Massless elastic membrane immersed in a 2D viscous incompressible fluid

\[ x = X(\theta, t) \]

\[ 0 \leq \theta \leq 2\pi \]

\( \theta \) = material coordinate

\( T(\theta, t) \) = tension in immersed boundary

\[ T(\theta, t) = \frac{\partial X/\partial \theta}{|\partial X/\partial \theta|} = \text{unit tangent to immersed boundary} \]

\[ F(\theta, t) \, d\theta = \text{force applied by arc d} \theta \text{ of immersed boundary to fluid} \]

Force balance on interval \((a, b)\)

\[ -T \quad \text{and} \quad \int_{a}^{b} F \, d\theta = \text{force of fluid on boundary} \]
\[ 0 = T \theta \bigg|_a^b - \int_a^b F \, d\theta \]

\[ = \int_a^b \left( \frac{\partial}{\partial \theta} (T \tau) - F \right) \, d\theta \]

Since \( a, b \) are arbitrary

\[ F = \frac{\partial}{\partial \theta} (T \tau) \]

\[ = \frac{\partial T \tau}{\partial \theta} + T \frac{\partial \tau}{\partial \theta} \]

\[ = \frac{\partial T}{\partial \theta} \tau + T C \, \eta \]

where

\[ C = \text{curvature} \]

\[ \eta = \text{unit normal to boundary} \]
In general, $T$ is some function of $\frac{\partial X}{\partial \theta}$

The special case

$$T = K \left| \frac{\partial X}{\partial \theta} \right|$$

is particularly simple. In that case

$$T^2 = K \left| \frac{\partial X}{\partial \theta} \right| \frac{\partial X / \partial \theta}{\partial X / \partial \theta} = K \frac{\partial X}{\partial \theta}$$

so

$$F = \frac{2}{\partial} (T^2) = K \frac{\partial^2 X}{\partial \theta^2}$$
Equations of motion of the whole system in immersed boundary form:

1. \[ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) + \nabla p = \mu \nabla^2 u + f \]

2. \[ \nabla \cdot u = 0 \]

3. \[ f(x, t) = \int_0^{2\pi} F(\theta, t) \delta \left( x - X(\theta, t) \right) d\theta \]

4. \[ \frac{\partial X}{\partial t}(\theta, t) = U(\theta, t) = u \left( X(\theta, t), t \right) \]
   \[ = \int \left( u(x, t) \delta \left( x - X(\theta, t) \right) \right) dx \]
   \[ \Omega \]

5. \[ F(\theta, t) = K \frac{\partial^2 X}{\partial \theta^2}(\theta, t) \]

where \( \Omega \) is the fluid domain.
Equations (1-2) are the Navier-Stokes equations of a viscous incompressible fluid. In these equations:

\[ \rho = \text{density}, \quad \mu = \text{viscosity} \]
\[ u(x, t) = \text{fluid velocity} \]
\[ p(x, t) = \text{fluid pressure} \]
\[ f(x, t) = \text{force density applied to fluid by immersed boundary} \]

Equations (3-4) are interface equations. We use the notation:

\[ \mathcal{I}(x) = \delta(x_1) \delta(x_2) \]

where \( x = (x_1, x_2) \) and \( \delta(x) \) is the Dirac delta function.

Equation (5) is the immersed boundary equation derived above.
**Spatial Discretized**

Let \( \{ e_1, e_2 \} \) be the standard basis of \( \mathbb{R}^2 \)

\[ \mathcal{J}_h = \{ x : x = h(j_1 e_1 + j_2 e_2), \text{ where } j_1 \text{ and } j_2 \text{ are integers} \} \]

\[ (D_\alpha) \phi(x) = \frac{\phi(x + h e_\alpha) - \phi(x - h e_\alpha)}{2h} \]

\[ D = (D_1, D_2) \]

\[ D \phi = (D_1 \phi, D_2 \phi) \sim \nabla \phi \]

\[ D \cdot u = D_1 u_1 + D_2 u_2 \sim \nabla \cdot u \]

\[ (L \alpha)(x) = \sum_{\alpha = 1}^{2} \frac{u(x + h e_\alpha) + u(x - h e_\alpha) - 2u(x)}{h^2} \sim \Delta u \]

\[ S(u) \phi = \frac{1}{2} u \cdot D \phi + \frac{1}{2} D \cdot (u \phi) \]

\[ (S^j u)(\alpha) = S(u) u_\alpha \]

Note that \( S^j u \sim u \cdot \nabla u \) if \( \nabla \cdot u = 0 \)
(1') \quad \rho \left( \frac{\partial u}{\partial t} + S(u) u \right) + D p = \mu \nabla u + f

(2') \quad \nabla \cdot u = 0

(3') \quad f(x,t) = \sum_{k=0}^{N-1} f(k\Delta \theta, t) \delta_h^N (x - X(k\Delta \theta, t)) \Delta \theta

(4') \quad \frac{\partial X}{\partial t} (k\Delta \theta, t) = \sum_{X \in Q_h} u(x,t) \delta_h^N (x - X(k\Delta \theta, t)) h^2

(5') \quad F(k\Delta \theta, t) = k \frac{X(k+1)\Delta \theta, t) + X((k-1)\Delta \theta, t) - 2X(k\Delta \theta, t)}{(\Delta \theta)^2}

where

\Delta \theta = \frac{2\pi}{N}

arithmetic in \( k \) is modulo \( N \)
Temporal Discretization: \( u^n(x) = u(x, n\Delta t) \) etc.

Step from \( n \to n+1 \) begins with predicting state from \( n \to n+\frac{1}{2} \):

\[
X^{n+\frac{1}{2}}(k\Delta \theta) = X^n(k\Delta \theta) + \Delta t \sum_{x \in \mathbb{X}_h} u^n(x) \delta_h(x - X^n(k\Delta \theta)) h^2
\]

\[
X^{n+\frac{1}{2}}(k\Delta \theta) = \frac{X^{n+\frac{1}{2}}((k+1)\Delta \theta) + X^{n+\frac{1}{2}}((k-1)\Delta \theta) - 2X^{n+\frac{1}{2}}(k\Delta \theta)}{(\Delta \theta)^2}
\]

\[
f^{n+\frac{1}{2}}(x) = \sum_{k=0}^{N-1} F^{n+\frac{1}{2}}(k\Delta \theta) \delta_h(x - X^{n+\frac{1}{2}}(k\Delta \theta)) \Delta \theta
\]

Solve for \( u^{n+\frac{1}{2}}, p^{n+\frac{1}{2}} \):

\[
p \left( \frac{u^{n+\frac{1}{2}} - u^n}{\Delta t/2} + S(u^n)u^n \right) + Dp^{n+\frac{1}{2}} = \mu \nabla \cdot u^{n+\frac{1}{2}} + f^{n+\frac{1}{2}}
\]

\[
D \cdot u^{n+\frac{1}{2}} = 0
\]
Complete k+1 step from \( n \rightarrow n+1 \) as follows:

\[
X^{n+1}(k\Delta \theta) = X^n(k\Delta \theta) + \Delta t \sum_{x \in g} u^{n+\frac{1}{2}}(x) h \int (X - X^{n+\frac{1}{2}}(k\Delta \theta))^2
\]

Solve for \( u^{n+1}, p^{n+\frac{1}{2}} \):

\[
p \left( \frac{u^{n+1} - u^n}{\Delta t} + S(u^{n+\frac{1}{2}})u^{n+\frac{1}{2}} \right) + D \cdot p^{n+\frac{1}{2}} = -\mu \left( \frac{u^n + u^{n+1}}{2} \right) + f^{n+\frac{1}{2}}
\]

\[
D \cdot u^{n+1} = 0
\]
In both the preliminary substep and the final substep, we need to solve linear systems of the form

\[(I - \frac{\Delta t}{\rho} \frac{\nabla^2}{\Delta t}) u + \frac{\Delta t}{\rho} \nabla g = w, \quad D \cdot u = 0\]

In the preliminary substep:

\[u = u^{n+\frac{1}{2}}, \quad g = \frac{1}{2} p^{n+\frac{1}{2}}, \quad w = u^n - \frac{\Delta t}{2} S(u^n) u^n + \frac{\Delta t}{\rho} f^{n+\frac{1}{2}}\]

In the final substep:

\[u = u^{n+1}, \quad g = p^{n+\frac{1}{2}}, \quad w = u^n - \Delta t S(u^{n+\frac{1}{2}}) u^{n+\frac{1}{2}} + \frac{\Delta t}{\rho} f^{n+\frac{1}{2}} + \frac{\Delta t}{2} \nabla \nabla \cdot u^n\]
Discrete Fourier Transform solution of the linear system for (4.18)

Let the fluid domain be \( \Omega = (0, L_0) \times (0, L_0) \) with periodic boundary conditions.

\[
h = L_0/N \\
\hat{J}_i, \hat{J}_2 = 0 \ldots N-1
\]

\( \mathcal{J}_h = \{ x : x = (j_1 h, j_2 h), j_\alpha \in \{0, \ldots, N-1\}, \alpha = 1, 2 \} \)

Arithmetic on \( \hat{j}_1, \hat{j}_2 \) is modulo \( N \).

For any function \( \Phi(x) \) defined for \( x \in \mathcal{J}_h \), let \( \Phi_{j_1, j_2} = \Phi(\hat{J}_1 h_{e_1} + \hat{J}_2 h_{e_2}) \).
Now define the Discrete Fourier Transform as follows:

\[ \Phi_{m_1, m_2} = \frac{1}{L_0^2} \sum_{x \in \mathbb{Z}^2} e^{-\frac{2\pi i}{L_0} (m_1 x_1 + m_2 x_2)} \phi(x) \]

\[ = \frac{1}{N^2} \sum_{j_1, j_2 = 0}^{N-1} e^{-\frac{2\pi i j_1}{N} (m_1 j_1 + m_2 j_2)} \Phi_{j_1, j_2} \]

It follows that

\[ \Phi_{j_1, j_2} = \sum_{m_1, m_2 = 0}^{N-1} e^{\frac{2\pi i j_1}{N} (m_1 j_1 + m_2 j_2)} \Phi_{m_1, m_2} \]

Which may also be written:

\[ \phi(x) = \sum_{m_1, m_2 = 0}^{N-1} e^{\frac{2\pi i}{L_0} (m_1 x_1 + m_2 x_2)} \Phi_{m_1, m_2} \]
Discrete Fourier Transform \( \hat{D}_\alpha \), \( \alpha = 1, 2 \)

\[
(\hat{D}_1 \varphi)(x) = \frac{\varphi(x + h e_1) - \varphi(x - h e_1)}{2h}
\]

\[
= \sum_{m_1, m_2 = 0}^{N-1} \left( e^{\frac{2\pi i}{L_0} (m_1 x + m_2 x_2)} - e^{\frac{2\pi i}{L_0} (m_1 x - m_2 x_2)} \right) \hat{\varphi}_{m_1, m_2}
\]

\[
= \sum_{m_1, m_2 = 0}^{N-1} \left( e^{\frac{2\pi i}{L_0} m_1 h} - e^{-\frac{2\pi i}{L_0} m_1 h} \right) e^{\frac{2\pi i}{L_0} (m_1 x + m_2 x_2)} \hat{\varphi}_{m_1, m_2}
\]

Therefore

\[
(\hat{D}_1)_{m_1, m_2} = \frac{2i \sin \left( \frac{2\pi}{L_0} m_1 h \right)}{2h} = \frac{i}{h} \sin \left( \frac{2\pi h}{L_0} m_1 \right)
\]

and similarly

\[
(\hat{D}_2)_{m_1, m_2} = \frac{i}{h} \sin \left( \frac{2\pi h}{L_0} m_2 \right)
\]

Note: As \( h \to 0 \),

\[
(\hat{D}_2)_{m_1, m_2} \to \frac{2\pi i}{L_0} m_1 \varphi = (\hat{\frac{\partial}{\partial x_1}})^\wedge
\]

\[
(\hat{D}_2)_{m_1, m_2} \to \frac{2\pi i}{L_0} m_2 \varphi = (\hat{\frac{\partial}{\partial x_2}})^\wedge
\]
Discrete Fourier Transform of $U$:

$$(L U)(x) \sum_{\alpha=1}^{2} \frac{U(x + he_{\alpha}) + U(x - he_{\alpha}) - 2U(x)}{h^2}$$

$$= \sum_{m_1, m_2 = 0}^{N-1} \left( \sum_{\alpha=1}^{2} \left( e^{\frac{2\pi i h m_\alpha}{L_0}} + e^{-\frac{2\pi i h m_\alpha}{L_0}} - 2 \right) \right) \cdot \left( e^{\frac{2\pi i L_0 (m_1 x_1 + m_2 x_2)}{L_0}} \right) \cdot \frac{1}{h^2}$$

Therefore,

$$\hat{L} = \sum_{\alpha=1}^{2} \left( 2 \cos \left( \frac{2\pi h m_\alpha}{L_0} \right) - 2 \right) \frac{1}{h^2}$$

$$= -\frac{4}{h^2} \sum_{\alpha=1}^{2} \sin \left( \frac{2\pi h m_\alpha}{L_0} \right) \frac{1}{h^2} \rightarrow -\frac{4\pi^2}{L_0^2} \sum_{\alpha=1}^{2} m_\alpha^2$$

$$= \Delta$$
The Discrete Fourier Transform of the equations satisfied by \( \hat{u}, \hat{q} \) is as follows:

\[
\left(1 - \frac{\Delta t}{\rho} \hat{\mathbf{L}}\right) \hat{u} + \frac{\Delta t}{\rho} \hat{\mathbf{D}} \hat{q} = \hat{w}
\]

\[
\hat{\mathbf{D}} \cdot \hat{u} = 0
\]

For each \( m, m_2 \) this is a system of 3 equations in 3 unknowns: \( \hat{u}_1, \hat{u}_2, \hat{q} \). The equations for different \( m, m_2 \) are not coupled to each other!

We can eliminate \( \hat{q} \) by applying \( \hat{\mathbf{D}} \) to both sides of the first equation and by making use of \( \hat{\mathbf{D}} \cdot \hat{u} = 0 \). The result is:

\[
\frac{\Delta t}{\rho} \hat{\mathbf{D}} \cdot \hat{\mathbf{D}} \hat{q} = \hat{\mathbf{D}} \cdot \hat{w}
\]

Which has the solution:

\[
\hat{q} = \frac{\hat{\mathbf{D}} \cdot \hat{w}}{\frac{\Delta t}{\rho} \hat{\mathbf{D}} \cdot \hat{\mathbf{D}}}
\]

Then:

\[
\hat{u} = \left(\hat{w} - \frac{\hat{\mathbf{D}} \left( \frac{\hat{\mathbf{D}} \cdot \hat{w}}{\hat{\mathbf{D}} \cdot \hat{\mathbf{D}}} \right)}{\left(1 - \frac{\Delta t}{\rho} \hat{\mathbf{L}}\right)}\right) / \left(1 - \frac{\Delta t}{\rho} \hat{\mathbf{L}}\right)
\]
Writing out the above more explicitly gives

\[ \hat{\delta}^{m_1, m_2} = \frac{i}{\hbar} \sin \left( \frac{2\pi}{N} m \right) \cdot \hat{W}^{m_1, m_2} \]

\[ \hat{W}^{m_1, m_2} = -\frac{\Delta t}{\rho \hbar^2} \sin \left( \frac{2\pi}{N} m \right) \cdot \sin \left( \frac{2\pi}{N} m \right) \]

\[ \hat{U}^{m_1, m_2} = \frac{\sin \left( \frac{2\pi}{N} m \right) \sin \left( \frac{2\pi}{N} m \right) \cdot \hat{W}^{m_1, m_2}}{\sin \left( \frac{2\pi}{N} m \right) \cdot \sin \left( \frac{2\pi}{N} m \right)} \]

\[ 1 + \frac{\Delta t}{2} \frac{\mu}{\rho} \frac{4}{\hbar^2} \sin \left( \frac{\pi}{N} m \right) \cdot \sin \left( \frac{\pi}{N} m \right) \]

where \( m = (m_1, m_2) \)

\[ \sin (\alpha m) = (\sin (\alpha m_1), \sin (\alpha m_2)) \]
The cases

\[(m_1, m_2) = (0, 0), (0, N/2), (N/2, 0), (N/2, N/2)\]

require special consideration. Going back to the original equations, we see that \( \hat{\xi} \) is undefined but plays no role at all, and that \( \hat{\xi} \) is given by

\[
\hat{\xi}_{m_1, m_2} = \frac{\hat{W}_{m_1, m_2}}{1 + \Delta t \cdot \frac{M}{2} \cdot \frac{4}{h^2} \cdot \sin\left(\frac{\pi m_1}{N}\right) \cdot \sin\left(\frac{\pi m_2}{N}\right)}
\]
Construct of $\delta_h$:

Let

$$\delta_h(x) = \frac{1}{h^2} \varphi\left(\frac{x_1}{h}\right) \varphi\left(\frac{x_2}{h}\right)$$

where $x = (x_1, x_2)$ and $\varphi$ has the following properties:

i) $\varphi$ is continuous

ii) $\varphi(r) = 0$ for $|r| \geq 2$

iii) $\sum_{i \text{ even}} \varphi(r-i) = \sum_{i \text{ odd}} \varphi(r-i) = \frac{1}{2}$, all $r$

iv) $\sum_i (r-i) \varphi(r-i) = 0$, all $r$

v) $\sum_i (\varphi(r-i))^2 = \zeta$, all $r$

Note: Unlike $i$, $r$ is a real variable.

"All $r$" means all real values for $r$. 
How to determine $\varphi(r)$:

Consider $0 \leq r \leq 1$. Then the nonzero $\varphi(r-i)$ are at most $\varphi(r-2), \varphi(r-1), \varphi(r), \varphi(r+1)$.

Therefore, conditions (iii')-(v) reduce to

$$\varphi(r-2) + \varphi(r) = \frac{1}{2}$$

$$\varphi(r-1) + \varphi(r+1) = \frac{1}{2}$$

$$(r-2)\varphi(r-2) + (r-1)\varphi(r-1) + r\varphi(r) + (r+1)\varphi(r+1) = 0$$

$$\left(\varphi(r-2)\right)^2 + \left(\varphi(r-1)\right)^2 + \left(\varphi(r)\right)^2 + \left(\varphi(r+1)\right)^2 = C'$$

To determine $C'$, set $r=0$. Then $\varphi(r-2)=0$, and the above equations reduce to

$$\varphi(0) = \frac{1}{2}$$

$$\varphi(-1) + \varphi(1) = \frac{1}{2} \quad \Rightarrow \quad \varphi(-1) = \varphi(1) = \frac{1}{4}$$

$$-\varphi(-1) + \varphi(1) = 0$$

$$C = \left(\varphi(-1)\right)^2 + \left(\varphi(0)\right)^2 + \left(\varphi(1)\right)^2 = \frac{1}{16} + \frac{1}{4} + \frac{1}{16} = \frac{3}{8}$$
With $C$ known, return to the case $0 \leq r \leq 1$. Make use of the first two equations to simplify the third one:

$$r \left( \varphi(r-2) + \varphi(r-1) + \varphi(r) + \varphi(r+1) \right)$$

$$= 2\varphi(r-2) + \varphi(r-1) - \varphi(r+1)$$

The fact that multiple $r$ is equal to 1. Thus, we have the system:

$$\varphi(r-2) + \varphi(r) = \frac{1}{2}$$
$$\varphi(r-1) + \varphi(r+1) = \frac{1}{2}$$

$$2\varphi(r-2) + \varphi(r-1) - \varphi(r+1) = r$$

$$(\varphi(r-2))^2 + (\varphi(r-1))^2 + (\varphi(r))^2 + (\varphi(r+1))^2 = \frac{3}{8}$$

Use the first 3 equations to express $\varphi(r-2), \varphi(r-1), \varphi(r+1)$ in terms of $\varphi(r)$:

$$\varphi(r-2) = \frac{1}{2} - \varphi(r)$$

$$\varphi(r-1) = \frac{1}{2} \left[ \frac{1}{2} + r - 2\varphi(r-2) \right] = \frac{1}{2} \left( r - \frac{1}{2} \right) + \varphi(r)$$

$$\varphi(r+1) = \frac{1}{2} \left[ \frac{1}{2} - r + 2\varphi(r-2) \right] = \frac{1}{2} \left( -r + \frac{3}{2} \right) - \varphi(r)$$
Substitute the above results into the sum-of-squares equation:

\[
\left(\frac{1}{2} - \varphi(r)\right)^2 + \left(\frac{1}{2}(r-\frac{1}{2}) + \varphi(r)\right)^2 + \left(\varphi(r)\right)^2
\]

\[
+ \left(\frac{1}{2}(-r + \frac{3}{2}) - \varphi(r)\right)^2 = \frac{3}{8}
\]

Collecting like powers of \(\varphi(r)\) yields:

\[
4(\varphi(r))^2 + (2r-3)\varphi(r) + \frac{1}{2}(r-1)^2 = 0
\]

\[
\varphi(r) = \frac{3-2r \pm \sqrt{(3-2r)^2 - 8(r-1)^2}}{8}
\]

\[
= \frac{3-2r \pm \sqrt{1+4r-4r^2}}{8}
\]

Recall that this holds only for \(0 \leq r \leq 1\).

Note that

\[
\varphi(0) = \frac{3 + 1}{8}, \quad \varphi(1) = \frac{1 + 1}{8}
\]

Since we previously found \(\varphi(0) = 1/2\) and \(\varphi(1) = 1/4\), we must choose the + root.
Then, for $0 \leq r \leq 1$, we have

$$\varphi(r) = \frac{3 - 2r + \sqrt{1 + 4r - 4r^2}}{8}$$

and also

$$\varphi(r-2) = \frac{4}{8} - \varphi(r) = \frac{1 + 2r - \sqrt{1 + 4r - 4r^2}}{8}$$

$$\varphi(r-1) = \frac{4r - 2}{8} + \varphi(r) = \frac{1 + 2r + \sqrt{1 + 4r - 4r^2}}{8}$$

$$\varphi(r+1) = \frac{6 - 4r}{8} - \varphi(r) = \frac{3 - 2r - \sqrt{1 + 4r - 4r^2}}{8}$$

Homework:

Plot $\varphi(r)$ and $\varphi'(r)$ for $-3 \leq r \leq 3$
Homework:

Evaluate the "6-point delta function," which is defined by

\( \phi(r) \) is continuous

\( \phi(r) = 0 \) for \( |r| \geq 3 \)

\[ \sum_{i \text{ even}} \phi(r-i) = \sum_{i \text{ odd}} \phi(r-i) = \frac{1}{2} \quad \text{all } r \]

\[ \sum_{i} (r-i) \phi(r-i) = 0 \quad \text{all } r \]

\[ \sum_{i} (r-i)^2 \phi(r-i) = 0 \quad \text{all } r \]

\[ \sum_{i} (r-i)^3 \phi(r-i) = 0 \quad \text{all } r \]

\[ \sum_{i} (\phi(r-i))^2 = C \quad \text{all } r \]

Plot \( \phi(r) \) and \( \phi'(r) \) for \(-4 \leq r \leq 4\)