemann conditions}$$

One can think of the functions

$$\phi(x, y)$$
$$\psi(x, y)$$

as establishing a mapping $(x, y) \to (\phi, \psi)$.

Any such mapping satisfying (*) gives a possible flow because the Cauchy-Riemann conditions imply $\nabla^2\phi = 0$ (and $\nabla^2\psi = 0$).

Moreover, suppose we have two mappings which satisfy (*)

$$(\xi, \eta) \to (x, y) \to (\phi, \psi) .$$

Then the induced mapping

$$(\xi, \eta) \to (\phi, \psi)$$

also satisfies (*) as one can show by direct substitution. Therefore if we have a flow in the x, y plane we can map it onto the ξ, η plane simply by finding a mapping $(\xi, \eta) \to (x, y)$ which satisfies (*).

The process of finding such mapping is made easy by the fact that functions of a complex variable satisfy the Cauchy-Riemann conditions. That is, if

$$w = \phi + i\psi$$
$$z = x + iy$$

then one can find a unique derivative $\frac{dw}{dz}$ (independent of the direction of dz) only if $\partial\phi/\partial x = \partial\psi/\partial y$ and $\partial\phi/\partial y = -\partial\psi/\partial x$.

For example the complex potential corresponding to a point vortex at the origin is

$$w = \phi + i\psi = \frac{K}{2\pi i} \log z \quad .$$

To see this write

$$z = re^{i\theta}$$

$$\log z = \log r + i\theta$$

$$w = \frac{K\theta}{2\pi} + \frac{-iK \log r}{2\pi}$$

thus

$$\phi = \frac{K\theta}{2\pi} \quad \psi = \frac{-K \log r}{2\pi} \quad .$$

The streamlines are circles, r = constant and the speed is $\left| \frac{\partial\psi}{\partial r} \right| = \frac{K}{2\pi r}$.

The flow described above with two image vortices and a main stream flow can be described by

$$w(\zeta) = u\zeta - \frac{K}{2\pi i} \log(\zeta-\zeta_o) + \frac{K}{2\pi i} \log(\zeta-\bar{\zeta}_o)$$

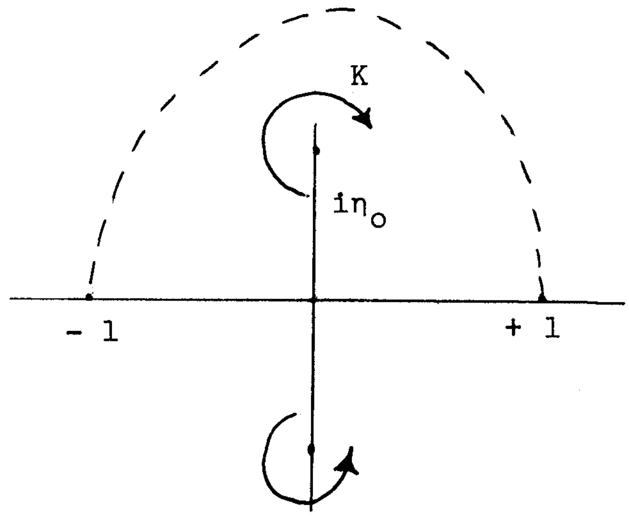
where

$$\zeta = \xi + i\eta$$

$$w = \phi + i\psi$$

$$\zeta_o = i\eta_o$$

$$\bar{\zeta}_o = -i\eta_o$$



ζ - plane

We choose K/u such that the dividing streamline meets the boundary at ±1, as shown. To do this compute

$$w'(\zeta) = u - \frac{K}{\pi}\frac{\eta_o}{\zeta^2+\eta_o^2} \quad .$$

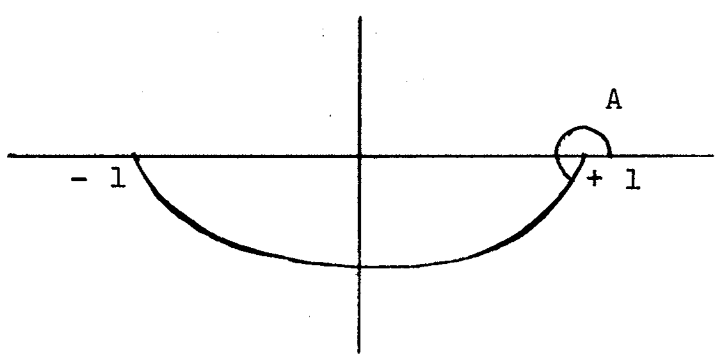
When ζ is real, this expression is real and equal to $\frac{\partial\phi}{\partial x}$, the x-component of velocity. Setting this to zero at ±1 we find

$$0 = \pi u(\eta_o + \eta_o^{-1}) - K \quad .$$

To transform the foregoing geometry into something that looks like an aortic sinus, apply the transformation

$$\zeta(z) = \frac{(z+1)^\alpha + (z-1)^\alpha}{(z+1)^\alpha - (z-1)^\alpha}$$

$\frac{1}{2} < \alpha < 1$



$A\alpha = \pi$

z - plane

Some properties of the transformation:

(1) Imaginary z-axis $\rightarrow$ Imaginary ζ-axis

(2) real line with $|z| > 1 \rightarrow$ real line with $|\zeta| > 1$

(3) Image of the real line with $|\zeta| > 1$ is a circular arc in the z-plane through ± 1. Then angle A is given by $A\alpha = \pi$. For $\alpha = \frac{2}{3}$ the arc is a semi-circle.

(4) As $|z| \rightarrow \infty \quad \zeta \rightarrow \frac{z}{\alpha}$

Thus, far from the sinus the streamlines become straight.

The flow in the z-plane will be described by the complex potential:

$$W(z) = w(\zeta(z)) \ .$$

We shall be especially interested in the motion of the point vortex at $\zeta = \zeta_o$. This has to be evaluated with care because the deformation of the boundaries produces an additional finite flow at ζ_o. Let

$$\zeta-\zeta_o = a(z-z_o) + b(z-z_o)^2 + \ldots$$
$$= (z-z_o)(a + b(z-z_o) + \ldots)$$
$$\log(\zeta-\zeta_o) = \log(z-z_o) + \log(a + b(z-z_o) + \ldots)$$

prototype point vortex — finite flow at $z = z_o$

No velocity at z_o

$$a = \zeta'\big|_o$$
$$b = \tfrac{1}{2}\zeta''\big|_o$$

Keeping only the finite part of the flow at z_o we have

$$W'(z_o) = u\zeta_o' - \frac{K}{2\pi i}\frac{b}{a} + \frac{K}{2\pi i}\frac{1}{\zeta_o-\bar{\zeta}_o}\zeta_o' = \left[u - \frac{K}{4\pi\eta_o}\left(\frac{\zeta}{\zeta'}\right)_o' \right]\zeta_o'$$

Setting this to zero and using the relation between K and u that keeps the dividing streamline at ±1, we have

$$0 = u - \frac{K}{4\pi\eta_o}\left(\frac{\zeta}{\zeta'}\right)_o'$$

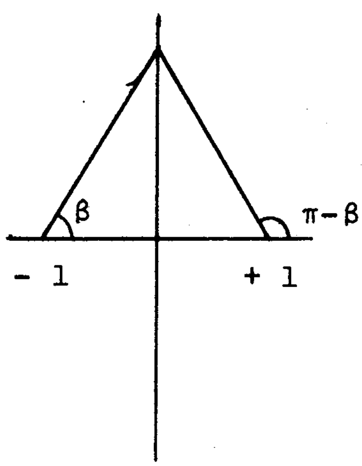
$$0 = u\pi(\eta_o+\eta_o^{-1}) - K \quad .$$

To solve these simultaneously with non-trivial K, u we need

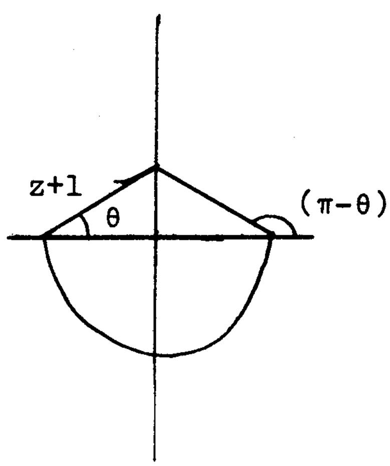
$$- 1 + (\eta_o+\eta_o^{-1})\frac{1}{4\eta_o}\left(\frac{\zeta}{\zeta'}\right)_o' = 0$$

or

$$\left(\frac{\zeta}{\zeta'}\right)_o' = \frac{4\eta_o^2}{1+\eta_o^2} = 4\sin^2\beta$$



ζ-plane



z-plane

Evaluate ζ on the imaginary axis

$$\rho = \frac{e^{i\alpha\theta} + e^{i\alpha(\pi-\theta)}}{e^{i\alpha\theta} - e^{-i\alpha(\pi-\theta)}}$$

$$= \frac{e^{i\alpha(\theta-\pi/2)} + e^{-i\alpha(\theta-\pi/2)}}{e^{i\alpha(\theta-\pi/2)} - e^{-i\alpha(\theta-\pi/2)}}$$

$$= \frac{\cos\alpha(\theta-\pi/2)}{i\sin\alpha(\theta-\pi/2)}$$

$$= i \operatorname{ctn} \alpha(\tfrac{\pi}{2} - \theta) \quad .$$

Let $\frac{\pi}{2} - \beta = \alpha(\frac{\pi}{2} - \theta)$. Then

$$\zeta = i \tan \beta$$

$$\zeta' = i \frac{\beta'}{\cos^2\beta}$$

$$\frac{\zeta}{\zeta'} = \frac{\sin\beta\cos\beta}{\beta'} = \frac{1}{2} \frac{\sin 2\beta}{\beta'}$$

$$\left(\frac{\zeta}{\zeta'}\right)' = \cos 2\beta - \frac{\beta''}{2(\beta')^2} \sin 2\beta$$

$$\beta' = + \alpha\theta'$$

$$\beta'' = + \alpha\theta''$$

$$\frac{\beta''}{2(\beta')^2} = + \frac{1}{2\alpha} \frac{\theta''}{(\theta')^2}$$

$$y = \tan\theta$$

$$\frac{dy}{d\theta} = \frac{1}{\cos^2\theta} \quad \frac{d\theta}{dy} = \cos^2\theta$$

$$\theta' = \frac{d\theta}{i\ dy} = -i\ \cos^2\theta$$

$$\theta'' = 2\ i\ \cos\theta\sin\theta\ \theta'$$

$$= 2\ \cos^3\theta\ \sin\ \theta$$

$$+\ \frac{1}{2\alpha}\ \frac{\theta''}{(\theta')^2} = -\ \frac{1}{\alpha}\ \frac{\sin\ \theta}{\cos\ \theta} = -\ \frac{1}{\alpha}\ \tan\ \theta$$

$$\left(\frac{\zeta}{\zeta'}\right)' = \cos\ 2\beta + \frac{1}{\alpha}\ \tan\ \theta\ \sin\ 2\beta$$

$$= 4\ \sin^2\beta$$

$$= \frac{4\ \sin\ \beta\ \sin\ \beta\ \cos\ \beta}{\cos\ \beta}$$

$$= 2\ \tan\ \beta\ \sin\ 2\beta$$

$$1 + \frac{1}{\alpha}\ \tan\ \theta\ \tan\ 2\beta = 2\ \tan\ \beta\ \tan\ 2\beta$$

$$1 = \tan\ 2\beta\ (2\ \tan\ \beta - \frac{1}{\alpha}\ \tan\ \theta)$$

For given α, θ, β are determined from

$$1 = \tan\ 2\beta\ (2\ \tan\ \beta - \frac{1}{\alpha}\ \tan\ \theta)$$

$$\frac{\pi}{2} - \beta = \alpha(\frac{\pi}{2} - \theta)$$

<u>Example</u> $\alpha = 1 \quad z \equiv \zeta$

$$\beta = \theta$$

$$1 = \tan\ 2\beta\ \tan\ \beta$$

$$2\beta + \beta = \pi/2$$

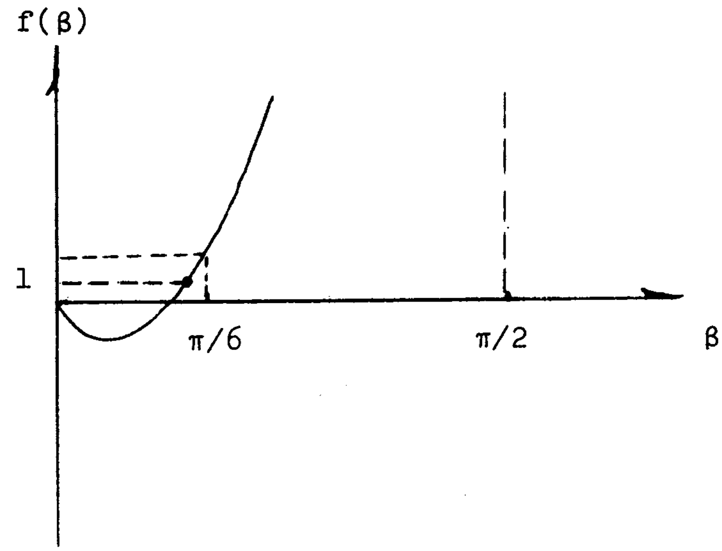
$$\beta = \pi/6$$

Example $\qquad \alpha = \frac{2}{3} \qquad \theta = \frac{\pi}{2} - \frac{3}{2}\left(\frac{\pi}{2} - \beta\right)$

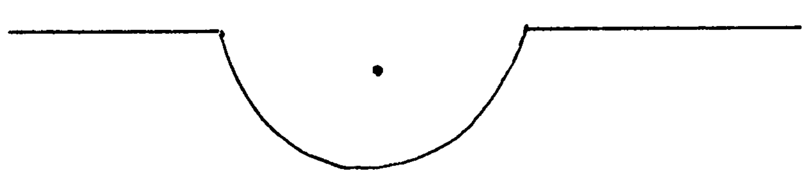
$$= -\frac{\pi}{4} + \frac{3}{2}\beta$$

$$1 = \tan 2\beta\left(2 \tan \beta - \frac{3}{2} \tan\left(\frac{3}{2}\beta - \frac{\pi}{4}\right)\right)$$

$$= f(\beta)$$



Note that the solution occurs at $\beta < \frac{\pi}{6}$ which implies $\theta < 0$, so the vortex is in the sinus.



This discussion of the dynamics of heart valve closure is incomplete, as it is limited to a discussion of point vortex motion in two dimensional flow with a simple geometry and no viscosity. The reader interested in this problem should also see the following references:

(1) Leonardo da Vinci

(O'Malley and Sanders: Leonardo da Vinci on the Human Body, N.Y. Henry Schuman 1952 pp. 216-326).

(2) Henderson and Johnson, Two Modes of Closure of the Heart Valves, Heart 4 69-82, 1912.

(3) Bellhouse and Talbot

Fluid Mechanics of the Aortic Valve, J. Fluid Mechanics 35: 721-735 (1969).

(4) Bellhouse

Fluid Mechanics of a Model Mitral Valve and Left Ventricle, Cardiovascular Research 6: 199-210 (1972).

Diffusion of Momentum (Viscosity)

This section begins our discussion of the dissipative properties of the flow of real fluids. The following derivation of the viscous force involves a simplified model of random molecular motion. We assume here that the molecule executes free flights of given duration at a given speed. The results are more general than the way in which they are derived.

Let $\underline{u}$ be the velocity field of a fluid. We consider the interaction of the fluid as a whole with a single molcule immersed in the fluid, and we suppose that the molecule (with mass m) moves in free flights of length r and duration τ. Later we shall contemplate a limit as $\tau \to 0$, $r \to 0$, $(r^2/6\tau) \to \nu$. This implies that $(r/\tau) \to \infty$ so the velocity of the free flights is much larger than the finite velocity of the fluid. At the ends of the time intervals τ the molecule "collides" with the fluid. By this we mean that the velocity after the collision has the form

$$\underline{v} = \frac{r}{\tau}\,\underline{a}_P + \underline{u}_P$$

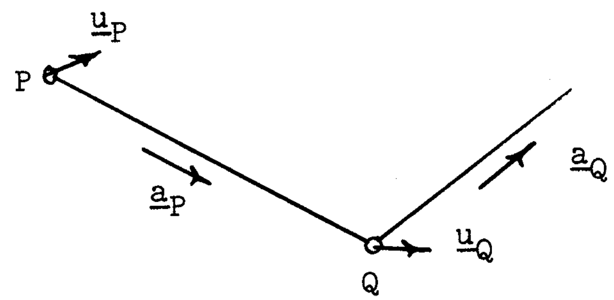
where

$$P = \text{point of collision}$$

$$\underline{u}_P = \text{fluid velocity at point } P$$

$$\underline{a}_P = \text{unit vector with random direction.}$$

Now consider two successive collisions of the molecule at points P and Q, which by hypothesis are separated by a distance r.



In drawing the vector $\underline{a}_P$ parallel to the path PQ we are making use of the relation

$$|\underline{u}_P| << r/\tau \quad .$$

The change in momentum experienced by the molecule at Q is given by

$$m(\frac{r}{\tau}\,\underline{a}_Q + \underline{u}_Q) - m(\frac{r}{\tau}\,\underline{a}_P + \underline{u}_P)$$

and the mean value of this quantity is

$$m(\underline{u}_Q - \underline{u}_P)$$

since the directions $\underline{a}_Q$ and $\underline{a}_P$ are random. More precisely, the change in momentum is

$$m(\underline{u}_Q(t + \tau) - \underline{u}_P(t))$$

where $t + \tau$ is the time of the collision in question. To compute the average value of this change in momentum, note that the point P could be anywhere on a sphere of radius r surrounding Q.

We therefore derive a formula relating the value of a function at a point to its average over a small sphere surrounding that point.

Let the point be the origin:

$$\phi(\underline{x}) = \phi_o + \sum_i x_i \left.\frac{\partial\phi}{\partial x_i}\right|_o + \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial^2\phi}{\partial x_i \partial x_j} + \ldots$$

Using the notation < > for the average over a sphere of radius r we have

$$\langle\phi(\underline{x})\rangle = \phi_o + \frac{r^2}{6}\sum_i \left.\frac{\partial^2\phi}{\partial x_i^2}\right|_o + \ldots$$

$$= \phi_o + \frac{r^2}{6}\nabla^2\phi\Big|_o + \ldots$$

where we have used

$$\langle x_i\rangle = 0$$
$$\langle x_i x_j\rangle = 0 \qquad i \neq j$$
$$\langle x_i^2\rangle = \frac{1}{3} r^2$$

Remark: The formula just derived becomes exact for the case $\nabla^2\phi = 0$ if this holds not only at the origin but throughout space. In that case $\langle\phi(\underline{x})\rangle = \phi_o$ for all spheres centered on the origin.

Therefore

$$\langle\underline{u}_P\rangle = \underline{u}_Q + \frac{r^2}{6}\nabla^2\underline{u}\Big|_Q + \ldots$$

where ∇^2 is understood to operate on each of the orthogonal components of $\underline{u}$ separately.

The mean change in momentum for the molecule of mass m which collides with the fluid at point Q at time $t + \tau$ is

$$m[\underline{u}(t+\tau) - \underline{u}(t) - \frac{r^2}{6}\nabla^2\underline{u} - \ldots]_Q = m\tau[\frac{1}{\tau}(\underline{u}(t+\tau)-\underline{u}(t)) - \frac{r^2}{6\tau}\nabla^2\underline{u} - \ldots]_Q$$

The change in momentum per unit volume, per unit time, at the fixed point Q in space is therefore

$$\rho[\tfrac{1}{\tau}(\underline{u}(t+\tau) - \underline{u}(t)) - \frac{r^2}{6\tau}\nabla^2\underline{u} - \ldots]_Q \quad .$$

Taking the limit as $\tau \to 0$, $r \to 0$, $(r^2/6\tau) \to \nu$ this becomes

$$\rho(\frac{\partial \underline{u}}{\partial t} - \nu\nabla^2\underline{u})_Q \quad .$$

This quantity replaces $\rho\frac{\partial \underline{u}}{\partial t}$ in the equation of motion of an ideal fluid and we have (Navier-Stokes equations)

$$\left.\begin{aligned} \rho(\frac{\partial \underline{u}}{\partial t} - \nu\nabla^2\underline{u}) &= -\nabla p - \rho\underline{u}\cdot\nabla\underline{u} \\ \nabla\cdot\underline{u} &= 0 \end{aligned}\right]$$

This result can be expressed more neatly by defining a new material time derivative in the presence of random molecular motion. Call this $\left.\frac{D}{Dt}\right|_B$, where the B stands for Brownian motion. The definition is

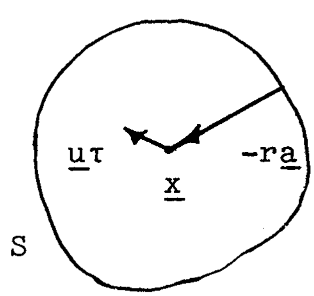
$$\left.\frac{D\phi}{Dt}\right|_B = \lim_{\tau\to 0} \left\langle \frac{\phi(\underline{x}+\underline{u}\tau, t+\tau) - \phi(\underline{x}-r\underline{a}, t)}{\tau} \right\rangle$$

where

$$\frac{r^2}{6\tau} = \nu$$

$\underline{a}$ = unit vector with random direction

< > = average over all directions of $\underline{a}$.



The quantity $\left.\frac{D\phi}{Dt}\right|_B$ represents the mean rate of change in ϕ seen by a particle which is at a random point of S at time t and at $\underline{x} + \underline{u}\tau$ at time $t + \tau$.

Again using

$$\langle\phi(\underline{x}-r\underline{a})\rangle = \phi(\underline{x}) + \frac{r^2}{6}\nabla^2\phi(\underline{x}) + \ldots$$

we have

$$\left.\frac{D\phi}{Dt}\right|_B = \frac{\partial\phi}{\partial t} + \underline{u}\cdot\nabla\phi - \nu\nabla^2\phi$$

or

$$\left.\frac{D}{Dt}\right|_B = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla - \nu\nabla^2$$

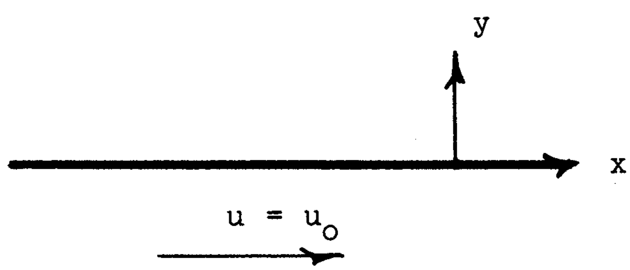
and the Navier-Stokes equations become

$$\left.\begin{aligned} \rho\left.\frac{D\underline{u}}{Dt}\right|_B &= -\nabla p \\ \nabla\cdot\underline{u} &= 0 \end{aligned}\right] .$$

Boundary Conditions and Boundary Layers in Viscous Flow

If we assume that exchange of momentum by diffusion is possible between the wall and the fluid, then we are forced to conclude that the fluid velocity is equal to that of the walls which bound the fluid at the surface of the wall. This no-slip condition is more restrictive than the conditions on the normal component of $\underline{u}$ that we prescribed previously for inviscid flow. In particular the no-slip condition rules out potential flow, unless the walls are going through some large tangential motions to keep up with the fluid. Nevertheless, if the flow is potential flow at t = 0, its departure from potential flow may remain small in significant regions of the flow field for significantly long times. This is because the effects of the wall on the fluid propagate away from the wall by diffusion (at least initially, later convection may become important). This point is illustrated by the following problem:

Let a fluid at rest be bounded by a plane wall which begins moving parallel to itself at t = 0 at the constant velocity u_o.



The fluid velocity is everywhere in the x-direction, the pressure is constant and the equations reduce to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad .$$

Seek a solution of the form

$$u = f(Y)$$

where

$$Y = y/(\nu t)^{1/2}$$

$$f'\left(\frac{\partial Y}{\partial t}\right) = \nu f''\left(\frac{\partial Y}{\partial y}\right)^2$$

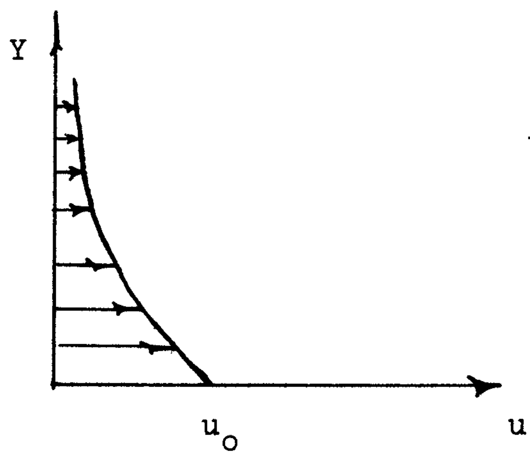
$$\frac{\partial Y}{\partial t} = -\frac{1}{2}(\nu t)^{-3/2}\nu$$

$$\left(\frac{\partial Y}{\partial y}\right)^2 = (\nu t)^{-1}$$

$$\boxed{-\frac{1}{2}Yf' = f''}$$

This equation has the solution

$$u = f(Y_1) = u_o(1 - \frac{1}{\pi^{1/2}}\int_0^{Y_1} e^{-Y^2}dY$$



velocity profile near the moving wall

Note that a given value of Y (and hence of u) represents a value of y which changes with time. Thus a particular value of u propagates away from the wall according to $y = Y(\nu t)^{1/2}$. The layer of fluid influenced by the wall has a thickness which grows like $(\nu t)^{1/2}$.

Estimate of boundary layer thickness in the human heart:

$$\nu = 0.04 \text{ cm}^2/\text{sec}$$

$$t < \text{time of heartbeat} < 1 \text{ sec}$$

$$(\nu t)^{1/2} < 0.2 \text{ cm}$$

(Diameter of mitral ring is about 3 cm)

Direct measurements of the flow profile* have confirmed that the profile is "flat" (i.e., uninfluenced by the wall) at distances greater than 0.2 cm. Measurements could not be made closer to the wall in the study cited.

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* Taylor and Wade

Flow through mitral valve during diastolic filling J. Physiol 200:73 P (1969).

Pattern of Flow around Atrioventricular Valves J. Physiol 207:71 P (1970).

Comparative Fluid Dynamics of Mammalian Hearts

When data** on mammalian hearts of different species from the dormouse to the elephant are compared, the following patterns emerge:

Let W = heart weight

L = length of ventricular cavity

T = time of a heart beat (reciprocal heart rate)

** Clark, A.J., Comparative Physiology of the Heart Cambridge, University Press 1927 p. 83.
Altman, P.L., Handbook of Circulation (Nat. Acad. Sci.) Saunders, 1959 p. 25, 81.
MacMahon, T., (personal communication)

Then

$$W \sim L^3$$
$$T \sim W^{0.28}$$

(The range of heart weights involved is a factor of 10^6.) From these relations we conclude that $T \sim L^{0.84}$, or since the data are hardly good enough to distinguish between a factor of 0.8 and 1, we conclude that roughly $T \sim L$. One could then make the hypothesis that the flows in these different species are scale models of each other (whether the equations of motion allow this will be investigated below), and conclude that (since ρ, ν are constant in the different species

(1) velocities will be equal

(2) pressures will be equal

(3) wall stress in the muscle will be equal (force/unit area)

(4) cardiac output (volume per unit time) will be proportional to L^2 or area.

The last two conclusions make especially good physiological sense because one expects the material strength of the muscles to be the same (on the basis of area), and because cardiac output is needed to supply oxygen for metabolism which tends to vary like surface area.

But can the flows actually be scale models of each other? To investigate this write the equations of motion in non-dimensional form through the substitutions (primes refer to dimensional quantities)

$$x' = Lx \qquad t' = Tt \qquad u' = \frac{L}{T} u$$

$$\frac{p'}{\rho} = \frac{L^2}{T^2} p \qquad R = \frac{L^2}{\nu T}$$

The Navier-Stokes equations become

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla p + \frac{1}{R} \nabla^2 \underline{u}$$

$$\nabla \cdot \underline{u} = 0$$

and a non-dimensional solution of these equations for some particular R generates a whole family of dimensional solutions by choosing L and T consistent with $R = \frac{L^2}{\nu T}$. Note that the solutions for different mammalian hearts are <u>not</u> related in this way since we have $L \sim T$, not $L^2 \sim T$. In fact the range of R for different mammalian hearts is a factor of 10^2, since $R \sim L^2/\nu T \sim L$.

In fact, the <u>relative</u> thickness of the boundary layer varies as $R^{-1/2}$, so one can conclude that there is a range of 10:1 in this parameter over the different species of mammal. Small mammals have relatively thick boundary layers.

Nevertheless, if the boundary layers are always thin compared to heart dimensions (as in man), then the flows will be approximately scale models of each other outside the boundary layers.

The similarity of the anatomy strongly suggests the truth of this hypothesis.

Vorticity

We have defined the circulation around a closed path Γ as

$$K = \oint_{\Gamma} \underline{u}\cdot d\underline{x} \quad .$$

From vector analysis we have

$$K = \iint_S (\nabla\times\underline{u})\cdot\underline{n}\, da$$

where S is any surface spanning the curve Γ, $\underline{n}$ is the unit normal to the surface, and da is the element of area in S. The vector

$$\underline{\omega} = \nabla\times\underline{u}$$

is called the vorticity. The components of $\underline{\omega}$ are

$$\omega_i = \frac{\partial}{\partial x_j} u_k - \frac{\partial}{\partial x_k} u_j$$

where (ijk) is a cyclic permutation of (123). From this formula it is obvious that $\underline{\omega} = 0$ is a necessary condition for the existance of a velocity potential ϕ, because if $\underline{u} = \nabla\phi$,

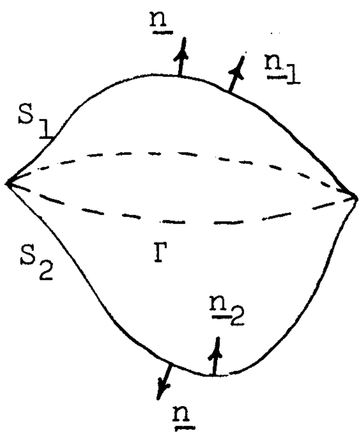
$$\omega_i = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\phi - \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}\phi = 0.$$

The fact that <u>any</u> surface S spanning the curve Γ gives the same value of K is remarkable. It is easy to see that this will hold if for all closed surfaces S_o

$$\oiint_{S_o} (\underline{\omega}\cdot\underline{n})da = 0$$

for, in that case one can combine any two surfaces spanning Γ into a single closed surface and show that

$$\iint_{S_1} \underline{\omega}\cdot\underline{n}_1 \; da = \iint_{S_2} \underline{\omega}\cdot\underline{n}_2 \; da$$



since $\displaystyle\iint_{S_1+S_2} (\underline{\omega}\cdot\underline{n}) \; da = 0$.

This shows that $\nabla\cdot\underline{\omega} = 0$, a fact which may be verified by direct calculation. The vorticity is the circulation per unit area, and it is equal to twice the local angular velocity. To see this for a rigid body rotation, note that in the plane normal to the axis of rotation $v = \dot{\theta}r$ where $\dot{\theta}$ = angular velocity. Then $K = 2\pi r\,\dot{\theta}r$ and $K/\pi r^2 = 2\dot{\theta}$. More generally, note that an arbitrary motion in the neighborhood of a point after the translation of the point itself has been subtracted away has the form: $\displaystyle\sum_{j=1}^{3} \left.\frac{\partial u_i}{\partial x_j}\right|_o x_j$.

But

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Thus one can think of the motion as being made of a symmetric and an antisymmetric part. The symmetric part of the motion has three orthogonal directions which are unchanged by the motion, it therefore contributes nothing to the rotation near the point in question.

The antisymmetric part is precisely a rigid body rotation at angular velocity $\underline{\dot{\theta}} = \frac{1}{2}\,\underline{\omega}$.

Equation of Motion for the Vorticity

Taking the curl of both sides of the Navier-Stokes equations we have

$$\nabla\times\frac{\partial \underline{u}}{\partial t} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = -\frac{1}{\rho}\,\nabla\times\nabla p + \nu\nabla\times\nabla^2\underline{u}$$

Since space derivatives commute with each other and with $\frac{\partial}{\partial t}$, we have

$$\nabla\times\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}\,(\nabla\times\underline{u}) = \frac{\partial}{\partial t}\,\underline{\omega}\ ,$$

$$\nabla\times\nabla^2\underline{u} = \nabla^2\ (\nabla\times\underline{u}) = \nabla^2\underline{\omega}\quad .$$

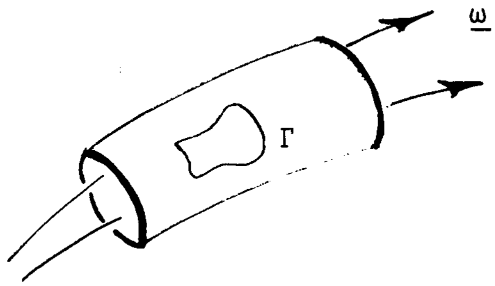
Since $\oint \nabla p = 0$ for all closed paths, $\nabla\times\nabla p = 0$. The term $\nabla\times(\underline{u}\cdot\nabla\underline{u})$ is more difficult to evaluate. Using the relations $\nabla\cdot u = 0\quad \nabla\cdot\underline{\omega} = 0$, one can show that

$$\nabla\times(\underline{u}\cdot\nabla\underline{u}) = \underline{u}\cdot\nabla\underline{\omega} - \underline{\omega}\cdot\nabla\underline{u}$$

Therefore we have

$$\frac{\partial\underline{\omega}}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}\quad .$$

The origin of the term $\underline{\omega}\cdot\nabla\underline{u}$ can be seen most clearly in an inviscid fluid for which we have conservation of the circulation around all closed paths moving with the fluid. Now consider a "vortex tube", i.e. a bundle of lines which are parallel to $\underline{\omega}$ at some instant.



A closed material path Γ on the surface of the vortex tube has zero circulation since $\underline{\omega} \cdot \underline{n} = 0$. Since it retains zero circulation, the material of the vortex tube remains a vortex tube. In addition the circulation K of the vortex tube (which is the same at all cross-sections since $\nabla \cdot \omega = 0$) remains constant in time. Now the volume of a section of the vortex tube also remains constant, since $\nabla \cdot \underline{u} = 0$. Therefore for a differential vortex tube of area da and length $d\ell$, the following quantities are constant in time:

$$da \;\; d\ell = \text{volume}$$

$$da \;\; \omega = \text{circulation}$$

It follows that ω is proportional to $d\ell$. Since the vector is parallel to the vector $d\underline{x}$ with length $d\ell$ along the axis of the tube we have

$$\underline{\omega} = \frac{\omega}{d\ell} \, d\underline{x}$$

where the factor $\frac{\omega}{d\ell}$ is independent of time (moving with the fluid). Applying the operator $\frac{D}{Dt}$ we have

$$\frac{D\omega}{Dt} = \frac{\omega}{d\ell} \, d \, \frac{D\underline{x}}{Dt} = \frac{\omega}{d\ell} \, d\underline{u} = \underline{\omega} \cdot \nabla \underline{u} \quad .$$

In summary, when vortex tubes are stretched, they also get thinner, and the vorticity has to go up to conserve the circulation.

Note that in two dimensional flow $\underline{\omega}$ is orthogonal to the plane of the flow everywhere, so that only its magnitude $\omega = |\underline{\omega}|$ is of significance. Also $\underline{\omega}\cdot\nabla\underline{u} = 0$, so that for two dimensional flow the equation of motion for the vorticity reduces to

$$\frac{D\omega}{Dt} = \frac{\partial\omega}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} = \nu\nabla^2\underline{\omega} \quad .$$

If we regard $\underline{u}$ as given, the equations for $\underline{\omega}$ are linear in $\underline{\omega}$, but such a separation is not usually possible since $\underline{\omega} = \nabla\times\underline{u}$. An expression for $\underline{u}$ in terms of $\underline{\omega}$ may be found as follows. Let $\underline{u} = \nabla\times\underline{A}$ with $\nabla\cdot\underline{A} = 0$, then $\underline{\omega} = \nabla\times\nabla\times\underline{A}$. But $\nabla\times\nabla\times\underline{A} = \nabla(\nabla\cdot A)-\nabla^2\underline{A}$ and $\nabla\cdot\underline{A} - 0$. Thus

$$\nabla^2\underline{A} = -\underline{\omega}$$

$$\underline{A}(\underline{x}) = \int \frac{\underline{\omega}(x')}{4\pi r}\, dv'$$

where

$$r = |\underline{x} - \underline{x}'|$$

$$\underline{u} = \nabla\times \int \frac{\underline{\omega}(x')}{4\pi r}\, dv' \quad .$$

Since the ∇ in the expression for $\underline{u}$ operates only on $\underline{x}$, $\underline{\omega}(\underline{x}')$ acts as a constant vector. But

$$\nabla\times(\phi\underline{a}) = \nabla\phi\times\underline{a}$$

when $\underline{a}$ is constant. Also

$$\nabla(\frac{1}{r}) = -\frac{1}{r^2}\nabla(r) = -\frac{1}{r^2}\frac{\underline{r}}{r} = -\frac{1}{r^3}\underline{r}$$

$$\underline{u} = -\frac{1}{4\pi}\int \frac{\underline{r}\times\underline{\omega}}{r^3}\,dv'$$

In the presence of boundaries a flow of the form $\underline{u} = \nabla\phi$ may have to be added in to satisfy the boundary conditions.

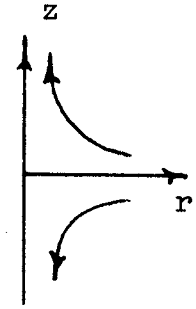
The Accumulation of Vorticity Near a Line

Consider the incompressible flow

$$u_z = \alpha z$$
$$u_x = -\frac{1}{2}\alpha x$$
$$u_y = -\frac{1}{2}\alpha y$$

In polar coordinates this becomes

$$u_z = \alpha z$$
$$u_r = -\frac{1}{2}\alpha r$$



This is a potential flow (zero vorticity), but consider what happens when we add on a radially symmetric component of vorticity pointing in the z-direction $\underline{\omega} = \omega(r)\hat{\underline{z}}$.

The flow associated with ω is in the θ-direction only, it contributes nothing to the convection or stretching of vorticity so we may write the equation for the vorticity in terms of the flows u_r, u_z defined above.

This means that we have in fact a linear problem to solve for ω of the form

$$\frac{\partial\omega}{\partial t} + u_r \frac{\partial\omega}{\partial r} = \frac{\partial u_z}{\partial z} + \nu\nabla^2\omega$$

$$\frac{\partial\omega}{\partial t} = \frac{\alpha}{2r} \frac{\partial}{\partial r}(r^2\omega) + \nu\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial\omega}{\partial r}\right) .$$

Setting $\partial\omega/\partial t = 0$ (steady flow) we find $\frac{\alpha r}{2\nu}\omega + \frac{\partial\omega}{\partial r} = 0$.

This has the solution

$$\omega = \omega_o e^{-\frac{\alpha}{4\nu} r^2}$$

The circulation around a circle of radius r is

$$K(r) = 2\pi \int_0^r r'\omega(r')dr'$$

and

$$K = \lim_{r\to\infty} K(r) = \pi \int_0^\infty 2rdr\, e^{-\frac{\alpha}{4\nu} r^2} \omega_o = 4\frac{\pi\omega_o\nu}{\alpha} .$$

Thus

$$\omega = \frac{\alpha K}{4\pi\nu} e^{-\frac{\alpha}{4\nu} r^2}$$

This distribution will evolve out of any initial distribution of axial vorticity with total circulation K. As $\nu \to 0$ the vorticity distribution tends to a line vortex of strength K.

Although this example may seem very special, conditions which are like this locally will occur often. Consider an arbitrary potential flow. In the neighborhood of a point which we take as origin

$$\phi = \phi_o + \sum_i \left.\frac{\partial \phi}{\partial x_i}\right|_o x_i + \frac{1}{2}\sum_{ij} \left.\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right|_o x_i x_j + \ldots$$

The velocity at the origin is determined by $\partial\phi/\partial x_i$ and the velocity in the neighborhood of the origin relative to the velocity at the origin is determined by $\partial^2\phi/\partial x_i \partial x_j$. Now it is always possible to choose orthogonal coordinates at the origin such that

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0 \qquad i \neq j$$

and we shall assume that this has been done. Then let $\lambda_i = \frac{\partial^2 \phi}{\partial x_i^2}$ and $\nabla^2\phi = 0$ yields $\sum_i \lambda_i = 0$.

The velocity field of the deformation near the point o is now given by $u_i = \lambda_i x_i$. The vorticity concentrating flow we considered above was precisely this with $\lambda_1 > 0$, $\lambda_2 = \lambda_3 = -\frac{1}{2}\lambda_1$. More generally, if $\lambda_1 > 0$ and $\lambda_2, \lambda_3 < 0$ then we can expect vortex stretching along the direction x_1 and a local concentration of vorticity of this type near the point 0.

Vortex Stretching in the Heart ?

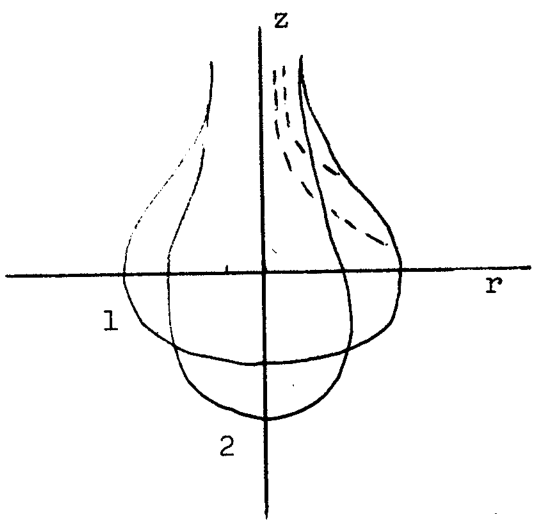
It is interesting that the flow pattern used to generate the vortex described above, namely

$$\phi = 2\lambda z^2 - \lambda x^2 - \lambda y^2 = 2\lambda z^2 - \lambda r^2$$

has actually been proposed as a model of flow in the heart.* The time dependence is put in by multiplying by a factor f(t) - recall that an arbitrary potential flow satisfies the equations of motion (even with viscosity) but usually will not satisfy no-slip conditions at the wall. Here Jones* ignores the no-slip condition because the error will be confined to a thin boundary layer. Moreover he assumes the flow and then calculates the normal motion of the boundary which is consistent with that flow.

Including the factor λ in f(t) we have

$$\phi = f(t)(2z^2 - x^2 - y^2)$$



The streamlines are constant though the speed along them may vary. An arbitrary initial shape for the heart can be drawn on these stream-lines (making it tangent at the out-flow keeps the outflow diameter constant) and the subsequent shapes can be found. The velocity $\underline{u}$ has the simple form

$$\underline{u} = f(t)(-2x, -2y, 4z)$$

---

* Jones, Robert T., "Blood Flow" Annual Review of Fluid Mechanics 1, 223-244 (1969).

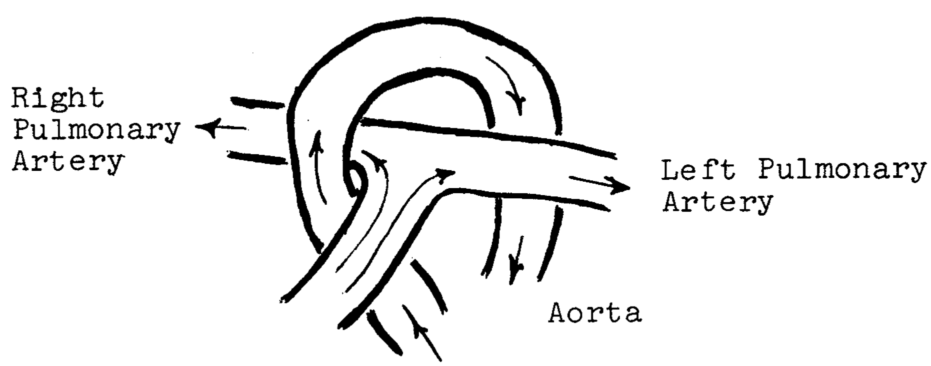
It follows that the configuration after time t is related to the configuration at t = 0 by a simple linear (and diagonal) transformation of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix} \begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix}$$

where

$$\alpha = e^{-\int_0^t f(t')dt'}$$

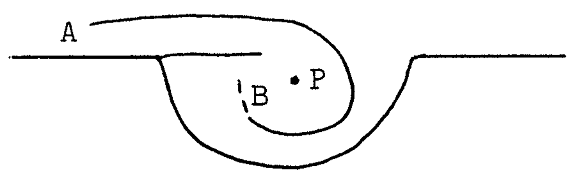
It is interesting to consider what happens if we take Jones' flow pattern and consider the possibility of superposition of axial vorticity on top of it as before. Such vorticity will be intensified by stretching motion of blood leaving the heart, and we can therefore expect some kind of spiralling motion of the blood as it leaves the heart, provided that the initial axial vorticity is non-zero. This will be enhanced by spiral contraction of the muscles, as is known to occur. It may be relevant to point out here that the aorta and pulmonary artery are sharply curved, and that these curves do not lie in a plane but form spirals around each other.



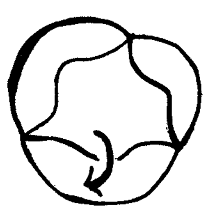
One ought to admit, however, that at the time when these vessels are being formed the heart has very different dimensions than in the adult. $L \cong ,.4$ cm $T \cong 1$ sec. and $L \cong 2(\nu T)^{1/2}$. Under these conditions viscous effects will be much more prominent than in the adult heart and it is questionable whether vortex stretching (which comes from the interval terms in the equations of motion) can play any role.

Vortex Stretching in the Aortic Sinus ?

Bellhouse and Talbot (references cited previously) find that dye injected near the wall in the ventricular outflow is swept into the aortic sinus in steady flow with the valve open. They do not comment on what happens to the streamline beyond B.



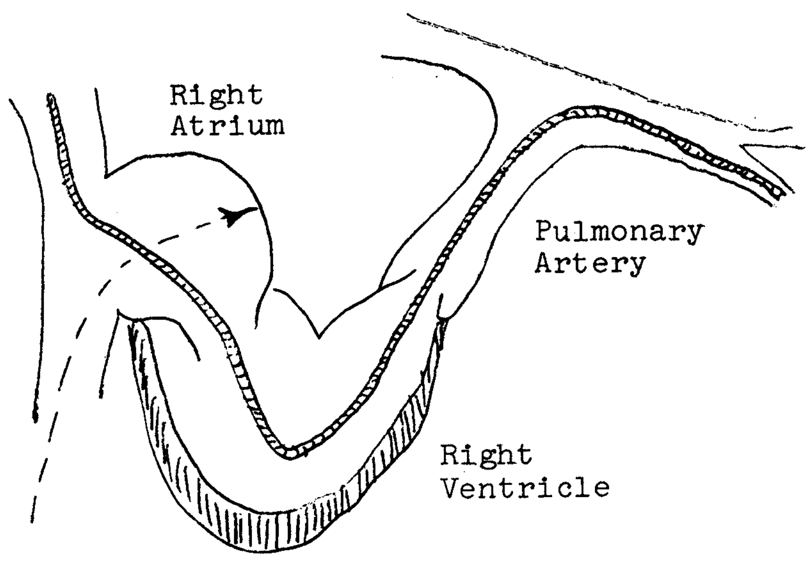
Clearly it must leave the sinus in a different plane to avoid crossing itself. That this is the case is suggested by a photograph taken by the same authors from downstream which shows the leaflets bent as though fluid were



entering the sinus at the center of the edge of each leaflet, with fluid leaving near the lines joining the different cusps. This suggests a flow in each sinus away from the plane of symmetry with concentration of vorticity at P.

Pressure and Flow Across Heart Valves

During cardiac catheterization the function of heart valves is assessed by recording pressure as a function of time upstream and downstream of the valve. Ideally this is done through simultaneous recording with two catheters, but more often the upstream and downstream pressures are recorded in rapid succession by withdrawal of the catheter through the valve. During right heart catheterization the catheter is advanced from a vein in the arm and follows the path of the blood. Entering the righ atrium from above it can be advanced downward through the right atrioventricular valve through the



ventricle and up into the pulmonary artery. When the catheter is wedged as far as it goes into the pulmonary arterial tree it establishes a still column of blood between itself and the pulmonary veins, and consequently it records the pulmonary venous pressure, which is roughly the same thing as left atrial pressure. A more direct route of access to the left atrium is by means of a trans-septal catheter (shown dotted on previous page) which is fed upward from a femoral vein and pierces the atrial septum at the region of the fossa ovalis, a thin portion of the atrial wall which in embryonic life was a flap valve permitting flow to pass from the right to the left atrium.

[It is worth noting that the two possible routes shown for the catheters also represent the paths taken by the blood in embryonic life: In the embryo the venous blood from the lower part of the body includes blood from the umbilical veins and is rich in oxygen; this is directed through the flap valve to the left side of the heart from which it goes preferentially to the head. Venous blood from the head and upper body goes preferentially to the right ventricle, pulmonary artery and from there to a special embryonic duct to the aorta and the lower part of the body.]

If necessary, left heart catheterization can also be performed, threading the catheter in the opposite direction to the flow of blood in the aorta, and through the aortic valve into the left ventricle.

Instantaneous flows across the valves are not usually measured (though this can be done in experimental animals), but the mean flow (cardiac output) can be measured from data on oxygen uptake: Let

V = rate of O_2 consumption measured at the mouth

C_a = arterial oxygen concentration

C_v = venous oxygen concentration

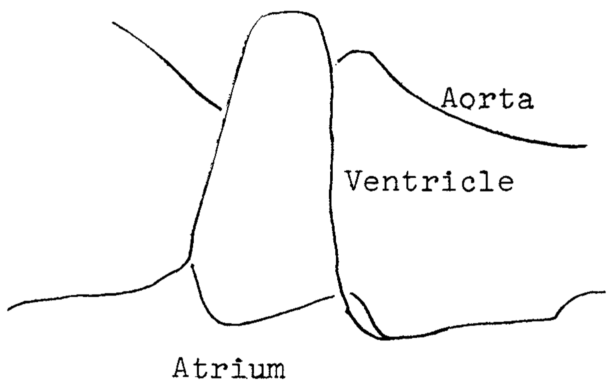
Q = cardiac output

Then $(C_a - C_v)Q = V$.

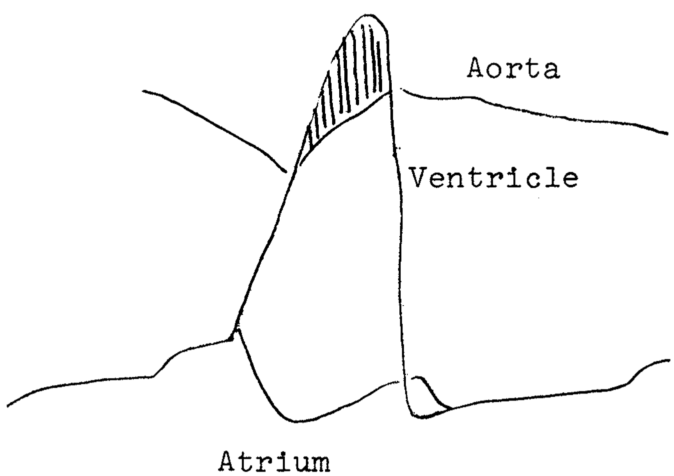
The functional defects of interest with respect to heart valves are stenosis (narrowing of the valve opening) and regurgitation (backflow when the valve should be closed). These defects may appear separately or coincide. When leaflets become calcified and inflexible they neither open nor close properly.

In the pressure records stenosis appears as a pressure difference between the upstream and downstream chambers when the valve should be open (see next page).

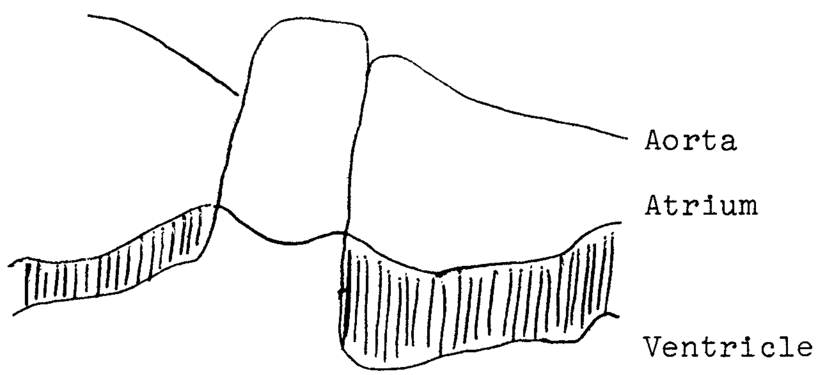
Regurgitation also can be observed in the pressure records. It appears as an abnormal tendency for the upstream and downstream pressures to approach each other when the valve should be closed. In the following, however, we shall be concerned primarily with stenosis.



Normal
Left Heart
Pressures

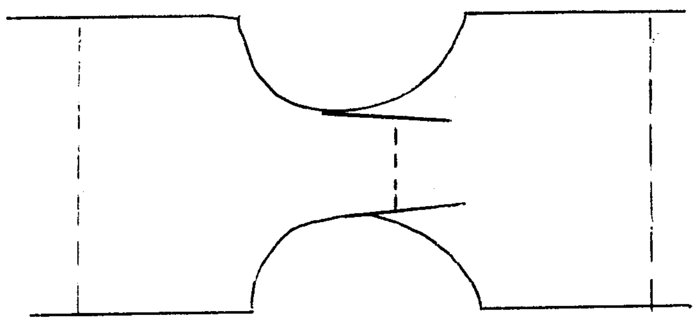


Aortic
Stenosis



Mitral
Stenosis

For proper interpretation of the pressure data, one needs a pressure-flow theory of the motion of blood through heart valves



For the configuration shown, we can write down an equation for the rate of change of kinetic energy in the fluid between 1 and 2. This equation

$$\frac{dT}{dt} = (p_1 - p_2)Q - D(Q)$$

where

T = kinetic energy in the fluid between 1 and 2.

Q = instantaneous flow (volume per unit time) through the system

$(p_1 - p_2)Q$ = rate of work done on the fluid in (1, 2) per unit time by the fluid external to this region.

$D(Q)$ = rate of dissipation (conversion of kinetic energy to heat) when the volume rate of flow is Q.

<u>Remarks:</u>

(1) Writing the rate of work as $(p_1-p_2)Q$ amounts to neglecting $\frac{1}{2}\rho u^2$ in comparison with p at the stations 1 and 2.

(2) It is not strictly correct to assume that the dissipation is uniquely determined by Q, since the dissipation depends on the details of the flow pattern which in turn might depend on how quantities are changing with time.

A guess at the form of D(Q) can be made as follows:

Let u_v be the velocity of the fluid between the valve leaflets at the point where the velocity is maximal and let A_v be the cross-sectional area at this point. This is less than the area at the valve ring because of the contraction of the jet. Then the flux of kinetic energy entering the downstream region, where most of the dissipation occurs, is

$$A\ u_v(\tfrac{1}{2}\ \rho\ u_v^2) \quad .$$

We assume that the rate of dissipation is a certain fraction (say α) of this kinetic energy flux. Then

$$\begin{aligned} D &= |\alpha A_v u_v(\tfrac{1}{2}\ \rho\ u_v^2)| \\ &= |\alpha Q\ \frac{\rho}{2A_v^2}\ Q^2| \\ &= |\ \alpha\ \frac{\rho}{2A_v^2}\ Q^3| \end{aligned}$$

where the absolute value sign is included because the dissipation is positive independent of the sign of Q.

In addition, it seems reasonable to assume that the kinetic energy T in the fluid is proportional to Q^2. We write this as $T = \frac{1}{2}\ kQ^2$, and note that k has dimensions of mass/area^2. This suggests that we write k in the form

$$k = k_o \frac{\rho L A}{A^2} = k_o \frac{\rho L}{A}$$

where

k_o = dimensionless constant

ρ = density of blood

L = distance between sites 1 and 2

A = characteristic cross-sectional area.

With these substitutions, we have

$$\frac{1}{2} k_o \frac{\rho L}{A} \frac{d}{dt} Q^2 = (p_1 - p_2)Q - \left| \frac{\alpha \rho}{2A_v^2} Q^3 \right|$$

which reduces to

$$k_o \frac{\rho L}{A} \frac{dQ}{dt} = (p_1 - p_2) - \frac{\alpha \rho}{2A_v^2} Q|Q|$$

when $Q > 0$ this has the form

$$\boxed{a \frac{dQ}{dt} + bQ^2 = \delta p}$$

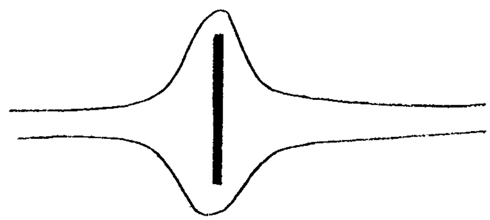
where

$$\delta p = p_1 - p_2 \quad .$$

Further fluid dynamical insight into the origins of this equation comes from the free-streamline theory of drag. In strictly irrotational flow there are no drag forces. But if surfaces of tangential discontinuity are allowed to separate two irrotational regions, a net drag force may result. This is a first approximation to representing the effect of a wake behind immersed bodies while circumventing the complicated details of rotational flow in the wake.

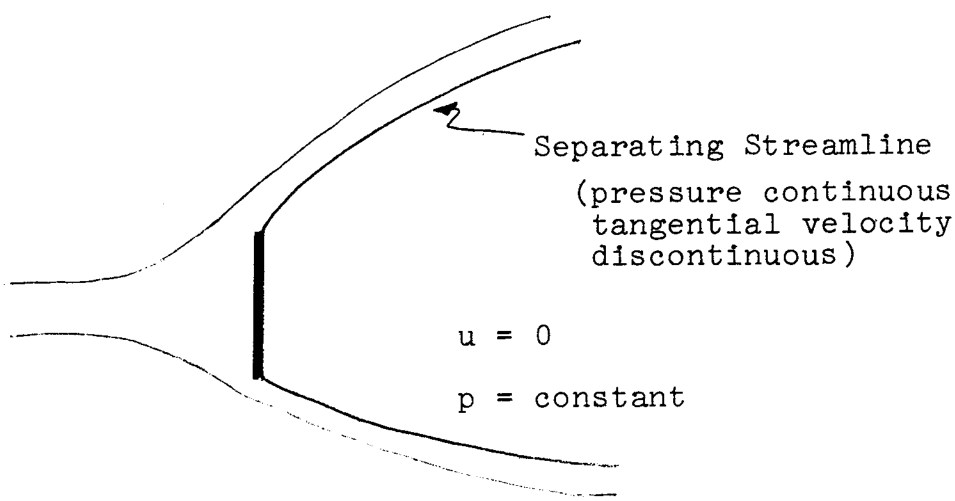
Example: A flat plate in a stream perpendicular to the plate.

Irrotational Flow:



Drag = 0

2 Regions of Irrotational Flow with Separating Streamlines:



$u = -\nabla\phi$

$p+\frac{1}{2}\rho u^2$ = constant

Drag $\neq$ 0

Remark:

Note that in the two pictures shown, the irrotational flow without separation has the property that one cannot tell which way the flow is going from a picture of the streamlines alone. This property is lost when the separating streamlines are included. This is the essential reason why there is no drag in the first case but there is drag in the second.

To analyse such a situation, one can use Bernoulli's theorem, which we now pause to derive. Let

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \nu\nabla^2\underline{u}$$

and let $\underline{u} = -\nabla\phi$ with $\nabla^2\phi = 0$. Note that $(\nabla\phi\cdot\nabla)\nabla\phi = \frac{1}{2}\nabla|\nabla\phi|^2 = \frac{1}{2}\nabla u^2$ where $u = |\nabla\phi|$. To see write the i^{th} component as follows

$$\sum_{\ell} \frac{\partial\phi}{\partial x_\ell}\frac{\partial}{\partial x_\ell}\frac{\partial\phi}{\partial x_i} = \sum_{\ell}\frac{\partial\phi}{\partial x_\ell}\frac{\partial}{\partial x_i}\frac{\partial\phi}{\partial x_\ell}$$

$$= \frac{1}{2}\frac{\partial}{\partial x_i}\sum_{\ell}\left(\frac{\partial\phi}{\partial x_\ell}\right)^2$$

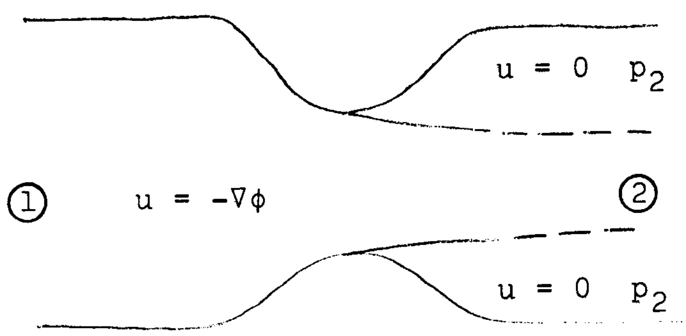
$$= \frac{1}{2}\frac{\partial}{\partial x_i}|\nabla\phi|^2 \quad .$$

Thus we have $\underline{u}\cdot\nabla\underline{u} = \frac{1}{2}|\nabla\underline{u}|^2$ (for irrotational flow only!), and

$$\nabla\left(-\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p\right) = 0$$

$$\boxed{\rho\frac{\partial\phi}{\partial t} = \frac{1}{2}\rho u^2 + p}$$

For the heart valve problem one can expect streamlines like this (in the free streamline theory).



Subtracting the equations for ϕ at sites (1) and (2)

$$\rho\frac{\partial}{\partial t}(\phi_1-\phi_2) = p_1-p_2 + \tfrac{1}{2}\rho u_1^2 - \tfrac{1}{2}\rho u_2^2 \quad .$$

Now write

$$\phi_1 - \phi_2 = \int_1^2 \underline{u}\cdot d\underline{x}$$

$$\cong Q \int_1^2 \frac{dx}{A(x)} \quad .$$

The integral $\int_1^2 \frac{dx}{A(x)}$ is a purely geometrical factor which we write as $\left\{\frac{L}{A}\right\}$. Now, neglecting u_1^2 in comparison with u_2^2 we obtain

$$\frac{\rho L}{A}\frac{dQ}{dt} = p_1-p_2 - \frac{1}{2}\rho u_2^2$$

$$= (p_1-p_2) - \frac{\rho}{2A_2^2}Q^2$$

Note that both formulas only apply to the case where the upstream and downstream value of $\frac{1}{2}\rho u^2$ can be neglected in comparison with this quantity near the valve. This will be true wherever the valve presents a significant obstruction to the flow. When it is false, $\delta p = 0$ is a good approximation for the open valve.

Comparing this result with the formula of page 138 we note that they are equivalent if we set $A_v = A_2$, $k_o = 1$, and $\alpha = 1$. The latter is presumably an upper bound for α, because the second formula assumes that the fluid outside the jet is at rest, a condition which tends to over-estimate the dissipation. Similarly $k_o = 1$ is probably a lower bound for k_o since $\frac{1}{2} k = \frac{1}{2} k_o \frac{\rho L}{A}$ is the ration between the kinetic energy and Q^2, and irrotational flow minimizes the kinetic energy.

<u>Experimental Confirmation</u>

Yellin* has set up experiments in which pulsatile flow of the form $Q = Q_o + Q_1 \sin \omega t$ is driven across a fixed stenosis (narrowing) in a rigid pipe. The pressure drop is measured. For the case $Q_o = 0$ we expect

$$\delta p = \begin{cases} a\omega Q_1 \cos \omega t + bQ_1^2 \sin^2 \omega t & Q > 0 \\ a\omega Q_1 \cos \omega t - bQ_1^2 \sin^2 \omega t & Q < 0 \end{cases}$$

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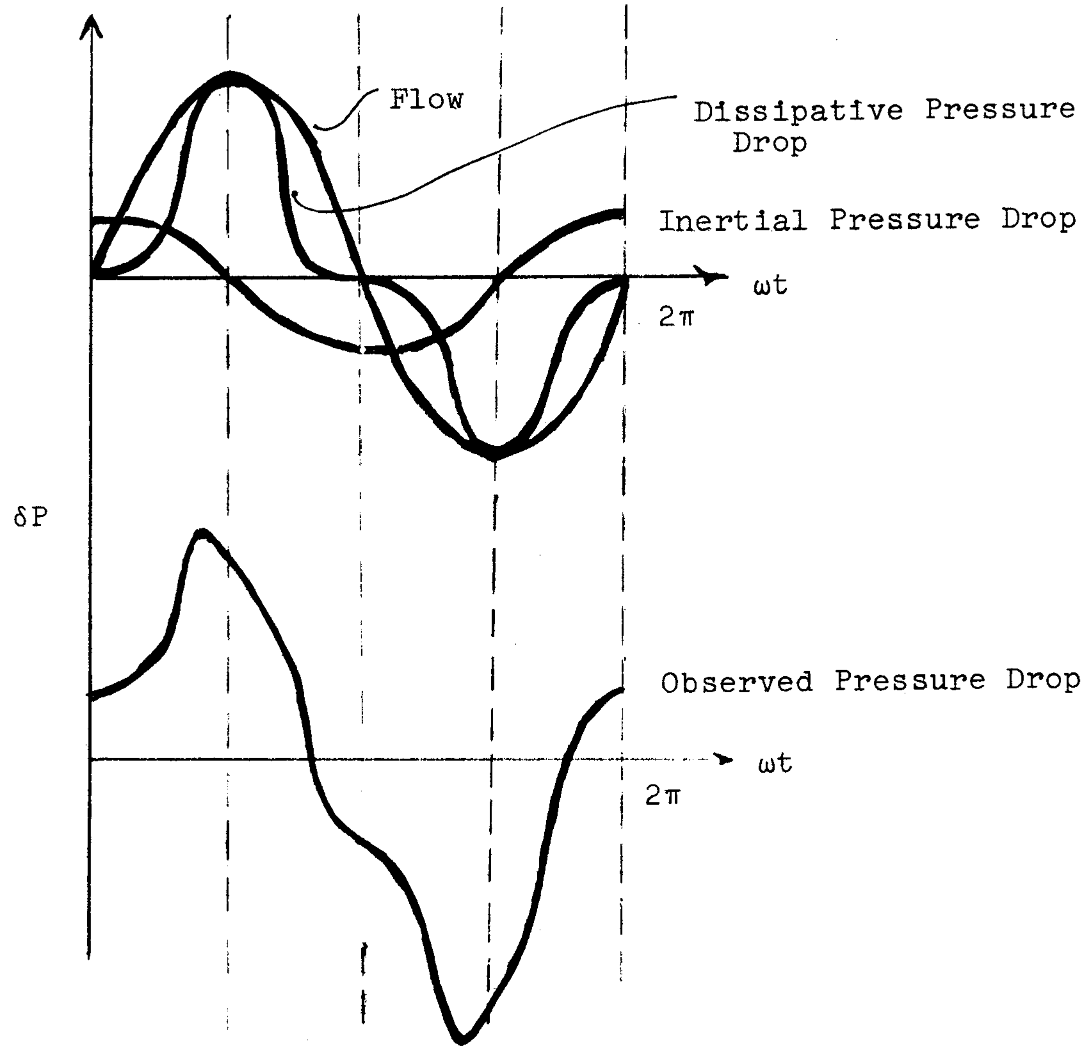
* Yellin and Peskin, "Large Amplitude Pulsatile Water Flow Across an Orifice" ASME preprint 73-WA Bio-12.



The inflection points which coincide in time with the zeros of the flow are indicative of the square-law character of the dissipative pressure drop, while the fact that the pressure drop is non-zero at these points shows the importance of inertia.

As a further test of the theory, the dissipative coefficient was evaluated in two ways. First, at the instant when $dQ/dt = 0$, one can evaluate b from $bQ^2 = \delta P$. Second, one can evaluate the same coefficient by integrating between two times when the flow is zero. In that case

$$\int_{t_1}^{t_2} \frac{dQ}{dt} \, dt = Q(t_2) - Q(t_1) = 0$$

and we have

$$b \int_{t_1}^{t_2} Q^2 \, dt - \int_{t_1}^{t_2} \delta P \, dt$$

or

$$b\overline{Q^2} = \overline{\delta P}$$

(This formula only holds for intervals of time such that $Q(t_1) = Q(t_2)$ and $Q > 0$ on (t_1, t_2)). The two values of b were equal.

Physiological Consequences:

(1) Pressure - Flow Dynamics

If the fluid is at rest just prior to valve opening, Q and δP become positive at the moment of valve opening. Because of the inertial term the peak value of δP occurs before peak flow, and there is an interval of time after δP has become negative when the flow is still positive. If there is any finite interval during which $\delta P = 0$, the flow decays hyperbolically (not exponentially) toward zero. To see this let Qt - a/b.

Then

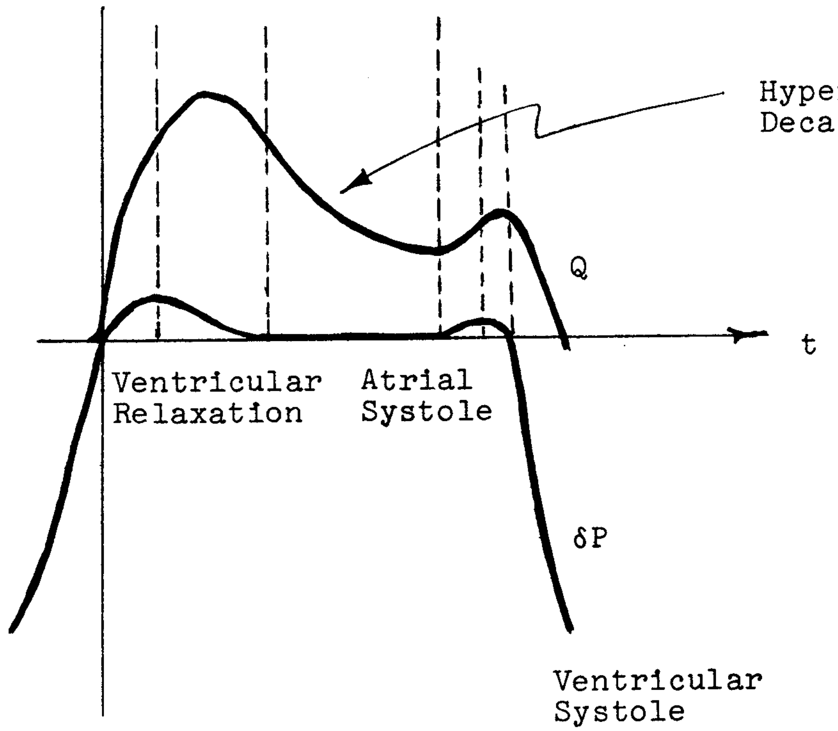
$$0 = \frac{d}{dt} (Qt) = \frac{dQ}{dt} t + Q$$

$$0 = \frac{dQ}{dt} (Qt) + Q^2 = \frac{a}{b} \frac{dQ}{dt} + Q^2$$

$$0 = a \frac{dQ}{dt} + bQ^2 \quad .$$

Note: To use the solution $Qt = a/b$, one has to define the origin of time in such a way that the initial conditions are satisfied. That is, if Q_o is known at some particular time, call this time t_o where $Q_o t_o = a/b$.

Example: Pressure-Flow curves for the mitral valve (inflow valve to left ventricle) look something like this:



(2) Pressure-flow dynamics during exercise.

During exercise cardiac output rises in proportion to heart rate, and the stroke volume (volume ejected by the heart beat) remains constant. To see whether or not the flow waveform for a valve should change shape, we should therefore look at whether a solution of $a \frac{dQ}{dt} + bQ^2 = \delta P$ admits a scaling of the form $t = \alpha t'$ $Q = \alpha^{-1}Q'$. This corresponds to a compression of the time axis by a factor α, and a stretching of the Q axis by a factor α^{-1}. It therefore leaves the volume, $\int Qdt$, ejected per beat unchanged. Making these substitutions, we find

$$\alpha^{-2} a \frac{dQ'}{dt'} + \alpha^{-2} bQ'^2 = \delta P$$

or

$$a \frac{dQ'}{dt'} + bQ'^2 = \delta P'$$

where

$$\delta P = \alpha^{-2} \delta P' \quad .$$

Thus the equation indeed admits such a scaling, **with** pressure changes that go as the square of the flow, and we can expect that to a first approximation the shape of the flow waveform for a given valve will be invariant with exercise.

Remark: The situation just described is in marked constrast to the case of the linear equation $a \frac{dQ}{dt} + bQ = \delta P$, for which the inertial term becomes more and more important with increasing frequency independent of the amplitude.

Remark #2: The foregoing result is only approximate, because changes in heart rate are not accompanied by stricly proportional changes in duration of the various parts of the cardiac cycle.

Remark #3: The proportionality between heart rate and cardiac ouput used above holds only in genuine exercise where the demands of the tissues for oxygen is high. If heartrate is artificially raised with a pacemaker, stroke volume falls proportionally and **cardiac output** is unchanged; control mechanisms responsible for this will be discussed later in the course.

(3) Evaluation of the dissipative coefficient **from** data obtained at cardiac catheterization.

In practice the flow is not measured in patients, and the following procedure is used to estimate the dissipative coefficient (and hence the severity of stenosis).* Let

Q_c = Cardiac Output

T = Duration of heartbeat

$\overline{Q}$ = Mean flow when valve is open

t = Duration of positive δP for the valve in question.

Then compute

$$\overline{Q} = \frac{T}{t} Q_c$$

and evaluate b from

$$b(\overline{Q})^2 = \overline{\delta P}$$

where the average is taken over the interval of time that δP is positive. There are three errors in the foregoing:

(1) $(\overline{Q})^2 \neq \overline{Q^2}$

(2) The formula $b\overline{Q^2} = \overline{\delta P}$ has theoretical justification <u>only</u> when the time interval (t_1, t_2) over which the averages are taken has the property $Q(t_1) = Q(t_2)$.

---

* Errors in this procedure are discussed below.

(3) The time t which is used to evaluate $\overline{Q}$ should be the interval when the valve is open. This is longer than the interval that $\delta P > 0$.

This shows the importance of finding a way to measure the flow in patients.

(4) Consequences to the patient of the square-law dependence of pressure on flow.

During exercise flow may increase by a factor of about 2, pressure drop by a factor of 4 and rate of dissipation of energy ($\overline{Q\delta P}$) by a factor of 8. In healthy individuals the resting pressure drop is so low that the increase is easily tolerated, but in valvular stenosis exercise becomes extremely difficult for the patient.

Aortic (outflow) stenosis:

Until the obstruction is severe excellent compensation is possible by thickening of heart walls. There is a tremendous increase in ventricular pressure, but aortic pulse is normal except for a slower rate of rise. With severe disease, chest pain, difficulty breathing, and fainting are symptoms, especially when exercise is attempted.

Mitral (inflow) stenosis:

Here symptoms appear earlier because there is no pumping chamber directly upstream of the valve to compensate for the condition. The increased pressure is felt in the pulmonary veins and capillaries, and there is a real danger of fluid being pressed out of the capillaries into the air spaces of the lung. First symptoms are difficulty breathing during exertion, and, if the stenosis progresses in severity, this happens at progressively lower levels of exercise.

Heart Sounds

The equation

$$a \frac{dQ}{dt} + bQ|Q| = \delta P$$

can be supplemented by other equations which show what happens when a heart valve is closed. We assume that the valve closes when Q becomes zero, but that Q can become slightly negative, displacing the closed valve leaflets because of their elasticity. Let V be the total volume displaced in this way. Then

$$V(t) = -\int_{t^*}^{t} Q(t')dt'$$

where t* is the time of valve closure, which we take to be the instant when Q becomes negative. The tensed valve leaflet exerts a force in the forward direction on the fluid which may be expressed as a pressure $\delta P_V = \frac{1}{C} V$, where C is the "compliance" of the valve. Thus we write

$$a \frac{dQ}{dt} + bQ|Q| = \delta P + \delta P_V$$

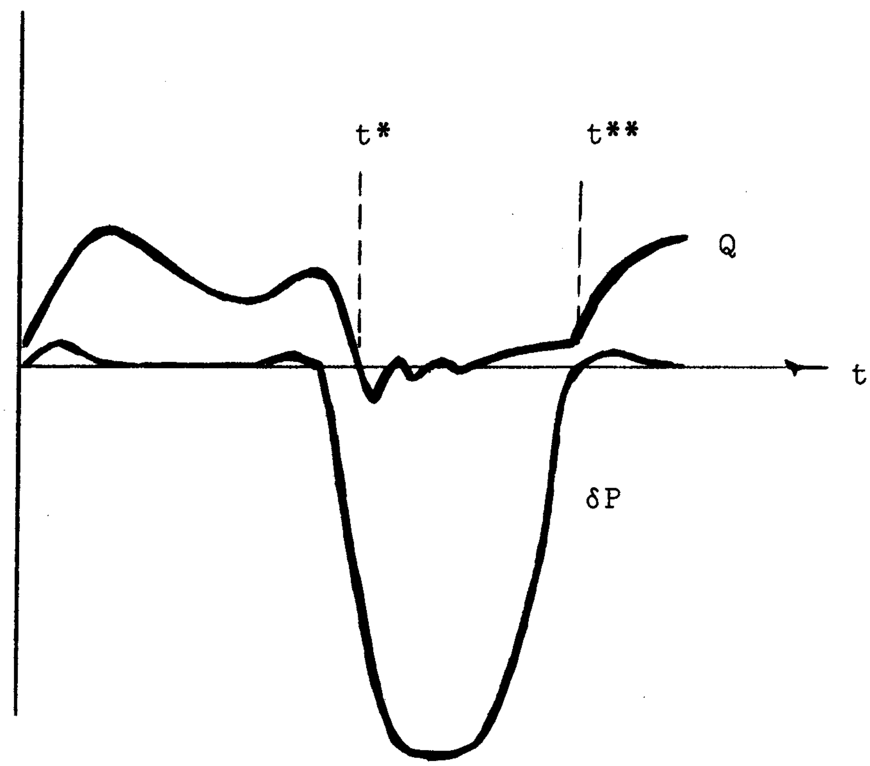
$$\delta P_V = \begin{cases} \frac{1}{C} \int_{t^*}^{t} - Q(t')dt' & t < t^{**} \\ 0 & t > t^{**} \end{cases}$$

$$\delta P = \text{given function of time}$$

where t^{**} is the time when the valve opens and is given by $V(t^{**}) = 0$, i.e.

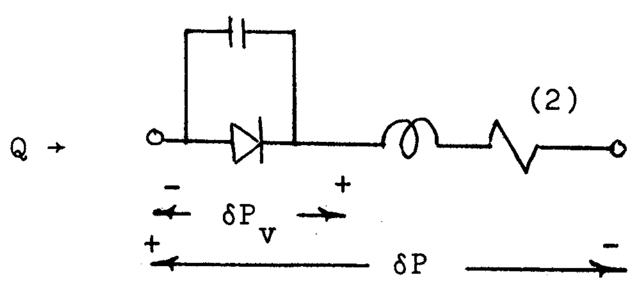
$$\int_{t^*}^{t^{**}} Q(t)dt = 0$$

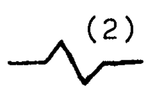
Here is a sketch of the solution for mitral flow:



Both the oscillation accompanying closure and the shoulder of forward flow prior to valve opening have been observed in flow measurements on the mitral valves of dogs. We believe that this oscillation is the source of the heart sound that is heard at the time of valve closure. The origin of this is the coupling of the mass of the fluid with the elasticity of the valve to form an harmonic oscillator. The observed oscillation is more damped than one would expect from the $bQ|Q|$ term as estimated from the forward flow. This suggests that tissue viscosity plays a role in damping the sound.

The equations outlined above for pressure-flow dynamics are summarized by the following electrical analog,



in which 

is the required non-linear dissipative element.

Numerical Analysis of Blood Flow in the Heart: Formulation of the Problem and Introduction to the Numerical Methods.

Motivation:

Approaches used previously in this course:

(i) Conformal mapping for aortic sinus.

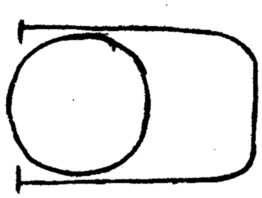
(ii) Assume flow out of the heart has the special form $\phi = f(t)(\frac{1}{2} r^2 - z^2)$

(iii) Pressure-flow analysis leading to the equation $\delta P = a \frac{dQ}{dt} + bQ^2$.

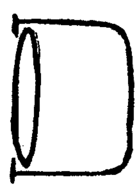
Such methods depend on special geometrical assumptions and fluid dynamical approximations. In particular, while they give some insight into what to expect they are inadequate for a detailed analysis of the flow pattern.

Such an analysis is required for the rational design of prosthetic valves.

Some valve designs which are now in use



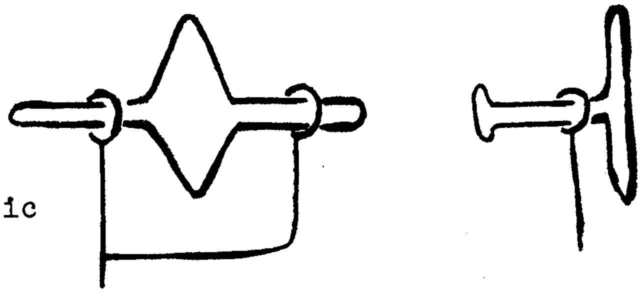
Ball Valve
(Starr - Edwards)



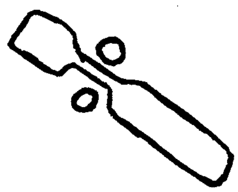
Caged Disc

Capetown Valves

Aortic



Mitral



Tilting Disc
(Bjork - Shiley)

Clinical problems with current prosthetic valves include:

(i) Formation of blood clots.

(ii) Destruction of blood cells.

(iii) Tissue ingrowth leading to interference with free valve motion.

(iv) Exercise intolerance.

These are partly fluid dynamical problems (materials considerations are also important):

(i) Clotting is related to regions of stasis and also (somewhat paradoxically) to wall damage produced by turbulence.

(ii) Cell damage is related to high rates of shear.

(iii) Tissue growth is partly controlled by flow lines.

(iv) Exercise intolerance is related to the elevated (square law) pressure drop across the valve during forward flow (see previous lecture) and to the backflow associated with the closure movements of the valve.

Many of these problems are reduced with valve transplants or with tissue valves constructed in place by the surgeon, but such valves do not always retain their flexibility over long periods of time. It is not yet clear whether improved treatment of transplanted tissues or improved design of prosthetic valves is the ultimate answer.

Formulation of the Problem*

The following formulation emphasizes <u>interaction</u> of valve and fluid. That is,

(1) Valve moves at local fluid velocity.

(2) Valve exerts forces on the fluid which modify the fluid motion. Such forces include the forces which stop the flow when the valve is closed and the forces which shear the flow to form vortices when the valve is open.

Because of (1) and (2), valve and fluid motion must be computed simultaneously, not independently.

Conditions at the valve-fluid interface:

<u>No slip</u>: $\frac{d\underline{x}}{dt} = \underline{u}(\underline{x})$

---

* Peskin, "Flow Patterns Around Heart Valves" J. Computational Physics <u>10</u>(2)252-272 (1972).

where $\underline{x}$ is the position of a material point of the valve leaflet and $\underline{u}$ is the fluid valocity field.

Force balance: Consider a material patch of the leaflet of mass m. Let $\underline{f}_1$ = force on the patch due to the rest of the leaflet, $\underline{f}_2$ = force on the patch due to the fluid. Then

$$m \frac{d^2\underline{x}}{dt^2} = \underline{f}_1 + \underline{f}_2 \quad .$$

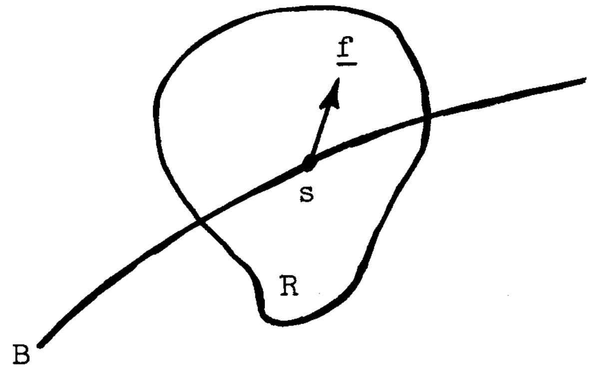
Note that

$\underline{f}_1$ depends on state of the tissue

$\underline{f}_2$ depends on state of the fluid.

When m = 0, we have $\underline{f}_1 = - \underline{f}_2$. In this situation, then, the material forces acting on the patch are "transmitted" directly to the fluid, and we do not need to evaluate the state of the fluid in the neighborhood of the patch to compute $\underline{f}_2$. In the following, we take m = 0 always, but our considerations will be applicable not only to infinitely thin zero-mass boundaries but also to neutrally buoyant immersed structures of finite thickness.

Boundary forces as a singular force-field: For simplicity consider a curved boundary in a two-dimensional fluid. Let s be a material parameter for the boundary curve $\underline{x}(s,t)$, and let $\underline{f}(s)ds$ be the force transmitted by an arc of length ds at s to the fluid. As shown above, $\underline{f}(s)$ is determined from the state of the boundary. Now consider the force on a region R of the fluid due to the presence of the boundary curve B.



The force is given by:

$$\int_{s\in(B\cap R)} ds\ \underline{f}(s) = \int_{s\in B} ds\ \underline{f}(s) \begin{Bmatrix} 1 & \underline{x}(s)\in R \\ & \\ 0 & \underline{x}(s)\notin R \end{Bmatrix}$$

$$\overset{*}{=} \int_{s\in B} ds\ \underline{f}(s) \int_{\underline{x}\in R} d\underline{x}\ \delta(\underline{x}-\underline{x}(s))$$

$$\overset{*}{=} \int_{\underline{x}\in R} d\underline{x} \left[\int_{s\in B} ds\ \underline{f}(s)\delta(\underline{x}-\underline{x}(s)) \right]$$

Since we are working in a two dimensional fluid $d\underline{x}$ is the area element in the fluid and $\delta(\underline{x}-\underline{x}(s))$ is a two dimensional impulse function centered at $\underline{x}(s)$.

The quantity

$$\underline{F}(\underline{x}) = \int_{s\in B} ds\ \underline{f}(s)\delta(\underline{x}-\underline{x}(s))$$

can be interpreted as the force density in the fluid due to the boundary. It is zero except on the boundary and infinite there

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* These manipulations are formal, but useful in motivating a numerical scheme.

(like a one-dimensional impulse), but its integral over any finite region of the fluid is finite.

Remark: Although $\underline{F}(\underline{x})$ is zero at points in the interior of the fluid. The *effects* of $\underline{F}(\underline{x})$ are propagated instantaneously throughout the incompressible fluid by the pressure field. The equation

$$\underline{F}(\underline{x}) = \int_{s\in B} ds\ \underline{f}(s)\,\delta(\underline{x}-\underline{x}(s))$$

can be thought of as a transformation from boundary to fluid variables. Similarly the no-slip condition can be re-written

$$\frac{d\underline{x}(s)}{dt} \int_{\underline{x}\in D} d\underline{x}\ \underline{u}(\underline{x})\,\delta(\underline{x}-\underline{x}(s))$$

where D = fluid domain.

The symmetry is not perfect, however, because the latter integral is over as many dimensions as the impulse function and this completely removes the singularity.

It is desireable to generalize the foregoing results in such a way that they do not depend on the choice of a particular coordinate system in the boundary or on the number of dimensions. In fact we would even like to include structures of finite thickness (like the muscular heart wall) as "boundaries". To accomplish this, proceed as follows.

Let $\{\underline{x}_k\}$ be a dense sequence of material sample points of the boundary. Specification of the points $\{\underline{x}_k\}$ is sufficient to determine the configuration in space of the boundary, the deformations of which are continuous. The force on a region R can be written

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{\underline{x}_k \in R \\ k \leq N}} \underline{f}_k = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \underline{f}_k \begin{Bmatrix} 1 & \underline{x}_k \in R \\ \\ 0 & \underline{x}_k \notin R \end{Bmatrix}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \underline{f}_k \int_R d\underline{x}\ \delta(\underline{x}-\underline{x}_k)$$

$$= \int_R d\underline{x} \left[\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \underline{f}_k\ \delta(\underline{x}-\underline{x}_k) \right]$$

which shows that

$$\underline{F}(\underline{x}) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \underline{f}_k\ \delta(\underline{x}-\underline{x}_k)$$

is the force-density in the fluid.

The interpretation of $\underline{f}_k$ can be given most clearly by writing

$$\mu(R) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \begin{Bmatrix} 1 & \underline{x}_k \in R \\ \\ 0 & \underline{x}_k \notin R \end{Bmatrix}$$

= fraction of the sample points which lies in the region R.

$\mu(R)$ forms a measure on the fluid domain*, and $\underline{f}$ is the intensity of the boundary force with respect to this measure.

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* $\mu(R) = 0$ when R does not contain any of the boundary!

Now that we have an expression for the force-field applied by the boundary to the fluid, we are in a position to proceed to a system of equations of motion.

Equation of motion:

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \eta\nabla^2\underline{u} + \underline{F} \tag{1}$$

$$\nabla\cdot\underline{u} = 0 \tag{2}$$

$$\underline{F}(\underline{x}) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\underline{f}_k\,\delta(\underline{x}-\underline{x}_k) \tag{3}$$

$$\frac{d\underline{x}_k}{dt} = \int_{\text{fluid}} dx\;\underline{u}(\underline{x})\,\delta(\underline{x}-\underline{x}_k) \tag{4}$$

$$\underline{f}_k = \underline{f}_k\,(\underline{x}_1,\,\underline{x}_2,\,\ldots\,) \tag{5}$$

(1) and (2) are fluid equations

(3) and (4) couple boundary and fluid

(5) expresses the material properties of the boundary. For active boundaries the relations (5) will vary with time.

Construction of a Numerical Scheme for the Navier-Stokes Equations:*

Motivation:

The Heat Equation in one Dimension (periodic domain)

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}$$

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* A.J. Chorin "On the Convergence of Discrete Approximations to the Navier-Stokes Equations", Mathematics of Computation 23: 341-353 (1969).

Discretization: Let $x = k\,\delta x$

$$t = n\,\delta t$$

$$u(x,t) = u_k^n$$

$$\frac{\partial^2 u}{\partial x^2} \to \frac{1}{(\delta x)^2}(u_{k+1} + u_{k-1} - 2u_k)$$

$$\frac{\partial u}{\partial t} \to \frac{u^{n+1} - u^n}{\delta t}$$

Then we can write down a numerical scheme of the following form

$$u_k^{n+1} = u_k^n + \frac{\nu\delta t}{(\delta x)^2}(u_{k+1}^* + u_{k-1}^* - 2u_k^*)$$

where

$$* = n \to \text{explicit scheme}$$

$$* = n+1 \to \text{implicit scheme} \quad .$$

Let $$\alpha = \frac{\nu\delta t}{(\delta x)^2}$$

$$(Au)_k = u_{k+1} + u_{k-1} - 2u_k \ .$$

Then

$$u^{n+1} = u^n + \alpha A u^* \quad .$$

Then we have

Explicit: $u^{n+1} = (I + \alpha A)u^n$

Implicit: $u^{n+1} = (I - \alpha A)^{-1}u^n$

The explicit scheme involves direct calculation while the implicit scheme involves solving a system of linear equations at each time step.

Since arbitrary u_k can be expanded in eigenfunctions of A, we study the behavior of these two schemes for such functions. We can do this without actually writing down the eigenfunctions.

Simply observe that $Au = \lambda u$ implies

$$u_{k+1} + u_{k-1} - 2u_k = \lambda u_k$$

$$u_{k+1} + u_{k-1} = (2 + \lambda)u_k \quad .$$

Multiply by u_k, sum over k, and note that $\sum u_k u_{k+1} = \sum u_k u_{k-1}$ (periodic domain). Then

$$2 \sum u_k u_{k+1} = (2 + \lambda) \sum u_k^2$$

$$1 + \frac{\lambda}{2} = \frac{\sum u_k u_{k+1}}{\sum u_k^2} \in [-1, 1]$$

$$-4 \leq \lambda \leq 0 \quad .$$

Note that on a periodic domain with an even number of points the values $\lambda = 0$ and $\lambda = -4$ are actually achieved. The corresponding eigenfunctions are $u_k = 1 \rightarrow \lambda = 0$, and

$$u_k = \left\{ \begin{array}{rl} 1 & k \text{ even} \\ -1 & k \text{ odd} \end{array} \right\} \rightarrow \lambda = -4 \quad .$$

For eigenfunctions u_λ the schemes reduce to

$$\text{Explicit:} \quad u_\lambda^{n+1} = (1 + \alpha\lambda) u_\lambda^n$$

$$\text{Implicit:} \quad u_\lambda^{n+1} = \frac{1}{1-\alpha\lambda} u_\lambda^n \quad .$$

Since $\lambda < 0$ the second scheme results in damping of all eigenfunctions independent of α, while the 1^{st} scheme yields growing oscillations of <u>at least</u> the $\lambda = -4$ eigenfunction unless $\alpha < \frac{1}{2}$.

<u>Heat Equation in one Dimension with Convection by a Given Flow v</u>

$$\frac{\partial u}{\partial t} + v \frac{\partial}{\partial x} u = \nu \frac{\partial^2 u}{\partial x^2}$$

The difference operator we require now is $Q(v)$

$$(Q(v)u)_k = \frac{v_k}{2\delta x} (u_{k+1} - u_{k-1}) - \frac{\nu}{(\delta x)^2} (u_{k+1} + u_{k-1} - 2u_k)$$

Let

$$A = - \frac{(\delta x)^2}{\nu} Q(v)$$

Then

$$A = (1-R_k)u_{k+1} + (1+R_k)u_{k-1} - 2u_k$$

where

$$R_k = \frac{v_k \delta x}{2\nu} \quad = \text{local Reynolds number based on mesh width.}$$

With this notation the implicit scheme is, exactly as before

$$(I - \alpha A) \quad u^{n+1} = u^n$$

or

$$[I + \delta t \; Q(v)] \quad u^{n+1} = u^n \quad .$$

This system becomes ill-behaved unless we set an upper bound on $|R_k|$. To see clearly what goes wrong consider the special case $R_k = R$ and $u^n = \begin{cases} 1 & k = 0 \\ 0 & k = 1,2,\ldots,N-1 \end{cases}$.

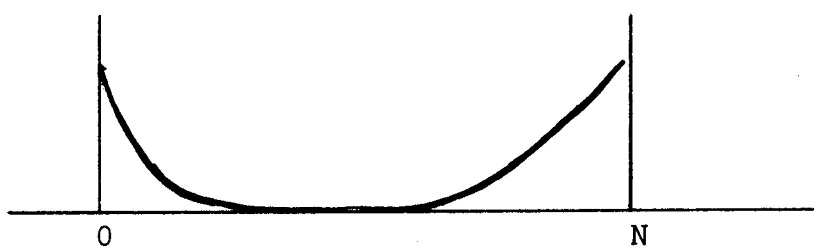
Recalling that the domain is periodic, we seek a solution of the form

$$u_k^{n+1} = c[e^{-\beta_1 k} + re^{-\beta_2(N-k)}]$$

where c will be chosen to satisfy the equation at $k = 0$, r will be chosen to make $u_o = u_N$, and where $e^{-\beta_1 k}$ and $e^{+\beta_2 k}$ will be solutions of the homogeneous equations at $k = 1,2,\ldots,N-1$ with $\beta_1, \beta_2 > 0$. The equation for r is

$$1 + re^{-\beta_2 N} = e^{-\beta_1 N} + r$$

$$r = \frac{1-e^{-\beta_1 N}}{1-e^{-\beta_2 N}} \cong 1$$



The equations for β_1, β_2 are obtained by substituting $u = e^{\beta k}$ in the homogeneous equation

$$\alpha(1-R)e^{\beta} + \alpha(1+R)e^{-\beta} - (2\alpha+1) = 0 \quad .$$

This has the form $ae^{\beta} + be^{-\beta} = c$, $\quad a = \alpha(1-R)$

$\quad b = \alpha(1+R)$

with

$$c > a + b \quad .$$

If $a > 0$ and $b > 0$ there are two roots, one negative and one positive, so we can construct the solution shown on previous page. But if a or b is negative there is only one such real root, because the function $ae^{\beta} + be^{-\beta}$ becomes monotonic. Thus we require $|R| < 1$ or

$$\left| \frac{v_k \delta x}{2\nu} \right| < 1$$

where R = Reynolds number based on mesh width.

To understand the meaning of this inequality for heart calculations, set $v = \frac{L}{T}$ and $x = hL$ where h is the non-dimensional mesh width. Then $\left(\frac{L^2}{\nu T}\right) h < 2$. We have seen that $L^2/\nu T$ varies widely over mammalian hearts, but is large for the human heart. Thus it will be easier to do the calculation for small mammals. If we call $\frac{(\nu T)^{1/2}}{L}$ the relative boundary layer thickness, then we have

$$h < 2 \text{ (relative boundary layer thickness)}^2 \quad .$$

<u>Fractional Step Method for Several Space Directions</u>.

Define a difference operator like Q(v) (above) for each space direction. Then (e.g. in two dimensions)

$$Q_x(v_x) \to v_x \frac{\partial}{\partial x} - \nu \frac{\partial^2}{\partial x^2}$$

$$Q_y(v_y) \to v_y \frac{\partial}{\partial y} - \nu \frac{\partial^2}{\partial y^2}$$

and

$$Q_x(v_x) + Q_y(v_y) \to v\cdot\nabla - \nu\nabla^2 \quad .$$

One would like to solve

$$(I + \delta t[Q_x(v_x) + Q_y(v_y)])u^{n+1} = u^n$$

but this is costly since differences in two space directions are involved. Alternatively one uses the following (fractional step) method:

$$\left.\begin{aligned} (I + \delta t\ Q_x(v_x))u^* &= u^n \\ (I + \delta t\ Q_y(v_y))u^{n+1} &= u^* \end{aligned}\right]$$

Substituting we find that

$$[I + \delta t\ Q_x(v_x)][I + \delta t\ Q_y(v_y)]u^{n+1} = u^n \quad .$$

This differs from the operator on the previous page by terms of order $(\delta t)^2$, provided that u is smooth so that $\frac{1}{(\delta x)^2}(u_{k+1} + u_{k-1} - 2u_k)$, and similarly $\frac{1}{2\delta x}(u_{k+1} - u_{k-1})$, remain bounded as δx, $\delta t \to 0$.

Navier-Stokes Equations*

In the foregoing, identify the flow v with the unknown u, which now becomes a vector $\underline{u}$. To keep the equations linear in each time step use $\underline{v} = \underline{u}^n$. If there is a known external force $\underline{F}$,

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* A.J. Chorin - see ref. page 88 .

it can be added on the right hand side. It remains to include the effects of pressure. This is done as an additional fractional step, and the pressure field is chosen to make $\underline{u}^{n+1}$ satisfy the equation of continuity. Thus

$$(I + \delta t\ Q_x(u_x^n))\underline{u}^* = \underline{u}^n + \delta t\ \underline{F}^n$$

$$(I + \delta t\ Q_y(u_y^n))\underline{u}^{**} = \underline{u}^*$$

$$\underline{u}^{n+1} = \underline{u}^{**} - \delta t\ G\ p$$

where p is chosen such that

$$D\underline{u}^{n+1} = 0$$

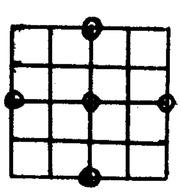
and where

$$G\ p = \frac{1}{2(\delta x)}\left([p_{i+1,j} - p_{i-1,j}],\ [p_{i,j+1} - p_{i,j-1}]\right)$$

$$D\ \underline{u} = \frac{1}{2(\delta x)}\left({}_xu_{i+1,j} - {}_xu_{i-1,j} + {}_yu_{i,j+1} - {}_yu_{i,j-1}\right) \quad .$$

Thus p must satisfy $0 = (D\underline{u}^{**}) - \delta t\ DGp$ where $D\underline{u}^{**}$ is a known function and

$$DGp = \frac{1}{4(\delta x)^2}(p_{i+2,j} + p_{i-2,j} + p_{i,j+2} + p_{i,j-2} - 4p_{ij})$$



so that DG is a discrete form of the Laplace operator.

Numerical Analysis of Blood Flow in the Heart: Boundary Representation and Numerical Stability

In the previous lecture, the problem was formulated as

$$\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \eta\nabla^2\underline{u} + \underline{F} \qquad (1)$$

$$\nabla\cdot\underline{u} = 0 \qquad (2)$$

$$\underline{F}(\underline{x}) = \lim_{N\to\infty}\sum_{k=1}^{N} \underline{f}_k\ \delta(\underline{x}-\underline{x}_k) \qquad (3)$$

$$\frac{d\underline{x}_k}{dt} = \int_{\text{fluid}} d\underline{x}\ \underline{u}(\underline{x})\delta(\underline{x}-\underline{x}_k) \qquad (4)$$

$$\underline{f}_k = \underline{f}_k\ (\underline{x}_1, \underline{x}_2, \ldots) \qquad (5)$$

A numerical scheme due to A.J. Chorin was discussed which can be used to solve Eqs. (1)-(2) (the fluid part of the problem) if the force-field $\underline{F}$ is regarded as known. Here we discuss the new features that are needed when equations (3), (4), (5) are included. This work is due to the present author, and is partly discussed in * but further progress since that paper will be included here.

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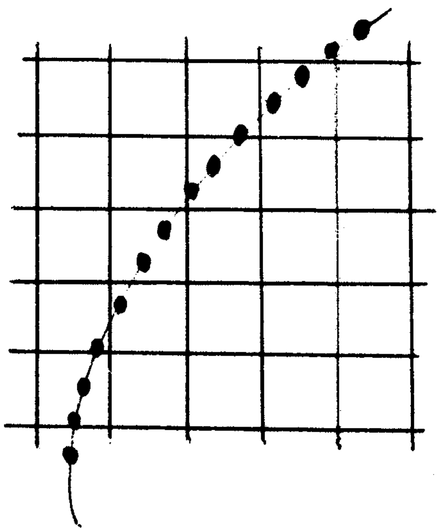
* Peskin, "Flow Patterns Around Heart Valves", J. Computational Physics 10(2)252-271 (1972).

Connecting the Representations of Boundary and Fluid:

Fluid quantities are stored on fixed mesh points (ij), while boundary quantities are stored at the moveable sample points k of the boundary. The boundary points need not coincide with fluid mesh points. The two representations need to be connected in order that:



(i) Forces computed from the state of the boundary can be applied to the fluid.

(ii) The boundary can be moved at the local fluid velocity.

Equations (3) and (4) give a hint how this is to be done. We must define a function $D_{ij}(\underline{x}_k)$ which will correspond to the two-dimensional* impulse $\delta(\underline{x} - \underline{x}_k)$. With such a function one could write instead of (3) and (4)

$$\underline{F}_{ij} = \frac{1}{N} \sum_{k=1}^{N} \underline{f}_k \; D_{ij}(\underline{x}_k) \tag{3'}$$

$$\underline{x}_k^{n+1} = \underline{x}_k^n + \delta t \sum_{ij} (\delta x)^2 \underline{u}_{ij} \; D_{ij}(\underline{x}_k^n) \tag{4'}$$

---

* Assuming here a two dimensional flow. Generalization to three dimensions is direct.

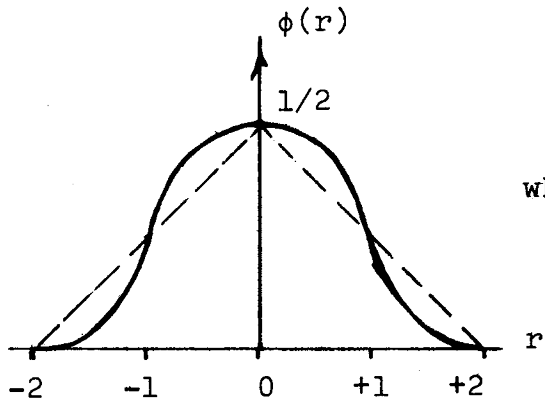
In the foregoing,

$$\delta t = \text{time step}$$

$$\delta x = \text{mesh width}$$

The function D_{ij} is constructed as follows: Let

$$\phi(r) = \begin{cases} \frac{1}{4}\left(1 + \cos\frac{\pi r}{2}\right) & |r| \leq 2 \\ 0 & |r| > 2 \end{cases}$$



Then let:

$$D_{ij}(\underline{x}) = h^{-2}\phi(x-i)\phi(y-j)$$

where

$$\underline{x} = (xh, yh)$$

and

$$h = \delta x$$

The function ϕ has the following properties (from which follow similar properties for D):

(i) $\int \phi(r)dr = 1$

(ii) $\phi(r) \geq 0$, and $\phi(r) = 0$ for $|r| \geq 2$

(iii) For all r

$$\sum_{k \text{ even}} \phi(r-k) = \sum_{k \text{ odd}} \phi(r-k) = \frac{1}{2}$$

$$\rightarrow \sum_{k} \phi(r-k) = 1$$

(iv) For all r

$$\sum_k \phi^2(r-k) = \frac{3}{8}$$

and therefore, for all r, s

(v)
$$\sum_k \phi^2(r-k) \geq \sum_k \phi(r-k)\phi(s-k)$$

The essential conditions in the foregoing are $\phi = 0$ for $|r| > 2$ and $\sum_k \phi(r-k) = 1$. From these alone it follows that equations (3') and (4') (page 97) can be interpreted as a local interpolation of the velocity field from the mesh to the boundary and a local distribution or spreading of the boundary forces onto the mesh, with conservation of the total force.

The other properties influence the smoothness of the results. In particular:

The stronger condition

$$\sum_{k\ \text{even}} \phi(r-k) = \sum_{k\ \text{odd}} \phi(r-k) = \frac{1}{2}$$

is imposed because the form of the Laplacian that appears in the equation for the pressure in Chorin's scheme couples only mesh points with the same parity (even or odd) in each coordinate. The mesh is therefore partitioned in a natural way into four "chains": (i even, j even), (i even, j odd), (i odd, j even), (i odd, j odd). Our condition guarantees that any given boundary point is coupled equally to all four chains.

The condition

$$\text{constant} = \sum_k \phi^2(r-k) \geq \sum_k \phi(r-k)\phi(s-k)$$

is motivated by considering the coupling that is indirectly introduced between two boundary points at r and s by coupling each of these to the mesh. One wants this to be maximal when r = s, and one wants this maximum to be independent of r. To see more clearly what is involved here, consider a one-dimensional periodic domain and an expression of the form

$$\begin{aligned}\tilde{K}(r,s) &= \sum_m \sum_n \phi(r-m)\phi(s-n)K(m-n) \\ &= \sum_p K(p) \sum_m \phi(r-m)\phi(s+p-m)\end{aligned}$$

where

$$p = m-n \quad .$$

Because of condition (v) we can assert that the coefficient of K(p) is maximal when r-s = p, independent of where r and s fall separately with respect to the mesh. To this extent, then, condition (v) makes the mesh "disappear". To see how one might go further in this direction, consider what would happen if

$$\text{(vi)} \qquad \sum_m \phi(r-m)\phi(s-m) = f(r-s) \quad .$$

Then we would have

$$\tilde{K}(r,s) = \sum_p K(p)f(r-s-p) \quad .$$

The latter expression depends only on r-s just as K depends only on m-n. Unfortunately, however, a condition like (vi) cannot be imposed when $\phi = 0$ outside a finite interval. To see why, pick $r-s = 4-\varepsilon$ where $0 < \varepsilon < 1$. Then by changing r, but holding r-s constant one can arrange that either one or zero mesh points m fall in the region where $\phi(r-m)\phi(s-m) > 0$. Thus $\phi(r-m)\phi(s-m)$ cannot be constant, although r-s is constant.

Boundary Forces - The Link Formalism

Here we make the form of the functions $\underline{f}_k(\underline{x}_1, \underline{x}_2, \ldots)$ more explicit by assuming that (when N is fixed) we can regard the forces as arising in straight line segments which connect specified pairs of boundary points.

Suppose that the link AB connects points A and B and let

$$\underline{x}_{AB} = \underline{x}_B - \underline{x}_A$$

$$R_{AB} = |\underline{x}_{AB}|$$

$$T_{AB}(R_{AB}) = \text{Tension in link AB}$$

$$\hat{a}_{AB} = \underline{x}_{AB}/R_{AB}$$

Then -

$$N^{-1}\underline{f}_A = \sum_{B=1}^{N} T_{AB}\,\hat{a}_{AB}$$

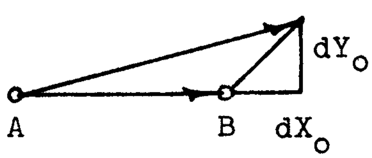
= force transmitted to the fluid by point A in the N-point representation.

We shall also need an expression for the changes in $\underline{f}_A$ that are produced by a perturbation of the points $\underline{x}_B$ $B = 1, 2, \dots, N$. These will be given by equations of the form

$$d\underline{f}_A = \sum_{B \neq A} Q_{AB} d\underline{x}_{AB} = \sum_{B \neq A} Q_{AB}(d\underline{x}_B - d\underline{x}_A)$$

$$= \Big(- \sum_{B \neq A} Q_{AB}\Big) d\underline{x}_A + \sum_{B \neq A} Q_{AB} d\underline{x}_B$$

where each Q_{AB}, $A \neq B$ is a 2×2 matrix which refers to the link AB. A specific formula for Q_{AB} can be derived by considering a rotation to a system of coordinates in which the x-axis is parallel to the link AB. In such a system (referred to as $^\circ$)



$$N^{-1} d\underline{f}_o = \begin{pmatrix} T' \, dX_o \\ T \, \dfrac{dY_o}{R} \end{pmatrix} = \begin{pmatrix} T' & 0 \\ 0 & T/R \end{pmatrix} \begin{pmatrix} dX_o \\ dY_o \end{pmatrix}$$

where $T' = dT/dR$.

Therefore in the general case

$$Q = N\tilde{S} \begin{pmatrix} T' & 0 \\ 0 & T/R \end{pmatrix} S$$

where S is a rotation between a general system of coordinates and the special system parallel to the link, and where $\tilde{S}S = I$.

Doing this for each link we have

$$Q_{AB} = N\tilde{S}_{AB} \begin{Bmatrix} T'_{AB} & 0 \\ 0 & T_{AB}/R_{AB} \end{Bmatrix} S_{AB}$$

and the matrices Q_{AB} satisfy the following

(i) Q_{AB} **is** positive definite if $T'_{AB} > 0$ and $T_{AB} > 0$

(ii) Q_{AB} is symmetric (<u>within</u> each 2×2 matrix)

(iii) $Q_{AB} = Q_{BA}$

(iv) $Q_{AB} = 0$ if $T_{AB} \equiv 0$, i.e., if there is no link between A and B.

Moreover, we have the following theorem on the stability of link structures. If $T_{AB} \geq 0$ and $T'_{AB} \geq 0$, then $\sum_A d\underline{x}_A \cdot d\underline{f}_A \leq 0$.

Proof: Let

$$(dx,\ df) = \sum_A d\underline{x}_A \cdot d\underline{f}_A \quad .$$

Then

$$(dx,\ df) = \sum_{A \neq B}\sum d\underline{x}_A \cdot Q_{AB}(d\underline{x}_B - d\underline{x}_A)$$

$$(dx,\ df) = \sum_{A \neq B}\sum d\underline{x}_B \cdot Q_{AB}(d\underline{x}_A - d\underline{x}_B)$$

$$(dx,\ df) = -\frac{1}{2}\sum_{A \neq B}\sum (d\underline{x}_{AB} \cdot Q_{AB} \cdot d\underline{x}_{AB})$$

But $d\underline{x}_{AB} \cdot Q_{AB}\ d\underline{x}_{AB} \geq 0$ so that $(dx,\ df) \leq 0$.

<u>Remark</u>: The hypotheses of the foregoing theorem exclude link structures under compression, which may indeed be unstable as in the case of a straight chain of links which would tend to

buckle under compression.* When compression is allowed, care must be exercised in the placement of the links so that such instability is avoided.

Remark #2: When implementing the foregoing on a computer it is not convenient to store all N^2 possible links. Instead only the links for which $T \neq 0$ are stored. Each link is given an index ℓ, and the end points of the links are stored in a link table:

$$\left.\begin{array}{l} k_A(\ell) \\ k_B(\ell) \end{array}\right\} = \text{indices of the points joined by link } \ell.$$

Thus we store in the computer the graph of the boundary, and this graph is unchanged as the boundary points move about in space. In this graph there is an edge between A and B when $Q_{AB} \neq 0$. Graphical considerations will be important in the efficient solution of the equations which yield numerical stability - see below.

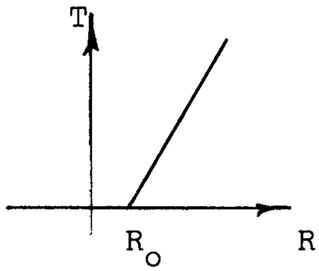
Properties of the Links:

The physical properties of the materials which bound the fluid enter into the mathematical formulation through the functions $T(R)$ which describe the links. Three types of links are

* As this example shows, such instability is physical, not numerical!

needed for the representation of the following three material types that arise:

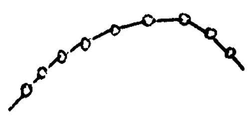
(i) Passive flexible leaflet (natural valve)



$$T = \begin{cases} K(R-R_o) & R \geq R_o \\ 0 & R < R_o \end{cases}$$

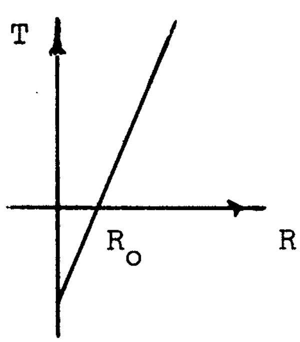
where K = Stiffness

R_o = Unstretched length



The links are connected as a single fiber, and there is no resistance to bending.

(ii) Prosthetic valve

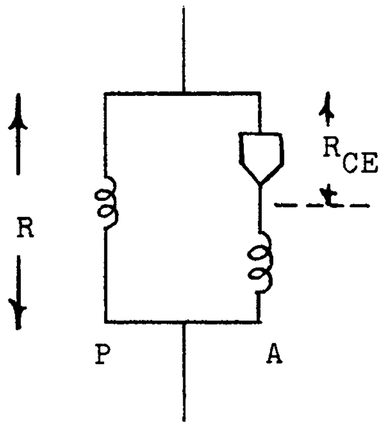


$$T = K(R-R_o) \qquad R > 0$$

The link resists compression as well as extension. Resistance to bending comes from the arrangement of the links. For example, to represent the cross-section of a disc valve we have used a "railroad bridge" structure.

(iii) Heart muscle

The model of heart muscle mechanics used in this work is based on *. Later in the course we shall examine the question of muscle mechanics in detail, considering both the experimental evidence for this model (which is a phenomenological model only) and also the possibility of introducing a model more closely related to the molecular mechanisms. This discussion will be limited to an uncritical presentation of the phenomenological model.



The model consists of 3 elements. Two springs (possibly non-linear) and a "contractile element".

Let

T_P, T_A = Passive and Active tensions.

Then

$$T = T_P(R) + T_A(R-R_{CE})$$

where the functions $T_P(R)$, $T_A(R-R_{CE})$ describe the properties of the two springs. The contractile element is described by an equation of motion of the form

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* Braumwald, Ross, Sonnenblick, Mechanisms of Contraction of the Normal and Failing Heart. Boston: Little Brown, 1968.

$$-\frac{dR_{CE}}{dt} = v(T_A, R, \alpha)$$

where α is a parameter that determines the state of activation of the muscle. These two equations plus a third equation describing the load suffice to determine the three unknowns T, R, R_{CE} if α is given. It is reasonable to take α as a given function of time since stimulation of the muscle.

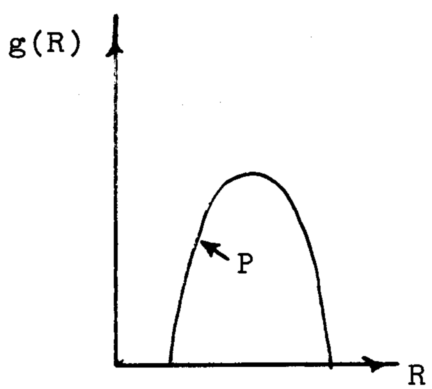
The function v has been well-characterized in skeletal muscle, where it is possible to stimulate the muscle continually and hold α constant. The form of v is $(v > 0)$

$$(v+v_o)(\tau+\tau_o) = v_o[\tau_o + \alpha(t)]$$

or

$$v = v_o \frac{\alpha(t)-\tau}{\tau_o+\tau}$$

where τ is given by $T_A = g(R)\tau$

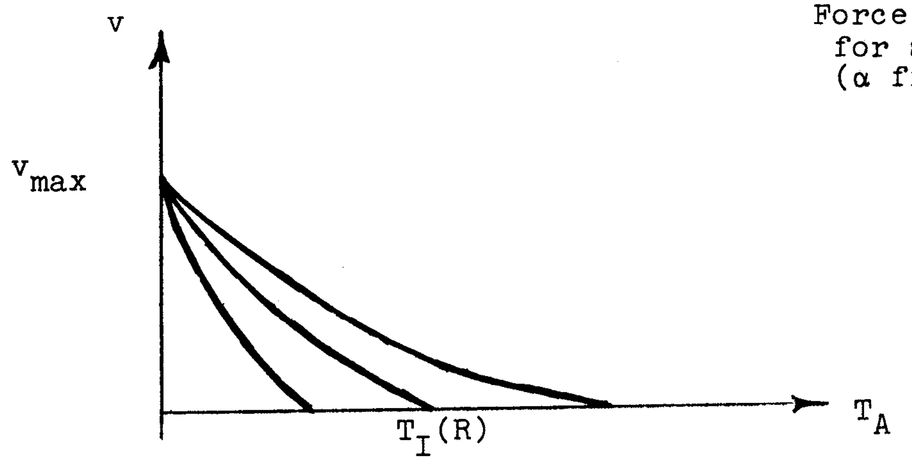


and where g(R) is a function like this.

(P marks the normal operating point of heart muscle: $g' > 0$)

Substituting for τ we have

$$v = v_o \frac{g(R)\alpha(t) - T_A}{g(R)\tau_o + T_A}$$



Force - velocity curve for several values of R. (α fixed)

When $T_A = 0$, $v = v_{max} = \frac{\alpha(t)}{\tau_0}$ independent of R. When $v = 0$, $T_A = T_I(R) = \alpha(t)g(R)$, the "isometric tension". Thus the form of g(R) is precisely the isometric tension as a function of length.

<u>Remark</u>: When $T_A >> T_I(R)$ the equation for v no longer agrees with experiment. (In particular, when $\alpha = 0$ the observation is $T_A \equiv 0$.) For $T_A >> T_I(R)$ the muscle is more easily stretched than the foregoing equation for v would indicate. In fact it appears to "yield" at $T \gtrsim 2T_I$.

We are now in a position to state how this model of muscle mechanics can be incorporated into our link formalism. Between each pair of boundary points which are to be connected by heart muscle we introduce <u>two</u> links corresponding to the two arms of the model. The first is a passive link like that used for natural valve. In the second we use the same type of link, but identify R_0 with R_{CE}, the length of the contractile element. This means that R_o changes with time, according to the equations that we have developed above. (Note that dR_o/dt depends on the tension T_A at any instant, and hence on what the fluid is doing to the ends of the link).

Remark: At present we have only used thin muscular walls for the heart chambers, but nothing in the present formulation excludes thick walls. In fact, the heart muscle is neutrally buoyant in blood, and its contractions are volume preserving (the muscle gets thicker as it shortens). Both of these features are automatically preserved by the present method, in which the muscle is regarded as a special region of the (incompressible) fluid where extra forces are applied.

Numerical Stability:

In the development of Chorin's scheme (previous lecture) we have seen that numerical stability was secured by using at each time step a sequence of implicit fractional steps. In the present case the force field F which appears in the fluid equations is not known, but depends on the boundary configuration. The use of the boundary configuration x^n to calculate F for the time step $n \to n+1$ leads to numerical instability, but if we precede each time step of Chorin's scheme with an implicit fractional step for the boundary forces, numerical stability can be secured.

Thus we write

$$\underline{u}^{*} = \underline{u}^{n} + \delta t\ F(\underline{x}^{*})$$

$$[I + \delta t Q_x(u_x)]\underline{u}^{**} = u^{*}$$

$$[I + \delta t Q_y(u_y)]\underline{u}^{***} = \underline{u}^{**}$$

$$\underline{u}^{n+1} = \underline{u}^{***} - \delta t G p$$

where x^{*} is the configuration

$$\underline{x}^{*} = \underline{x}^{n} + \delta t\ \underline{u}^{*} .$$

Solving the pair of equations

$$\underline{u}^{*} = \underline{u}^{n} + \delta t\ \underline{F}(\underline{x}^{*})$$

$$\underline{x}^{*} = \underline{x}^{n} + \delta t\ \underline{u}^{*}$$

constitutes the first fractional step.

Written out explicitly this pair of equations is

$$\underline{u}_{ij}^{*} = \underline{u}_{ij}^{n} + \delta t\ \frac{1}{N} \sum_{k} D_{ij}(\underline{x}_k^n)\underline{f}_k(\underline{x}_1^{*} \ldots \underline{x}_N^{*})$$

$$\underline{x}_k^{*} = \underline{x}_k^{n} + \delta t \sum_{ij} (\delta x)^2\ D_{ij}(\underline{x}_k^n)\underline{u}_{ij}^{*} .$$

Eliminating $\underline{u}_{ij}^{*}$ we have

$$\underline{x}_k^{*} = \underline{x}_k^{o} + \frac{(\delta t)^2(\delta x)^2}{N} \sum_{\ell} \sum_{ij} D_{ij}(\underline{x}_k^n) D_{ij}(\underline{x}_\ell^n)\underline{f}_\ell(\underline{x}_1^{*} \ldots \underline{x}_N^{*})$$

where

$$\underline{x}_k^{o} = \underline{x}_k^{n} + \delta t \sum_{ij} (\delta x)^2 D_{ij}(\underline{x}_k^n)\underline{u}_{ij}^{n}$$

= known quantity at beginning of time step.

Let

$$K_{k\ell} = \sum_{ij} D_{ij}(\underline{x}_k^n) D_{ij}(\underline{x}_\ell^n) \quad .$$

Because of the manner in which D_{ij} was constructed, we have the following:

$$K_{k\ell} \leq \frac{9}{64} (\delta x)^{-4}$$

with equality holding when $k = \ell$, and

$$K_{k\ell} = 0 \text{ when } \max_{p=1,2} \left| x_{kp}^n - x_{\ell p}^n \right| \geq 4(\delta x) \quad .$$

As an approximation, then, we make the replacement

$$K_{k\ell} \rightarrow \delta_{k\ell} \frac{9}{64} (\delta x)^{-4} \beta$$

where β is a parameter of order 1.

This replacement eliminates all reference to the fluid mesh (for the first fractional step) and we get the system

$$\underline{x}_k^* = \underline{x}_k^o + \lambda \underline{f}_k(\underline{x}_1^* \ldots \underline{x}_N^*)$$

where

$$\lambda = \frac{9}{64} \frac{(\delta t)^2}{N(\delta x)^2} \beta \quad .$$

This is a non-linear fixed point problem on the boundary.

The solution of this problem can be accomplished by Newton's method. Let $\underline{\xi}_k^m$ be the m^{th} guess for $\underline{x}_k^*$. Then $\underline{\xi}_k^{m+1}$ is found by solving the linear system of equations which results when the functions $\underline{f}_k$ are expanded about $\underline{\xi}_k^m$. Dropping the subscripts, we have, to first order

$$f(\xi^{m+1}) = f(\xi^m) + \frac{df}{dx}\Big|^m (\xi^{m+1} - \xi^m)$$

and

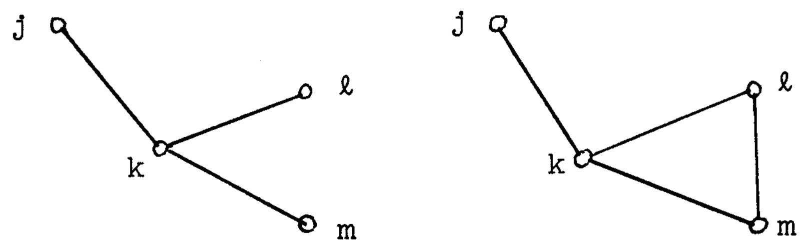
$$\xi^{m+1} = x^o + \lambda f(\xi^m) + \lambda \frac{df}{dx}\Big|^m (\xi^{m+1} - \xi^m)$$

or

$$\left\{I - \lambda \frac{df}{dx}\Big|^m\right\}(\xi^{m+1} - \xi^m) = [x^o + \lambda f(\xi^m) - \xi^m] \quad .$$

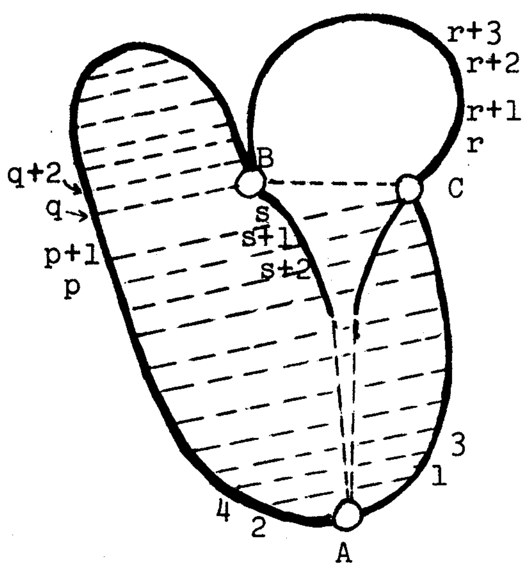
In the foregoing $\frac{df}{dx}\Big|^m$ is a $2N\times 2N$ matrix with components of the form $\partial f_{kp}/\partial x_{\ell q}$ $p, q = 1, 2$. Fixing k and ℓ, $k \neq \ell$ we get the 2×2 matrix $Q_{k\ell}$ which was evaluated under Link Formalism (above). From the properties derived there we have the following: That matrix $(I - \lambda \frac{df}{dx})$ is symmetric, and it is positive definite if the link structure is stable. Moreover it is a sparse matrix, whose graph is precisely the link structure itself. These considerations suggest solving the equation for ξ^{m+1} by the Cholesky factorization.* Moreover, to preserve sparsity as much as possible we seek a numbering of the unknowns in which, except for a small number of special boundary points, points k and ℓ are linked only if $|k-\ell|$ is small. The special points (a "separation set", see Read, Graph Theory and Computing) are then numbered last. The importance of this is that during the factorization, when a node k is eliminated, all of the neighbors of k whose subscripts are greater than k become coupled to each other.

* See Isaacson and Keller, Analysis of Numerical Methods, New York. Wiley 1966.



$j < k < \ell < m$

It follows that if all of the links $(k\ell)$ $k < \ell$ have $|k-\ell| \leq b$ or $\ell \in S$, the separation set, then this property is retained during the factorization provided that the points of S are numbered last. An appropriate numbering for a heart valve calculation is indicated here:



$S = \{A, B, C\}$

$|k-\ell| \leq 2$ or $\ell \in S$

Summary of the Numerical Scheme

(1) Solve:

$$\underline{x}_k^* = \underline{x}_k^o + \lambda \underline{f}_k(\underline{x}_1^* \ldots \underline{x}_N^*)$$

(2) Let

$$\underline{F}_{ij}^* = \frac{1}{N} \sum_{k=1}^{N} D_{ij}(\underline{x}_k^n) \underline{f}_k(\underline{x}_1^* \ldots \underline{x}_N^*)$$

$$\underline{u}_{ij}^* = \underline{u}_{ij}^n + \delta t \underline{F}_{ij}^*$$

(3) Solve:

$$[I + \delta t Q_x(u_x^n)]\underline{u}^{**} = \underline{u}^*$$

$$[I + \delta t Q_y(u_y^n)]\underline{u}^{***} = \underline{u}^{**}$$

$$\underline{u}^{n+1} = \underline{u}^{***} - \delta t G p$$

$$D\underline{u}^{n+1} = 0$$

(4) Evaluate:

$$\underline{x}_k^{n+1} = \underline{x}_k^n + \delta t \sum_{ij} (\delta x)^2 D_{ij}(\underline{x}_k^n) \underline{u}_{ij}^{n+1}$$

This completes the time step.

Heart Sounds and Murmurs

Observations:

(1) Valve closure is heard as a sound.

(2) Valve opening is normally silent.

(3) Murmurs are heard when flow is driven through an abnormally narrowed opening. The time variation of the amplitude of the murmur reflects (in a qualitative way) the amplitude of the pressure drop or flow across the narrowed valve.

(4) Murmurs are heard best in the circulation itself just downstream of the obstruction or at a point on the body surface near that part of the circulation which is downstream of the obstruction. They are heard poorly within the circulation at other points (e.g. upstream of the obstruction).*

Theory of Sound

In the following we shall consider the nature of sound in an almost incompressible fluid. Begin with the Navier-Stokes equations for the compressible case supplemented by an equation of state for the medium of the form $\rho = \rho(p)$.

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* Yellin, E.L. "Hydraulic Noise in Submerged and Bounded Liquid Jets" in Biomedical Fluid Mechanics Symposium ASME pp. 209-221.

Neglecting

(1) non-linear terms

(2) dissipation

$$\left.\begin{aligned} \rho \frac{\partial \underline{u}}{\partial t} &= -\nabla p \\ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) &= 0 \\ \rho &= \rho(p) \end{aligned}\right]$$

If

$$\rho \cong \rho_o$$

Then

$$\left.\begin{aligned} \rho_o \frac{\partial \underline{u}}{\partial t} &= -\nabla p \\ \rho_o' \frac{\partial p}{\partial t} + \rho_o \nabla\cdot\underline{u} &= 0 \end{aligned}\right]$$

Let

$$\underline{u} = \nabla\phi$$

then

$$p = -\rho_o \frac{\partial \phi}{\partial t} \quad .$$

And

$$-\rho_o'\rho_o \frac{\partial^2\phi}{\partial t^2} + \rho_o\nabla^2\phi = 0$$

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0 \quad \underline{\text{The wave equation}}$$

$$c = \text{sound speed} = \left[\frac{d\rho}{dp}\bigg|_o\right]^{-1/2}$$

Consider a solution of this of the form

$$\phi = \frac{-f(t-r/c)}{4\pi r} \quad .$$

As will appear below, this represents the vibrations of the medium produced by a source at the origin with a volume rate of flow given by f(t). The velocity and pressure fields are

$$\underline{v} = \hat{n} \left[\frac{f'(t-r/c)}{4\pi rc} + \frac{f(t-r/c)}{4\pi r^2} \right]$$

$$p = \rho_o \frac{f'(t-r/c)}{4\pi r}$$

The instantaneous power radiated through a shell of radius r is

$$P = \int pv \, ds = 4\pi r^2 \left[\frac{\rho_o (f')^2}{16\pi^2 r^2 c} + \frac{\rho_o f' f}{16\pi^2 r^3} \right]$$

and the instantaneous kinetic energy density is given by

$$T = \frac{1}{2} \rho_o v^2 = \frac{1}{2} \rho_o \left[\frac{(f')^2}{16\pi^2 r^2 c^2} + \frac{2ff'}{16\pi^2 r^3 c} + \frac{f^2}{16\pi^2 r^4} \right]$$

Now assume that f(t) is periodic. Then letting $\bar{\ }$ denote mean values we have $\overline{f'f} = 0$, and therefore

$$\bar{P} = \frac{\overline{\rho_o (f')^2}}{4\pi c} \quad \text{independent of } r$$

and

$$\bar{T} = \frac{1}{2} \rho_o \frac{\overline{(f')^2}}{16\pi^2 r^2 c^2} + \frac{1}{2} \rho_o \frac{\overline{f^2}}{16\pi^2 r^4}$$

$$= \qquad T_1 \qquad + \qquad T_2$$

Note the following

$$\overline{P} = 4\pi r^2 c(2T_1)$$

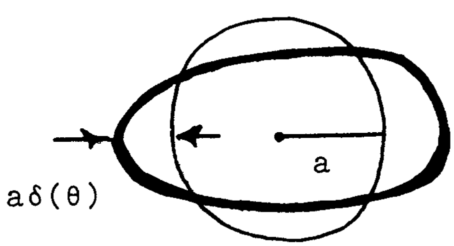
$$\lim_{c\to\infty} \overline{P} = 0$$

$$\lim_{c\to\infty} \overline{T} = T_2 = \frac{1}{2}\rho_o \frac{\overline{f^2}}{16\pi^2 r^4} \quad .$$

The latter quantity falls off rapidly with distance (r^{-4}) and represents the <u>non</u>-radiated part of the kinetic energy. In the limit $c\to\infty$, there is no radiated power, but there is still a vibration in the fluid which is sometimes referred to as "pseudo-sound". Note that the properties of this vibration can be found from incompressible fluid dynamics.

<u>Vibrational Modes for the Heart and Great Vessels</u>

As a first approach to this problem consider the vibrations of the aorta (just after the closure of the valve) in the plane of an aortic cross-section. We assume fluid of density ρ (inside <u>and</u> outside), a membrane under constant tension T supported by a resting internal pressure $P_o = T/a$, where a is the radius.



$$\underline{r} = a[1 + \delta(\theta)]\underline{\hat{r}}$$

Curvature of the membrane

$$K = |d\underline{\tau}/ds| \qquad \text{where } \underline{\tau} = \text{unit tangent}$$

$$s = \text{arc length}$$

$$= \frac{|d\underline{\tau}/d\theta|}{|d\underline{r}/d\theta|}$$

We calculate K to first order in δ only.

$\hat{\underline{r}}$ = unit vector in radial direction

$\hat{\underline{\theta}}$ = unit vector in circumferential direction

$$d\hat{\underline{r}}/d\theta = \hat{\underline{\theta}}$$

$$d\hat{\underline{\theta}}/d\theta = -\hat{\underline{r}}$$

$$\frac{d\underline{r}}{d\theta} = a\{[1+\delta(\theta)]\hat{\underline{\theta}} + \delta'(\theta)\hat{\underline{r}}\}$$

$$\left|\frac{d\underline{r}}{d\theta}\right| = a\{1 + \delta(\theta)\}$$

$$\underline{\tau} = \frac{d\underline{r}/d\theta}{|d\underline{r}/d\theta|} = \hat{\underline{\theta}} + \delta'(\theta)\hat{\underline{r}}$$

$$\frac{d\underline{\tau}}{d\theta} = [-1+\delta''(\theta)]\underline{r} + \delta'(\theta)\hat{\underline{\theta}}$$

$$\left|\frac{d\underline{\tau}}{d\theta}\right| = 1 - \delta''(\theta)$$

$$K = \frac{|d\underline{\tau}/d\theta|}{|d\underline{r}/d\theta|} = \frac{1}{a}\left[\frac{1-\delta''(\theta)}{1+\delta(\theta)}\right] = \frac{1}{a}[1-\delta''(\theta)-\delta(\theta)]$$

CALCULATIONS ON THIS PAGE ARE TO FIRST ORDER IN δ ONLY!

Equilibrium at the membrane:

Let

$$p_1 + P_o = \text{inner pressure}$$

$$p_2 = \text{outer pressure}$$

$$P_o + p_1 - p_2 = \frac{T}{a}\,(1-\delta(\theta)-\delta''(\theta))$$

$$(p_1-p_2) = -\,(\delta(\theta)+\delta''(\theta))P_o\ .$$

In the fluid

$$\rho\,\frac{\partial\phi_1}{\partial t} + p_1 = 0$$

$$\rho\,\frac{\partial\phi_2}{\partial t} + p_2 = 0$$

$$\rho\,\frac{\partial}{\partial t}\,(\phi_1-\phi_2) + (p_1-p_2) = 0$$

$$\rho\,\frac{\partial^2}{\partial t^2}\,(\phi_1-\phi_2) - P_o\,\frac{\partial}{\partial t}\,(\delta+\frac{d^2}{d\theta^2}\,\delta) = 0\ .$$

Note that $a\,\frac{\partial}{\partial t}\,\delta = \left.\frac{\partial\phi}{\partial r}\right|_1 = \left.\frac{\partial\phi}{\partial r}\right|_2$ = normal velocity at the membrane (same on both sides)

$$\boxed{a\rho\,\frac{\partial^2\phi}{\partial t^2} - P_o(1+\frac{\partial^2}{\partial\theta^2})\frac{\partial\phi}{\partial r} = 0}_{\,r=a}$$

In the **fluid** $\nabla^2\phi = 0$ inside and out.

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\phi}{\partial\theta^2} = 0\ .$$

Let

$$\phi = r^k e^{in\theta} .$$

Then

$$k^2 = n^2 .$$

Use $k > 0$ inside, $k < 0$ outside. In the following assume $n > 0$.

The solutions

$$\phi_1 = f(t)\left(\frac{r}{a}\right)^n e^{in\theta}$$

$$\phi_2 = - f(t)\left(\frac{r}{a}\right)^{-n} e^{in\theta}$$

satisfy

$$\left.\frac{\partial\phi}{\partial r}\right|_a = f(t)na^{-1}e^{in\theta} .$$

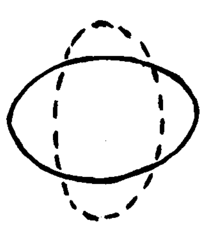
The boundary condition becomes

$$2\rho a \frac{\partial^2 f}{\partial t^2} - P_o na^{-1}(1-n^2)f = 0$$

$$\frac{\partial^2 f}{\partial t^2} + \omega^2 f = 0$$

$$\omega^2 = \frac{n(n^2-1)P_o}{2\rho a^2} .$$

The first mode with a non-zero frequency is $n = 2$. It looks like this:



To estimate the frequency use

$$P_o = 100\text{mmHg} = 1.33\times10^5 \ \frac{\text{dynes}}{\text{cm}^2}$$

$$\rho = 1 \ \text{gm/cm}^3$$

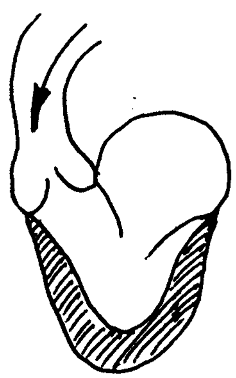
$$a = 1.5 \ \text{cm}$$

$$\frac{P_o}{2\rho a^2} = 3\times10^4 \ \text{sec}^{-2}$$

n	ω^2	ω(radians/sec)	$\frac{\omega}{2\pi}$ cycles/sec
2	18×10^4	425	68
3	72×10^4	850	135
4	180×10^4	1350	215

The figures in the last column are in the appropriate range for heart sounds. (A heart sound is a brief event with a broad frequency spectrum). R. Burridge has worked out the frequencies in the spherical case with similar results.

One may wonder how these circular modes of vibration would be excited at the closure of the aortic valve. The fact that the aorta is curved is suggestive in this regard: Even though



the vibration of the valve itself is not explicit in the foregoing, it should be clear that both valve and wall will be vibrating and that valve closure is the event which initiates the sound.

Murmurs

Heart murmurs seem to be produced by turbulent flow which arises when blood is forced through narrow openings in stenotic or incompetent valves, or through holes between the different heart chambers that occur in congenital heart disease. The mathematical description of turbulent flow is an unfinished subject, but at least the broad outlines exist for a mathematical theory of the origins of turbulence. This is the theory of hydrodynamical stability.* In this theory a given solution of the Navier-Stokes equations is examined to determine whether small perturbations superimposed on the given solution will grow or decay. The possibility of growth (instability) arises because of the coupling between the given flow and the disturbance. This coupling comes from the non-linear terms in the Navier-Stokes equations. For example, consider the two-dimensional flow between parallel planes under the influence of a (possibly time-dependent) body force F(t) in the x direction. The undisturbed flow has the form $u = u_o(y,t)$ which obeys the equation

$$\frac{\partial u_o}{\partial t} = \nu \frac{\partial^2 u_o}{\partial y^2} + T(t) \quad .$$

Then we seek a more general flow of the form

$$u = u_o + u'$$
$$v = v'$$
$$p = p' \ .$$

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* Landau, Lifshitz, Fluid Mechanics

* Schlichting, Boundary Layer Theory

* Lin, The Theory of Hydrodynamical Stability

Substituting in the Navier-Stokes equations and neglecting terms with products of the small ' quantities, we obtain

$$\frac{\partial u'}{\partial t} + u_o(y)\frac{\partial u'}{\partial x} + v' \frac{\partial u_o}{\partial y} + \frac{\partial p'}{\partial x} = \nu\left\{\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right\}$$

$$\frac{\partial v'}{\partial t} + u_o(y)\frac{\partial v'}{\partial x} + \frac{\partial p'}{\partial y} = \nu\left\{\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right\}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \quad .$$

One can reduce this to an equation for the vorticity of the disturbance in the following way. Let

$$w' = \frac{\partial}{\partial x} v' - \frac{\partial}{\partial y} u' \ .$$

Then

$$\boxed{\frac{\partial w'}{\partial t} + u_o(y) \frac{\partial w'}{\partial x} = \nu \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} \right) + v' \frac{\partial^2 u_o}{\partial y^2}}$$

The pair of equations

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

$$-\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} = w'$$

yield

$$\boxed{\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} = \frac{\partial w'}{\partial x}}$$

One can now consider a disturbance with the form of a wave:

$$w' = \hat{w}(y,t)e^{i\alpha x}$$

$$v' = \hat{v}(y,t)e^{i\alpha x}$$

(At this point it is also customary to include an exponential time factor, which is avoided here so that the case where u_o depends on time can be included.) These substitutions yield the pair of equations:

$$\frac{\partial^2 \hat{v}}{\partial y^2} - \alpha^2 \hat{v} = i\alpha\hat{w} \tag{1}$$

$$\frac{\partial \hat{w}}{\partial t} + i\alpha u_o(y)\hat{w} = \nu\left(\frac{\partial^2}{\partial y^2} - \alpha^2\right)\hat{w} + \hat{v}\,\frac{\partial^2 u_o}{\partial y^2} \tag{2}$$

The boundary conditions are

(1) $\hat{v} = 0$ at $y = \pm a$. This can be used as a boundary condition for solving (1) when w is known.

(2) $\frac{\partial \hat{v}}{\partial y} = 0$ at $y = \pm a$. (This comes from $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ and $u \equiv 0$ at $y = \pm a$.)

The boundary condition (2) imposes a constraint on w. To see this, write the solution of (1) in the form

$$\hat{v}(y) = \int_{-a}^{a} i\alpha\hat{w}(y_o)K(y,y_o)dy_o$$

where $K(y,y_o)$ is the solution of

$$\left.\begin{aligned} \frac{\partial^2 K}{\partial y^2} - \alpha^2 K &= \delta(y-y_0) \\ K(a,y_o) = K(-a,y_o) &= 0 \end{aligned}\right\}$$

Then, letting $K' = \frac{\partial K}{\partial y}$ we have

$$0 = \int_{-a}^{a} \hat{w}(y_o) K'(-a,y_o) dy_o$$

$$0 = \int_{-a}^{a} \hat{w}(y_o) K'(a,y_o) dy_o$$

These two integral constraints are the "boundary conditions" for equation (2).