Improved Volume Conservation

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We construct a new difference operator $\underline{D}$ to replace the standard central difference operator. The new $\underline{D}$ is "tuned" to the regularized delta function δ_h. It is defined as follows (in the two-dimensional case)

For $\underline{x} \in g_h$

$$\left(\underline{D} \cdot \underline{u}\right)(\underline{x}) = \frac{1}{h^2} \int_{B(\underline{x})} \left(\nabla \cdot \underline{U}\right) d\underline{X}$$

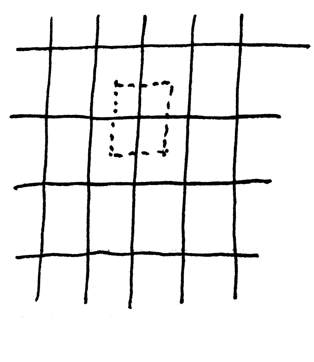
where

$$B(\underline{x}) = \left(x_1 - \frac{h}{2}, x_1 + \frac{h}{2}\right) \times \left(x_2 - \frac{h}{2}, x_2 + \frac{h}{2}\right)$$

$$\underline{x} = (x_1, x_2)$$

$$\underline{U}(\underline{X}) = \sum_{\underline{x}' \in g_h} \underline{u}(\underline{x}')\, \delta_h(\underline{X} - \underline{x}')\, h^2$$

Thus, $\underline{D} \cdot \underline{u}$ at a grid point is the average of $\nabla \cdot \underline{U}$ over a box surrounding that grid point



We have

$$(\underline{D}\cdot\underline{u})(\underline{x}) = \frac{1}{h^2}\sum_{\underline{x}'\in g_h} \underline{u}(\underline{x}')\cdot\int_{B(\underline{x})}(\nabla\delta_h)(\underline{X}-\underline{x}')\,d\underline{X}\; h^2$$

Let $\underline{X}' = \underline{X} - \underline{x}$. Then $d\underline{X}' = d\underline{X}$

and $\underline{X}\in B(\underline{x}) \Leftrightarrow \underline{X}'\in B(\underline{0})$

Then

$$(\underline{D}\cdot\underline{u})(\underline{x}) = \frac{1}{h^2}\sum_{\underline{x}'\in g_h} \underline{u}(\underline{x}')\cdot\int_{B(\underline{0})}(\nabla\delta_h)(\underline{X}'+\underline{x}-\underline{x}')\,d\underline{X}'\, h^2$$

$$= \sum_{\underline{x}'\in g_h} \underline{u}(\underline{x}')\cdot\underline{\mathcal{D}}(\underline{x}-\underline{x}')\,h^2$$

where

$$\underline{\mathcal{D}}(\underline{x}) = \frac{1}{h^2}\int_{B(\underline{0})}(\nabla\delta_h)(\underline{X}+\underline{x})\,d\underline{X}$$

$$\underline{D}_1(\underline{x}) = \frac{1}{h^2} \int_{B(\underline{0})} \frac{\partial}{\partial X_1} \delta_h(\underline{X} + \underline{x})\, d\underline{X}$$

$$= \frac{1}{h^4} \iint_{-h/2}^{h/2} \frac{\partial}{\partial X_1} \varphi\left(\frac{X_1 + x_1}{h}\right) \varphi\left(\frac{X_2 + x_2}{h}\right) dX_1\, dX_2$$

Let $x_1 = j_1 h$, $x_2 = j_2 h$, $\underline{D}_{j_1 j_2} = \underline{D}(\underline{x})$

Also, make the change of variables

$$X_1 = h r_1, \quad X_2 = h r_2, \quad \frac{\partial}{\partial X_1} = \frac{1}{h}\frac{\partial}{\partial r_1}$$

$$(\underline{D}_1)_{j_1 j_2} = \frac{1}{h^2}\, \frac{1}{h} \int_{-1/2}^{1/2} \frac{\partial}{\partial r_1} \varphi(r_1 + j_1)\, dr_1 \int_{-1/2}^{1/2} \varphi(r_2 + j_2)\, dr_2$$

$$= \frac{1}{h^3}\left[\varphi\left(\tfrac{1}{2} + j_1\right) - \varphi\left(-\tfrac{1}{2} + j_1\right)\right] \int_{-1/2}^{1/2} \varphi(r_2 + j_2)\, dr_2$$

$$= \frac{1}{h^3}\, \gamma_{j_1}\, \omega_{j_2}$$

where (since φ is even)

$$\gamma_j = -\varphi(j-\tfrac{1}{2}) + \varphi(j+\tfrac{1}{2})$$

$$\omega_j = \int_{-1/2}^{1/2} \varphi(j+r)\,dr$$

Similarly, we have

$$(\mathcal{D}_2)_{j_1 j_2} = \frac{1}{h^3}\,\omega_{j_1}\,\gamma_{j_2}$$

Since $\varphi(r)=0$ for $|r| \geq 2$, we have

$$\gamma_j = \omega_j = 0 \quad \text{for } |j| \geq 3$$

The nonzero values occur at $j=-2,-1,0,1,2$

Thus $\underline{\mathcal{D}}$ has a 5×5 stencil. Since φ is even

$$\gamma_j = -\gamma_{-j}$$

$$\omega_j = \omega_{-j}$$

Thus, we need only consider $j=0,1,2$

The values of γ_j can be read off by inspection from the values of φ

$$\gamma_0 = -\varphi(-\tfrac{1}{2}) + \varphi(\tfrac{1}{2}) = 0$$

$$\gamma_1 = -\varphi(\tfrac{1}{2}) + \varphi(\tfrac{3}{2})$$

$$= -\varphi(\tfrac{1}{2}) + \left(\tfrac{1}{2} - \varphi(-\tfrac{1}{2})\right)$$

$$= -2\varphi(\tfrac{1}{2}) + \tfrac{1}{2}$$

$$= -\frac{3 - 2(\frac{1}{2}) + \sqrt{1 + 4(\frac{1}{2}) - 4(\frac{1}{2})^2}}{4} + \frac{1}{2}$$

$$= -\frac{3 - 1 + \sqrt{1 + 2 - 1}}{4} + \frac{1}{2}$$

$$= \frac{-\sqrt{2}}{4}$$

$$\gamma_2 = -\varphi\left(\tfrac{3}{2}\right) + \cancel{\varphi\left(\tfrac{5}{2}\right)} = -\left(\tfrac{1}{2} - \varphi\left(-\tfrac{1}{2}\right)\right) = -\left(\tfrac{1}{2} - \varphi\left(\tfrac{1}{2}\right)\right)$$

$$= -\left(\frac{1}{2} - \frac{2 + \sqrt{2}}{8}\right) = -\left(\frac{2 - \sqrt{2}}{8}\right)$$

The different values of ω are related in the following way:

$$\sum_{j\ \text{even}} \omega_j = \int_{-1/2}^{1/2} \sum_{j\ \text{even}} \varphi(j+r)\,dr$$

$$\omega_{-2} + \omega_0 + \omega_2 = \int_{-1/2}^{1/2} \frac{1}{2}\,dr = \frac{1}{2}$$

$$\boxed{\omega_0 + 2\omega_2 = \frac{1}{2}} \Rightarrow \boxed{\omega_2 = \frac{1}{4} - \frac{1}{2}\omega_0}$$

$$\sum_{j\ \text{odd}} \omega_j = \int_{-1/2}^{1/2} \sum_{j\ \text{odd}} \varphi(j+r)\,dr$$

$$\omega_{-1} + \omega_1 = \int_{-1/2}^{1/2} \frac{1}{2}\,dr = \frac{1}{2}$$

$$2\omega_1 = \frac{1}{2}$$

$$\boxed{\omega_1 = \frac{1}{4}}$$

Thus we need only evaluate

$$\omega_f = \int_{-1/2}^{1/2} \varphi(r)\,dr = 2\int_0^{1/2} \varphi(r)\,dr$$

$$= \frac{1}{4}\int_0^{1/2}\left(3-2r+\sqrt{1+4r-4r^2}\right)dr$$

$$= \frac{1}{4}\left(\frac{3}{2}-\frac{1}{4}+\int_0^{1/2}\sqrt{1+4r-4r^2}\,dr\right)$$

$$= \frac{1}{4}\left(\frac{5}{4}+\int_0^{1/2}\sqrt{1+4r-4r^2}\,dr\right)$$

But

$$\int_0^{1/2}\sqrt{1+4r-4r^2}\,dr = \int_0^{1/2}\sqrt{2-1+4r-4r^2}\,dr$$

$$= \int_0^{1/2}\sqrt{2-(1-2r)^2}\,dr = \int_0^{1/2}\sqrt{1-\left(\frac{1-2r}{\sqrt{2}}\right)^2}\,\sqrt{2}\,dr$$

Let $x = \dfrac{1-2r}{\sqrt{2}}$, $dx = -\sqrt{2}\,dr$

$r=0 \Leftrightarrow x = 1/\sqrt{2}$, $r=\frac{1}{2} \Leftrightarrow x=0$

$$= \int_0^{1/\sqrt{2}}\sqrt{1-x^2}\,dx = \int_0^{\pi/4}(\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\left((\cos\theta)^2 + (\sin\theta)^2 + (\cos\theta)^2 - (\sin\theta)^2\right) d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\left(1 + \cos 2\theta\right) d\theta$$

$$= \frac{\pi}{8} + \frac{1}{4}\sin\frac{2\pi}{4} = \left(\frac{\pi}{8} + \frac{1}{4}\right)$$

Therefore

$$\omega_0 = \frac{1}{4}\left(\frac{5}{4} + \frac{\pi}{8} + \frac{1}{4}\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + \frac{\pi}{8}\right)$$

$$= \frac{3}{8} + \frac{\pi}{32}$$

and then

$$\omega_2 = \frac{1}{4} - \frac{1}{2}\omega_0 = \frac{1}{4} - \frac{3}{16} - \frac{\pi}{64}$$

$$= \frac{1}{16} - \frac{\pi}{64}$$

In summary, we have

j	γ_j	ω_j
-2	$+\frac{2-\sqrt{2}}{8}$	$\frac{1}{16}-\frac{\pi}{64}$
-1	$+\frac{\sqrt{2}}{4}$	$\frac{1}{4}$
0	0	$\frac{3}{8}+\frac{\pi}{32}$
1	$-\frac{\sqrt{2}}{4}$	$\frac{1}{4}$
2	$-\frac{2-\sqrt{2}}{8}$	$\frac{1}{16}-\frac{\pi}{64}$

Next, we evaluate the discrete Fourier transform of the new difference operator $\underline{D}$.

From the formula

$$(\underline{D}\cdot\underline{u})(\underline{x}) = \sum_{\underline{x}'\in g_h} \underline{u}(\underline{x}')\cdot \mathcal{D}(\underline{x}-\underline{x}')h^2$$

we read off the recipe for applying D_1 or D_2 to a scalar grid function, say $s(\underline{x})$:

$$(D_\alpha s)(\underline{x}) = \sum_{\underline{x}'\in g_h} s(\underline{x}')\,\mathcal{D}_\alpha(\underline{x}-\underline{x}')h^2$$

$$= \sum_{\underline{x}'\in g_h} s(\underline{x}-\underline{x}')\,\mathcal{D}_\alpha(\underline{x}')h^2$$

or

$$(D_\alpha s)_{j_1,j_2} = \sum_{k_1,k_2=0}^{N-1} s_{j_1-k_1,j_2-k_2}\,\frac{1}{h}(\mathcal{D}_\alpha)_{k_1,k_2}$$

For example if $\alpha = 1$

$$(D_1 s)_{j_1,j_2} = \sum_{k_1,k_2=0}^{N-1} s_{j_1-k_1,\,j_2-k_2} \,\frac{1}{h}\gamma_{k_1}\,\omega_{k_2}$$

$$= \sum_{k_1,k_2=0}^{N-1} \sum_{m_1,m_2=0}^{N-1} \hat{s}_{m_1,m_2}\, e^{\frac{2\pi i}{N}\left(m_1(j_1-k_1)+m_2(j_2-k_2)\right)} \frac{1}{h}\gamma_{k_1}\,\omega_{k_2}$$

$$= \sum_{m_1,m_2=0}^{N-1} e^{\frac{2\pi i}{N}(m_1 j_1 + m_2 j_2)}\, \hat{s}_{m_1,m_2} \left(\sum_{k_1=0}^{N-1} e^{-\frac{2\pi i}{N} m_1 k_1}\frac{1}{h}\gamma_{k_1}\right)$$

$$\left(\sum_{k_2=0}^{N-1} e^{-\frac{2\pi i}{N} m_2 k_2}\,\omega_{k_2}\right)$$

$$= \sum_{m_1,m_2=0}^{N-1} e^{\frac{2\pi i}{N}(m_1 j_1 + m_2 j_2)}\, \hat{s}_{m_1,m_2}\left(\frac{N\hat{\gamma}_{m_1}}{h}\right)\left(N\hat{\omega}_{m_2}\right)$$

Thus $$\left(\hat{D}_1\right)_{m_1,m_2} = \left(\frac{N\hat{\gamma}_{m_1}}{h}\right)\left(N\hat{\omega}_{m_2}\right)$$

and similarly $$\left(\hat{D}_2\right)_{m_1,m_2} = \left(N\hat{\omega}_{m_1},\ \frac{N\hat{\gamma}_{m_2}}{h}\right)$$

It remains to evaluate the two factors that appear in $\hat{D}_1$ and $\hat{D}_2$:

$$\frac{N}{h}\hat{\gamma}_m = \frac{1}{h}\sum_{k=-2}^{2} e^{-\frac{2\pi i}{N}km}\,\gamma_k$$

$$= \frac{2i}{h}\left(+\frac{2-\sqrt{2}}{8}\sin\left(\frac{2\pi\, 2m}{N}\right) + \frac{\sqrt{2}}{4}\sin\frac{2\pi m}{N}\right)$$

$$= \frac{2i}{h}\left(+\frac{2-\sqrt{2}}{4}\sin\left(\frac{2\pi m}{N}\right)\cos\left(\frac{2\pi m}{N}\right) + \frac{\sqrt{2}}{4}\sin\left(\frac{2\pi m}{N}\right)\right)$$

$$= +\frac{2i}{h}\sin\left(\frac{2\pi m}{N}\right)\left[\frac{1}{4}\right]\left[(2-\sqrt{2})\cos\left(\frac{2\pi m}{N}\right)+\sqrt{2}\right]$$

$$= +\frac{i}{h}\sin\left(\frac{2\pi m}{N}\right)\left[\left(1-\frac{1}{\sqrt{2}}\right)\cos\left(\frac{2\pi m}{N}\right)+\frac{1}{\sqrt{2}}\right]$$

$$= \frac{i}{h}\sin\frac{2\pi m}{N}\left[\left(\cos\frac{\pi m}{N}\right)^2-\left(\sin\frac{\pi m}{N}\right)^2+\frac{1}{\sqrt{2}}\,2\left(\sin\frac{\pi m}{N}\right)^2\right]$$

$$= \frac{i}{h}\sin\frac{2\pi m}{N}\left[\left(\cos\frac{\pi m}{N}\right)^2+(\sqrt{2}-1)\left(\sin\frac{\pi m}{N}\right)^2\right]$$

$$N\hat{\omega}_m = \sum_{k=-2}^{2} e^{-\frac{2\pi i}{N}km}\,\omega_k$$

$$= \left(\frac{3}{8}+\frac{\pi}{32}\right) + \frac{1}{4}\,2\cos\left(\frac{2\pi m}{N}\right) + \left(\frac{1}{16}-\frac{\pi}{64}\right)2\cos\frac{2\pi\, 2m}{N}$$

$$= \left(\frac{3}{8}+\frac{\pi}{32}\right) + \frac{1}{2}\cos\frac{2\pi m}{N} + \left(\frac{1}{8}-\frac{\pi}{32}\right)\left[\left(\cos\frac{2\pi m}{N}\right)^2 - \left(\sin\frac{2\pi m}{N}\right)^2\right]$$

$$= \frac{3}{8} + \frac{1}{2}\cos\frac{2\pi m}{N} + \frac{1}{8}\left[2\left(\cos\frac{2\pi m}{N}\right)^2 - 1\right]$$

$$+ \frac{\pi}{32}\left(1 - \left(\cos\frac{2\pi m}{N}\right)^2 + \left(\sin\frac{2\pi m}{N}\right)^2\right)$$

$$= \frac{1}{4} + \frac{1}{2}\cos\frac{2\pi m}{N} + \frac{1}{4}\left(\cos\frac{2\pi m}{N}\right)^2$$

$$+ \frac{\pi}{32}\,2\left(\sin\frac{2\pi m}{N}\right)^2$$

$$= \frac{1}{4}\left(1+\cos\frac{2\pi m}{N}\right)^2 + \frac{\pi}{16}\left(\sin\frac{2\pi m}{N}\right)^2$$

$$= \left(\cos\frac{\pi m}{N}\right)^4 + \frac{\pi}{16}\left(\sin\frac{2\pi m}{N}\right)^2$$

$$= \left(\cos\frac{\pi m}{N}\right)^4 + \frac{\pi}{16}\left(2\sin\frac{\pi m}{N}\cos\frac{\pi m}{N}\right)^2$$

$$= \left(\cos\frac{\pi m}{N}\right)^2\left[\left(\cos\frac{\pi m}{N}\right)^2 + \frac{\pi}{4}\left(\sin\frac{\pi m}{N}\right)^2\right]$$

Therefore

$$\left(\hat{D}_1\right)_{m_1,m_2} = \frac{i}{h}\left(\sin\frac{2\pi m_1}{N}\right)\left[\left(\cos\frac{\pi m_1}{N}\right)^2 + \left(\sqrt{2}-1\right)\left(\sin\frac{\pi m}{N}\right)^2\right]$$
$$\left(\cos\frac{\pi m_2}{N}\right)^2\left[\left(\cos\frac{\pi m_2}{N}\right)^2 + \frac{\pi}{4}\left(\sin\frac{\pi m_2}{N}\right)^2\right]$$

$$\left(\hat{D}_2\right)_{m_1,m_2} = \frac{i}{h}\left(\sin\frac{2\pi m_2}{N}\right)\left[\left(\cos\frac{\pi m_2}{N}\right)^2 + \left(\sqrt{2}-1\right)\left(\sin\frac{\pi m_2}{N}\right)^2\right]$$
$$\left(\cos\frac{\pi m_1}{N}\right)^2\left[\left(\cos\frac{\pi m_1}{N}\right)^2 + \frac{\pi}{4}\left(\sin\frac{\pi m_1}{N}\right)^2\right]$$

$$(\hat{D}_1)_{m_1,m_2} = 0 \quad \text{when } (m_1 = 0 \text{ OR } m_1 = N/2 \text{ OR } m_2 = N/2)$$

$$(\hat{D}_2)_{m_1,m_2} = 0 \quad \text{when } (m_2 = 0 \text{ OR } m_2 = N/2 \text{ OR } m_1 = N/2)$$

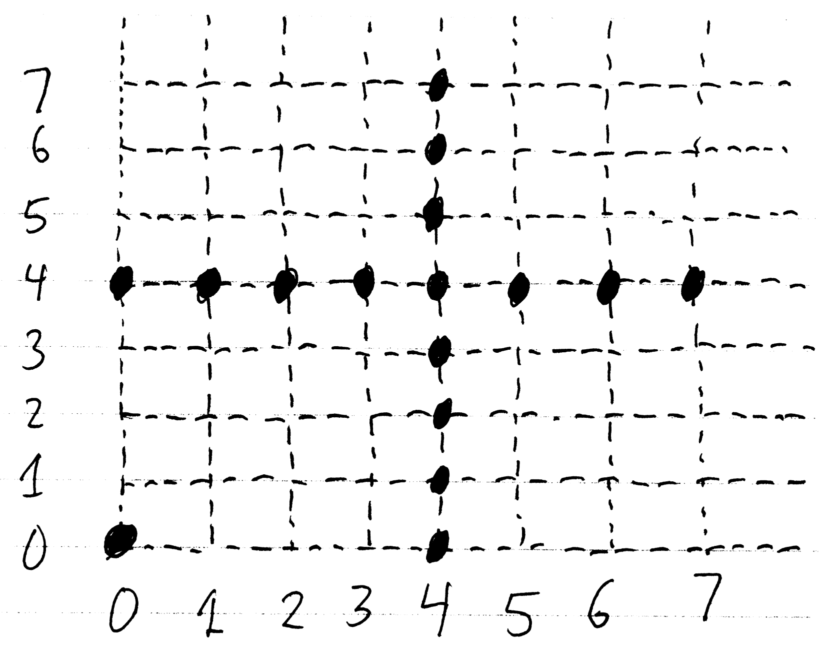
Note that

$$m_1 = N/2 \Rightarrow (\hat{D}_1 = 0 \text{ AND } \hat{D}_2 = 0)$$

$$m_2 = N/2 \Rightarrow (\hat{D}_1 = 0 \text{ AND } \hat{D}_2 = 0)$$

Therefore, the cases in which $\underline{\hat{D}} = 0$ are

$$(m_1, m_2) = (0,0), \quad m_1 = N/2, \quad m_2 = N/2$$



Acknowledgement

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Thanks also to Cowen for doing all of the evaluations independently as a check on the results derived above.