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ABSTRACT

This paper describes an analytic and a numerical method for the aortic sinus problem. Both methods are based on the dynamics of point vortices, and both exploit a particular conformal mapping from a model aortic sinus to the upper half plane. The analytic description is based on an isolated point vortex in equilibrium with a free stream. This inviscid model is used to study the stability of the aortic sinus vortex and to elucidate the mechanism of aortic valve closure, but it cannot be used to study the formation of the sinus vortex and it gives a somewhat incorrect picture of the flow pattern. These difficulties are overcome by the introduction of a numerical method for the aortic sinus problem with fluid viscosity. We use Chorin's vortex method combined with conformal mapping. The conformal mapping approach gives an explicit formula for the vortex velocities and it resolves the singularities associated with the corners of the domain. This method is then used to study the formation of the sinus vortex and to confirm the predictions of the point vortex model with respect to the role of the vortex in valve closure.—**Peskin, C. S., and A. W. Wolfe.** The aortic sinus vortex. *Federation Proc.* 37: 2784–2792, 1978.

There have been many excellent studies of the aortic sinus vortex, beginning of course with the work of Leonardo da Vinci (3) and including the elegant experiments of Bellhouse and Talbot (1), the detailed numerical studies of Gillani and Swanson (4), and the recent combined experimental and theoretical work of van Steenhoven et al. (10).

The present paper is therefore intended only as a modest contribution to an area in which much excellent work has already been done. The approach that we describe differs from the earlier work not so much in the results as in the method by which the problem is attacked. Since this paper is part of a symposium on computational methods, we have also used this opportunity to include expository material on vortex dynamics in general and on the vortex method of A. J. Chorin (2) in particular. The combination of the vortex method with conformal mapping that is described in this paper is new, however.

This paper describes an analytic and a numerical method for the aortic sinus problem. Both methods are based on the dynamics of point vortices, and both exploit a particular conformal mapping from a model aortic sinus to the upper half plane.

The analytic description of the sinus vortex is based on the simplest possible model: an isolated, inviscid point vortex in equilibrium with a free stream. Despite its simplicity, this model sheds light on the stability of the aortic sinus vortex, and it allows us to draw useful conclusions about the mechanism of aortic valve closure.

The point vortex model does not include viscosity, and it tells us nothing about the mechanism of formation of the aortic sinus vortex. To remedy this situation, we use a numerical method, the vortex method of A. J. Chorin (2), in which the fluid is represented by a collection of modified point vortices that are convected by the fluid and undergo a random walk representing viscous diffusion. Since this numerical method is based on the dynamics of point vortices, many of the formulas of the analytic part of this paper can be used directly in the numerical method.

1. POINT VORTEX IN THE AORTIC SINUS

The influence of boundaries on a point vortex

A point vortex is a singularity in a two-dimensional, inviscid, incompressible flow. An isolated point vortex in the plane has circular streamlines, and the speed of the fluid has the form $K/2\pi r$, where K is the strength (circulation) of the point vortex and r is the distance from its center. $K > 0$ corresponds to counterclockwise rotation. The point vortex may also be described by the complex velocity potential

$$\Phi = \phi + i\psi = \frac{K}{2\pi i} \log (z - z_0) \qquad (1.1)$$

where ϕ is the real velocity potential, ψ is the stream function, and z_0 is the position of the singularity in the complex plane.

It is obvious by symmetry that a point vortex in an unbounded fluid does not move itself. In the presence of boundaries, however, this symmetry is lost, and the vortex acquires a velocity. The computation of the vortex motion is complicated by the singularity in the fluid velocity field at the vortex itself. A procedure that circumvents this difficulty is outlined in the following.

Consider an arbitrary domain Ω in the z-plane, z

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$= x + iy$, and assume that we know a conformal mapping $\zeta = f(z)$ which maps Ω onto the upper half ζ-plane, $\zeta = \xi + i\eta$. Let the flow in the z-plane be given by the complex velocity potential

$$\Phi(z) = U\mathrm{f}(z) + \frac{K}{2\pi i}\log(\mathrm{f}(z) - \mathrm{f}(z_0)) - \frac{K}{2\pi i}\log(\mathrm{f}(z) - \overline{\mathrm{f}(z_0)}) \quad (1.2)$$

where z_0 is a point in Ω and where the overbar denotes complex conjugate.

In the special case $\mathrm{f}(z) = z$ (see Fig. 1), the domain Ω is the upper half plane, the first term represents a uniform flow in the x direction with velocity U, the second term represents a vortex of strength K at $z = z_0$, and the third term represents an image vortex with strength $-K$ at the point $\bar{z}_0$. Note that the singularity associated with the third term lies outside of Ω. This term gives the non-singular potential flow that is needed to satisfy the boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ on the border of Ω. In the general case, the flow given by *equation 1.2* consists of a potential flow in Ω with a point vortex of strength K at $z = z_0$. If the mapping has no singularity in the interior of Ω, then this point vortex is the only interior singularity of the flow.

To compute the velocity of the vortex itself, we write the singular term as a sum of a symmetrical part and a continuous part:

$$\frac{K}{2\pi i}\log(\mathrm{f}(z) - \mathrm{f}(z_0)) = \frac{K}{2\pi i}\log(z - z_0) + \frac{K}{2\pi i}\log\frac{\mathrm{f}(z) - \mathrm{f}(z_0)}{z - z_0} \quad (1.3)$$

Thus

$$\Phi(z) = \frac{K}{2\pi i}\log(z - z_0) + \Phi_0(z) \quad (1.4)$$

where

$$\Phi_0(z) = U\mathrm{f}(z) + \frac{K}{2\pi i}\log\frac{\mathrm{f}(z) - \mathrm{f}(z_0)}{z - z_0} - \frac{K}{2\pi i}\log(\mathrm{f}(z) - \overline{\mathrm{f}(z_0)}) \quad (1.5)$$

The vortex velocity may now be computed from Φ_0, since the first term in *eq. 1.4* is symmetrical around the vortex and does not contribute to its motion. The vortex velocity (u,v) is found by setting

$$u - iv = \frac{d\Phi_0}{dz}(z_0) \quad (1.6)$$

In evaluating this derivative, it is convenient to use

$$\frac{\mathrm{f}(z) - \mathrm{f}(z_0)}{z - z_0} = \mathrm{f}'(z_0) + \frac{1}{2}(z - z_0)\mathrm{f}''(z_0) + \cdots \quad (1.7)$$

Then

$$\frac{d\Phi_0}{dz}(z_0) = U\mathrm{f}'(z_0) + \frac{K}{2\pi i}\left[\frac{1}{2}\frac{\mathrm{f}''(z_0)}{\mathrm{f}'(z_0)} - \frac{\mathrm{f}'(z_0)}{\mathrm{f}(z_0) - \overline{\mathrm{f}(z_0)}}\right] \quad (1.8)$$

The term involving $\mathrm{f}''(z_0)$ may be thought of as arising from the curvature of the boundary, since it is absent when $\mathrm{f}(z)$ is a linear function and the border of Ω is a straight line.

These formulas assign a definite vortex velocity (u,v) to each possible position z_0 of the point vortex. Thus, they define a velocity field which should not, however, be confused with the velocity field of the fluid. The possible trajectories of the vortex are the integral curves of this velocity field. That is, they are the solutions $z_0(t)$ of

$$\frac{dz_0}{dt} = u + iv = \overline{\frac{d\Phi_0}{dz}(z_0)} \quad (1.9)$$

A conformal mapping for the aortic sinus problem

Let Ω be the union of the upper half plane and the interior of a circle through the points $z = \pm 1$. There is a one-parameter family of circles through this pair of points, and we define Ω uniquely by specifying a parameter α in the interval $[\frac{1}{2},1]$ such that π/α is the interior angle of Ω at $z = \pm 1$. For example, when $\alpha = \frac{2}{3}$, the interior angle is $3\pi/2$, and Ω is made up of the upper half plane and a half disc.

The part of Ω that lies below the real axis represents the aortic sinus itself and the rest of Ω represents the nearby part of the aorta. (We assume that the fluid dynamics of one aortic sinus is not much influenced by the others, and we treat the aorta as an unbounded free stream.)

It can be verified that the following conformal mapping takes Ω onto the upper half plane

$$\zeta = \mathrm{f}(z) = B(A(B(z))) \quad (1.10)$$

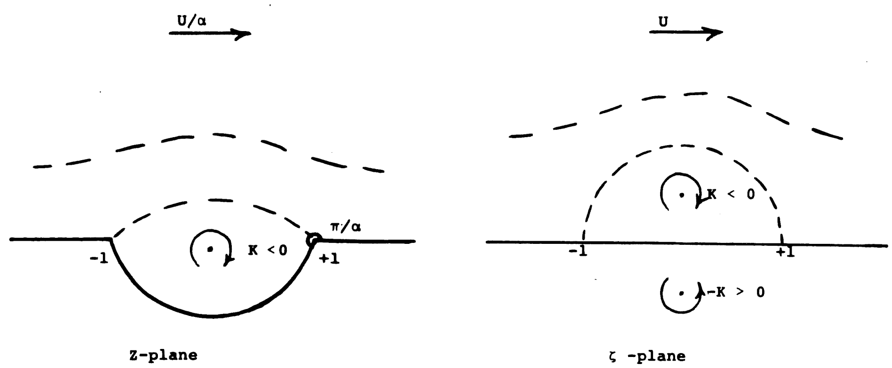
where

$$B(z) = \frac{z + 1}{z - 1} \quad (1.11)$$

$$A(z) = z^{\alpha} \quad (1.12)$$

Note that B has the property $B = B^{-1}$ so that $\alpha = 1$ yields $\zeta = z$. It is also clear that the points $z = \pm 1$ and $z = \infty$ are fixed points of the mapping for any α. It can be shown that $\mathrm{f}'(z) \to 1/\alpha$ as $|z| \to \infty$. This shows that the mapping reduces to a simple scaling far from the sinus.

Figure 1. Sketch of the equilibrium flow configuration with an isolated point vortex in the aortic sinus. The flow is described by *equation 1.2* and the conformal mapping $\zeta = \mathrm{f}(z)$ is described by *equations 1.10–1.12*.



Equilibrium of the sinus vortex with a uniform stream

We now ask the following question: In the context of inviscid, incompressible fluid dynamics, is there a steady flow consisting of a stationary point vortex in the aortic sinus with a free stream at ∞ and with a separating streamline that connects the upstream border of the sinus with the downstream border?

The assumption that streamline separation occurs at the upstream border with reattachment at the downstream border cannot be motivated within the context of inviscid fluid dynamics. The point of separation is determined by the effects of fluid viscosity in the boundary layer. The numerical computations discussed in section 2 show, however, that this assumption is valid for the steady aortic sinus flow pattern in the presence of fluid viscosity, and we simply impose it as a constraint here.

The condition that the points of separation and reattachment be at $z = \pm 1$ implies, first, that the vortex must lie along the imaginary axis, $x = 0$. Accordingly, let the position be $z_0 = iy_0$. Our conformal mapping takes the imaginary axis into itself, a fact which we express by writing $f(iy) = i\eta(y)$, so the image of the vortex in the ζ plane is $\zeta_0 = i\eta(y_0)$. The separation condition then implies

$$U + \frac{K\eta(y_0)}{\pi(1 + (\eta(y_0))^2)} = 0 \qquad (1.13)$$

which is a relationship between the free stream velocity, the vortex position, and the vortex strength. (This relationship is most easily found by considering the flow in the ζ plane and imposing the condition that the velocity be zero at the point of separation.)

We now consider the condition that the vortex be at rest. This is found by setting the right-hand side of *eq. 1.8* equal to zero. We simplify the resulting expression, however, using $f(iy) = i\eta(y)$. The result is

$$U\eta'(y_0) + \frac{K}{4\pi}\left\{-\frac{\eta''(y_0)}{\eta'(y_0)} + \frac{\eta'(y_0)}{\eta(y_0)}\right\} = 0 \qquad (1.14)$$

Equations 1.13–1.14 may be thought of as a pair of linear equations in the unknowns U, K. There are nontrivial solutions if the determinant is zero. This condition may be written $F'(y_0) = 0$, where

$$F(y) = \frac{\eta(y)}{(1 + \eta^2(y))^2\eta'(y)} \qquad (1.15)$$

Thus the equilibrium points are the stationary points of F. It is easy to see that there is at least one such stationary point. To show this, note that $\eta' > \delta > 0$ (for some δ) and that the domain of F is $y_{\min} \leq y \leq \infty$ where $\eta(y_{\min}) = 0$. It then follows from *eq. 1.15* that $F(y_{\min}) = F(\infty) = 0$ and that $F(y)$ is positive and bounded on $y_{\min} < y < \infty$. Thus F has at least one maximum, at which $F' = 0$ because F has a continuous derivative. This proof of the existence of the equilibrium point uses only the symmetry of Ω about the imaginary axis and the fact that η' is bounded away from zero.

In practice, the equilibrium point(s) can be found numerically. (We worked with $(\log F)' = 0$ and used the method of bisection.) Once $\eta_0 = \eta(y_0)$ has been found, we can express K in terms of U by solving either *eq. 1.13* or *1.14*. For example, from *1.13*

$$K = -\frac{\pi(1 + \eta_0^2)}{\eta_0}U \qquad (1.16)$$

For a given free stream velocity, we determine in this way an equilibrium position and strength for a point vortex in the aortic sinus. The equilibrium position is independent of the free stream velocity, and the equilibrium strength is proportional to the free stream velocity. When the flow is from left to right ($U > 0$), the vortex rotates in the clockwise direction ($K < 0$).

Stability

The next natural question is whether the equilibrium found above is stable or not. Only a partial answer to this question can be given in the context of inviscid point vortex dynamics. For example, the strength of the vortex is always independent of time, and there is no mechanism available for generating new vortices or destroying old ones, so the only variable we can study is the position of the vortex. By studying the vortex trajectories, we can hope to learn how a vortex will behave if it is perturbed away from the equilibrium position found above.

A useful tool in the study of vortex trajectories is Routh's stream function (7), which is defined, for the present problem, as follows:

$$S(z,\bar{z}) = \frac{1}{2i}\left\{U(f(z) - \bar{f}(\bar{z})) + \frac{K}{4\pi i}\log[f'(z)\bar{f}'(\bar{z})] - \frac{K}{4\pi i}\log[(f(z) - \bar{f}(\bar{z}))(\bar{f}(\bar{z}) - f(z))]\right\} \qquad (1.17)$$

where $\bar{f}$ is the analytic function defined by $\bar{f}(\bar{z}) = \overline{f(z)}$. It can then be checked that

$$2i\frac{\partial S}{\partial z} = u - iv = \frac{d\bar{z}}{dt} \qquad (1.18)$$

$$-2i\frac{\partial S}{\partial \bar{z}} = u + iv = \frac{dz}{dt} \qquad (1.19)$$

so that, along a vortex trajectory,

$$\frac{dS}{dt} = \frac{\partial S}{\partial z}\frac{dz}{dt} + \frac{\partial S}{\partial \bar{z}}\frac{d\bar{z}}{dt} = 0 \qquad (1.20)$$

We conclude that Routh's stream function is constant along each vortex trajectory. The existence of such a function means that vortex equilibriums are of only two types. In the first case, Routh's stream function has a maximum or a minimum, and the vortex trajectories form closed curves around the equilibrium point. Such an equilibrium point is called *stable* because a sufficiently small displacement of the vortex away from equilibrium leaves it on a closed orbit around the equilibrium point. In the second case, Routh's stream function has a saddle point. Such an equilibrium point is called *unstable* because a typical perturbation (however small) puts the vortex on a trajectory that carries it arbitrarily far away from equilibrium. This classification of equilibrium points is invariant under a change of sign of the time, as is appropriate for inviscid dynamics.

Stability may be investigated analytically, or it may be studied by plotting the vortex trajectories with the aid of Routh's stream function. We have used the latter method (Fig. 2). For the aortic sinus problem we find one equilibrium point on the midline of the sinus which is stable for all positive sinus depths, although the region of stability becomes smaller as the sinus becomes shallower. (By region of stability, we mean the region that is filled with closed vortex trajectories.) There are also unstable equilibriums near the upstream and downstream borders of the sinus. The trajectories through these points delimit the region of stability of the stable equilibrium point. This stability analysis makes precise the idea that the vortex is "trapped" in the aortic sinus.

The mechanism of aortic valve closure

The role of the sinus vortex in aortic valve closure has been much discussed, ever since Leonardo da Vinci predicted correctly that the valve would close under the influence of the vortex while the flow is still in the forward direction but decreasing.

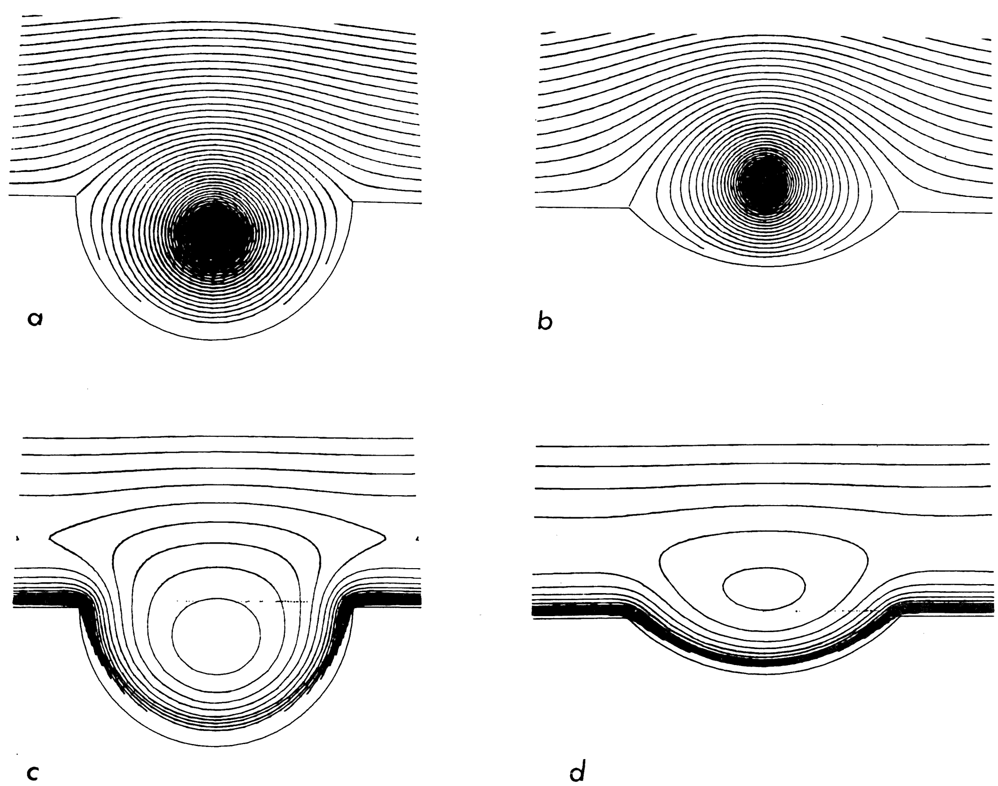
Precisely how does the vortex accomplish this trick? To study this question, we consider what happens to the equilibrium configuration of the point vortex with a free stream if the free stream is suddenly turned off (but not reversed). In the equilibrium configuration we assume that the aortic valve leaflet, which is anchored to the upstream border of the sinus, lies along the separating streamline. After the free stream has been turned off, we think of the valve leaflet as being moved by the fluid towards closure without altering the fluid flow pattern.

When the free stream is suddenly turned off, the resulting flow pattern is described by the same equations as before but with $U = 0$. This change has two dramatic consequences. First, the character of the fluid streamlines changes instantaneously. In the equilibrium situation there were closed streamlines in the sinus and streamlines connecting $x = \pm\infty$ in the aorta. When the free stream is shut off, the closed streamlines associated with the vortex fill up the entire domain. In particular, they cross the valve leaflet, which therefore acquires a velocity that drives the leaflet towards its closed position. (The familiar rule that streamlines run parallel to boundaries is not correct for boundaries in motion. Indeed, the only way that an immersed boundary can move, other than tangent to itself, is for streamlines to cross the boundary. This follows from the definition of a streamline as a line tangent to the fluid velocity field.)

The second effect of setting $U = 0$ is that the vortex position ceases to be an equilibrium point. It is easy to see that the potential flow associated with U tended to move the vortex to the right; in the equilibrium configuration this was balanced by the self-velocity of the vortex associated with the presence of boundaries. When U is set equal to zero, the vortex starts to move to the left (upstream) towards the aortic valve leaflet. Thus the vortex, the streamlines of which are moving the valve towards closure, moves itself in the same direction as the valve is moving. The vortex actually chases the leaflet towards the closed position!

Although these effects were derived under the highly idealized conditions of point vortex dynamics in an inviscid fluid with sudden interruption of the main stream, it is very likely that the events of aortic valve closure actually occur as described here. Van Steenhoven et al.

Figure 2. Equilibrium streamlines (a,b) and vortex trajectories (c,d) for an isolated point vortex in the aortic sinus. The separating streamline bulges out into the aorta, especially in a shallow sinus (b). This unrealistic feature is removed when fluid viscosity is included in the model (see Fig. 3). The closed vortex trajectories in (c,d) indicate that the vortex is trapped by the sinus. In (a,c) $\alpha = 2/3$, $U = 1$, $K = -3.0151\pi$. In (b,d) $\alpha = 0.8$, $U = 1$, $K = -2.5130\pi$.



(10) have recently recorded the motion of the vortex as described above in physical model experiments on aortic valve dynamics, and they have given a physical explanation of this effect in terms of the pressure gradient developed during deceleration of the free stream. Our own numerical work (section 2) also confirms that these mechanisms of aortic valve closure are still applicable with diffuse sinus vorticity and with nonzero fluid viscosity.

Inadequacy of the point vortex model

Although inviscid point vortex dynamics give some insight into questions of stability and the mechanism of valve closure, the inviscid analysis tells us nothing about the process of formation of the aortic sinus vortex. Moreover, the point vortex model gives a flow configuration that approximates the true aortic sinus vortex only very roughly. In particular, the separating streamline bulges out into the aorta in the point vortex model; in reality it is nearly flat. Also, the center of the vortex is on the midline of the sinus in the model. The real vortex is shifted downstream.

These difficulties are overcome by including fluid viscosity in the model. The Navier-Stokes equations must then be solved numerically. We shall use a method, however, that incorporates much of the mathematical structure associated with the point vortex analysis. In particular, the method combines the vortex method of A. J. Chorin (2) with conformal mapping.

2. THE AORTIC SINUS PROBLEM WITH FLUID VISCOSITY

The vortex method of A. J. Chorin (2)

This is a grid-free numerical method for the incompressible Navier-Stokes equations in two space dimensions. The equations to be solved are as follows. In the domain Ω:

$$\omega_t + \mathbf{u}\cdot\nabla\omega = \nu\Delta\omega \qquad (2.1)$$

$$\operatorname{curl}\mathbf{u} = \omega\hat{\mathbf{z}} \qquad (2.2)$$

$$\operatorname{div}\mathbf{u} = 0 \qquad (2.3)$$

On the boundary $\partial\Omega$:

$$\mathbf{u} = 0 \qquad (2.4)$$

In the foregoing, $\mathbf{u}$ is the fluid velocity, ω is the (scalar) vorticity, ν is the kinematic viscosity, and $\hat{\mathbf{z}}$ is a unit vector normal to the plane of the flow.

In Chorin's vortex method, the fluid is described by a collection of modified point vortices. A modified point vortex of unit strength at the origin has the velocity field

$$\mathbf{u}_0(\mathbf{x}) = \frac{1}{2\pi\ \mathrm{Max}\,(|\mathbf{x}|, r_0)}\,\frac{\hat{\mathbf{z}}\times\mathbf{x}}{|\mathbf{x}|} \qquad (2.5)$$

The constant r_0 is called to cut-off length. For $|\mathbf{x}| \geq r_0$, this velocity field corresponds exactly to that of a point vortex. For $|\mathbf{x}| \leq r_0$, the streamlines are still circular, but $|\mathbf{u}_0| = \frac{1}{2}\pi r_0$, independent of $\mathbf{x}$. Thus the velocity of a modified point vortex is bounded, but it is discontinuous at the origin. In the formulas that follow, we adopt the convention $\mathbf{u}_0(0) = 0$.

Now suppose we have a collection of modified point vortices at positions $\mathbf{x}_j$ with strengths K_j. In an unbounded fluid the velocity field would be

$$\mathbf{u}_f(\mathbf{x}) = \sum_j K_j\mathbf{u}_0(\mathbf{x} - \mathbf{x}_j) \qquad (2.6)$$

In the presence of boundaries, $\mathbf{u}_f$ violates both the normal and the tangential components of the boundary condition $\mathbf{u} = 0$. The normal boundary condition can be satisfied (without changing the vorticity) by adding a potential flow. Thus, set

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_f(\mathbf{x}) + \nabla\phi \qquad (2.7)$$

where

$$\Delta\phi = 0 \quad \text{in} \quad \Omega \qquad (2.8)$$

$$\mathbf{n}\cdot\nabla\phi = -\mathbf{n}\cdot\mathbf{u}_f \quad \text{on} \quad \partial\Omega \qquad (2.9)$$

The velocity field $\mathbf{u}(\mathbf{x})$ given by *eq. 2.7* still violates the tangential boundary condition $\mathbf{u}\cdot\boldsymbol{\tau} = 0$. This is remedied by a device that appears to be purely formal, which has important consequences. We declare that *eq. 2.7* defines $\mathbf{u}(\mathbf{x})$ only for $\mathbf{x}$ in the interior of Ω. At the boundary, we set $\mathbf{u}(\mathbf{x}) = 0$ by definition. This implies that we have a δ-function layer of vorticity along the boundary. The circulation per unit length of this vortex layer is $-\mathbf{u}\cdot\boldsymbol{\tau}$, where $\mathbf{u}$ is the limiting velocity as the boundary is approached from the interior and $\boldsymbol{\tau}$ is the unit tangent to the boundary oriented so that the domain is to the left of $\boldsymbol{\tau}$. This vortex layer can be represented by modified point vortices in the following (approximate) way. Introduce modified vortices at intervals of length h along the boundary with circulation equal to the local value of

$$-2(\mathbf{u}\cdot\boldsymbol{\tau})h \qquad \sout{-2(\mathbf{u}\cdot\boldsymbol{\tau})/h} \qquad (2.10)$$

The factor 2 can be explained by noting that half the vorticity of the modified vortices falls outside the domain and therefore has no effect on the flow. (Another explanation of the factor 2 will appear below.) It is important to notice that the modified vortices give a much better representation of the vortex layer than a corresponding array of point vortices. This is the principal reason for the introduction of the cut-off length.

The vortex method may now be described as follows. After n time steps, we have a collection of vortices with positions $\mathbf{x}_k^n$ and strengths K_k. This vortex configuration is updated in two steps. First, new vortices are created along the boundary to satisfy $\mathbf{u}\cdot\boldsymbol{\tau} = 0$, as described above. Then, all of the vortices are moved according to

$$\mathbf{x}_k^{n+1} = \mathbf{x}_k^n + \Delta t[\sum_j K_j\mathbf{u}_0(\mathbf{x}_k^n - \mathbf{x}_j^n) + (\nabla\phi^n)(\mathbf{x}_k^n)] + (4\nu\Delta t)^{1/2}\mathbf{d}_k^n, \qquad (2.11)$$

where the $\mathbf{d}_k^n$ are independent Gaussian random variables in the plane with $E[\mathbf{d}_k^n] = 0$ and $E[|\mathbf{d}_k^n|^2] = 1$. The term involving these random variables in *eq. 2.11* corresponds to the viscous (diffusion) term in the vorticity transport equation.

Those vortices that leave the domain during the implementation of *eq. 2.11* are dropped from the computation. This includes roughly half the newly created vortices and compensates for the factor 2 in *eq. 2.10*. This explanation of the factor 2 is complementary to the previous explanation, since those vortices that diffuse into the domain now have all of their vorticity within Ω.

This completes the description of one time step. Note that the vortex strengths have not changed except that some vortices have been created and some vortices have been dropped from the calculation.

We have not yet specified, however, how the potential flow associated with a vortex configuration is actually computed. In simple cases this flow can be found by introducing a finite system of images for each vortex. An important domain for which this procedure works is the exterior of a circle. In the general case, one has to use a numerical method such as an integral equation method as in ref. 2 or the capacitance matrix method (9). There is, however, a class of problems in which a finite system of images will not suffice but where an explicit formula for the vortex velocity can still be obtained by conformal mapping. Aside from the convenience of having an explicit formula, the conformal mapping approach also has substantial advantages when the domain has corners.

The vortex method combined with conformal mapping

As in section 1, suppose that we have available a conformal mapping $\zeta = f(z)$ that takes the domain Ω onto the upper half plane. The generalization of *eq. 1.2* for a collection of point vortices in Ω is

$$\Phi(z) = Uf(z) + \sum_j \left\{\frac{K_j}{2\pi i}\log(f(z) - f(z_j)) - \frac{K_j}{2\pi i}\log(f(z) - \overline{f(z_j)})\right\} \quad (2.12)$$

If we focus attention on the kth vortex, we can separate its singular, symmetrical part, and write

$$\Phi(z) = \frac{K_k}{2\pi i}\log(z - z_k) + \Phi_k(z) \quad (2.13)$$

where

$$\Phi_k(z) = Uf(z) + \frac{K_k}{2\pi i}\log\frac{f(z) - f(z_k)}{z - z_k} - \frac{K_k}{2\pi i}\log(f(z) - \overline{f(z_k)}) + \sum_{j\neq k}\left\{\frac{K_j}{2\pi i}\log(f(z) - f(z_j)) - \frac{K_j}{2\pi i}\log(f(z) - \overline{f(z_j)})\right\} \quad (2.14)$$

By the same reasoning as in section 1, this gives a formula for the velocity of the kth vortex

$$u_k - iv_k = \frac{d\Phi_k}{dz}(z_k) = Uf'(z_k) + \frac{K_k}{2\pi i}\left[\frac{f''(z_k)}{2f'(z_k)} - \frac{f'(z_k)}{f(z_k) - \overline{f(z_k)}}\right] + f'(z_k)\sum_{j\neq k}\frac{K_j}{2\pi i}\left[\frac{1}{f(z_k) - f(z_j)} - \frac{1}{f(z_k) - \overline{f(z_j)}}\right] \quad (2.15)$$

Thus, we have used the conformal mapping to get an explicit formula for the velocity of each point vortex in the presence of all of the others. This formula includes the effects of the normal boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$.

The conformal mapping is also helpful in the implementation of the vorticity creation algorithm. The fundamental reason for this is that the vorticity created along some arc of the boundary depends on an integral of the form

$$\int_a^b (\mathbf{u}\cdot\boldsymbol{\tau})ds = \phi_b - \phi_a \quad (2.16)$$

which is invariant under a conformal mapping. Accordingly, we can implement the vorticity creation algorithm in the ζ plane. This has several advantages. First, the boundary is simply the real axis and it is very easy to specify a set of points for vorticity creation and to compute the slip velocity in the ζ plane at these points. The formula for this slip velocity is derived from *eq. 2.12* after making the change of variables $\zeta = f(z)$, $\zeta_j = f(z_j)$. The result is

$$s(\xi) = U + \sum_j \frac{K_j}{2\pi i}\left\{\frac{1}{\xi - \zeta_j} - \frac{1}{\xi - \bar{\zeta}_j}\right\} = U + \sum_j \frac{K_j}{2\pi i}\frac{Im(\zeta_j)}{|\xi - \zeta_j|^2} \quad (2.17)$$

The most important advantage of implementing the vorticity creation process in the ζ plane, however, is that we completely avoid the singularities associated with the corners of the domain in the z plane. When the interior angle at a corner is greater than π, the slip velocity in the z plane is infinite at the corner for almost every distribution of vorticity. This singularity is removed by the mapping to the ζ plane. By spacing the points for vorticity creation uniformly along the ξ axis of the ζ plane, we automatically concentrate their images in the z plane near those corners of the domain where the viscous fluid can be expected to develop boundary layer separation and vortex shedding. This is precisely where high resolution is needed.

We are now ready to give a statement of the vortex method with conformal mapping. Let there be M sites for vorticity creation along the ξ axis of the ζ plane. Call these ξ_m, $m = 1, 2, \ldots, M$. Let $\xi_{m+1} - \xi_m = h$, with $\xi_1 = -(M + 1)h/2$. After n time steps, the number of vortices will be nM, their strengths will be given by K_k, $k = 1, 2, \ldots, nM$, and their corresponding positions will be $\zeta_k{}^n = f(z_k{}^n)$. This vortex configuration is updated as follows. First, create M new vortices with labels given by

$$k = nM + m, \quad m = 1, 2, \ldots, M. \quad (2.18)$$

The positions of these vortices are given by

$$\zeta_k^n = f(z_k^n) = \xi_m \quad (2.19)$$

and their strengths are

$$K_k = -2s^n(\xi_m)h \quad (2.20)$$

where s^n is given by the form of *eq. 2.17* that is appropriate for modified point vortices. This form is

$$s^n(\xi) = U + \sum_{j=1}^{nM}\frac{K_j}{\pi}\frac{Im(\zeta_j{}^n)}{|\xi - \zeta_j{}^n|\,\mathrm{Max}(|\xi - \zeta_j{}^n|, \rho_0)} \quad (2.21)$$

which reduces to *eq. 2.17* when $\rho_0 = 0$. The constant ρ_0 is the cut-off length of the modified point vortex. Note that we implement this cut-off in the ζ-plane.

The next step is to move all of the vortices one random step. It is important that this random step be implemented in the z (physical) plane. Otherwise the viscosity will be effectively a function of position determined by the mapping. Thus, for $k = 1, 2, \ldots, (n + 1)M$, let

$$z_k{}^* = z_k{}^n + (4\nu\Delta t)^{1/2}d_k{}^n \quad (2.22)$$

where the $d_k{}^n$ are independent, Gaussian, random variables in the complex plane with $E[d_k{}^n] = 0$ and $E[|d_k{}^n|^2] = 1$. Those vortices that leave the domain during the random step are dropped from the calculation by setting their strengths to zero. Vortices with zero strength are ignored in all subsequent calculations. They may also be overwritten to save storage.

Finally, the vortices are moved at the velocity given by *eq. 2.15*. This formula has to be modified, however, to take into account the cut-off length ρ_0. The result is

$$z_k^{n+1} = z_k^n + \Delta t w_k^* \qquad (2.23)$$

where

$$\overline{w_k^*} = U f'(z_k^*) + \frac{K_k}{2\pi i}\left[\frac{f''(z_k^*)}{2f'(z_k^*)} - \frac{f'(z_k^*)}{\tilde{\rho}_{kk}\exp(i\tilde{\theta}_{kk})}\right] + f'(z_k^*) \times \sum_{\substack{j=1 \\ j\neq k}}^{(n+1)M} \frac{K_j}{2\pi i}\left[\frac{1}{\rho_{jk}\exp(i\theta_{jk})} - \frac{1}{\tilde{\rho}_{jk}\exp(i\tilde{\theta}_{jk})}\right] \qquad (2.24)$$

$$\rho_{jk} = \mathrm{Max}\,(|f(z_k^*) - f(z_j^*)|, \rho_0) \qquad (2.25)$$

$$\tilde{\rho}_{jk} = \mathrm{Max}\,(|f(z_k^*) - \overline{f(z_j^*)}|, \rho_0) \qquad (2.26)$$

$$\theta_{jk} = \arg\,(f(z_k) - f(z_j)) \qquad (2.27)$$

$$\tilde{\theta}_{jk} = \arg\,(f(z_k) - \overline{f(z_j)}) \qquad (2.28)$$

This completes the description of one time step.

In the vortex method combined with conformal mapping, a substantial part of the computation is done in the ζ plane. It should be emphasized, therefore, that we do not simply compute a solution in the ζ plane and then map the results into the z plane afterwards. Such a procedure would be completely incorrect, since the Navier-Stokes equations are not invariant under a conformal mapping.

The shape of the domain in the z plane influences the computation in three important ways. First, the random walk is implemented directly in the z plane. Second, the vortex has a self velocity term involving $f''(z)$ which would not be present if the physical domain were the upper half ζ plane. Third, quite apart from this self-velocity, the motions of fluid particles in the z-plane are not what one would expect by computing these motions in the ζ plane and then mapping the result back into the z-plane. That is, the time dependence of the motion is different. To see this, consider just the flow given by the complex velocity potential $\Phi = Uf(z) = U\zeta$. Fluid particles in the z plane move according to $dz/dt = \overline{d\Phi/dz} = \overline{Uf'(z)}$. Therefore, the image in the ζ plane of a fluid particle in the z plane satisfies $d\zeta/dt = f'(z)\,dz/dt = U|f'(z)|^2$. If we regard Φ as the complex velocity potential of a physical flow in the ζ plane, however, the fluid particles all have velocity U. For all of these reasons it is essential that the shape of the domain in the z-plane should influence the computation.

Results

We have used the vortex method with conformal mapping to study the aortic sinus vortex. In these numerical experiments the fluid is at rest until $t = 0$. The free stream velocity is suddenly turned on at $t = 0$, held constant until $t = T$, and then set equal to zero for $t > T$. This rectangular pulse is not intended to be realistic, but it gives us the opportunity to study the effects of the sudden onset and termination of flow. We choose our unit of length so that the corners of the aortic sinus are at $z = \pm 1$, and we choose our unit of time so that the parameter $U = 1$. (The free stream velocity is U/α.) The Reynolds number was set equal to 1,000 or 5,000; the results are essentially independent of Reynolds number in this range. The parameter α, which determines the sinus depth, was set to various values between ⅔ and 1. When $\alpha = ⅔$ the boundary of the sinus is a semicircle; when $\alpha = 1$ the boundary is flat so there is no sinus at all. It was checked that the results are independent of the numerical parameters by varying the time step and the distance between the sites of vortex creation. Agreement of the flow patterns was satisfactory as judged by the streamline plots; it was especially good during the early part of the run where the dramatic changes of the configuration of the flow occur.

The principal results are as follows (Fig. 3). At $t = 0^+$ the flow is potential flow and the streamlines go from $x = -\infty$ to $x = +\infty$. The streamlines that are close to the boundary follow the contours of the sinus and it is only at large values of y that the streamlines become essentially straight. This point is emphasized because there is a common misconception that the initial flow pattern in this situation consists of a uniform flow in the aorta with a state of rest in the sinus. Although such a flow pattern is, in fact, a solution of the steady inviscid equations of motion, there is no way to reach it instantaneously from a state of rest by the application of an impulsive pressure gradient.

When the flow is started abruptly, streamline separation occurs essentially instantaneously at the upstream and downstream borders of the sinus. The streamline that separates from the upstream border reattaches along the wall of the sinus. At first, the point of reattachment is near the point of separation, but the reattachment point moves rapidly downstream along the wall of the sinus until it encounters the downstream corner. The flow pattern then stabilizes with a vortex in the aortic sinus and a separating streamline that divides the sinus from the aorta. This streamline is nearly flat; it marks the open position of the aortic valve leaflet. After the flow pattern stabilizes in this way, less dramatic changes still occur. In particular, the sinus vortex becomes more diffuse and its center shifts away from the midline of the sinus in the downstream direction. Cinefilms of the motions of the point vortices on which the computations are based also reveal that the late flow pattern is not quite steady. Instead, puffs of vorticity are periodically carried away downstream in the aorta. Most of these phenomenons were also seen in the numerical computations of Gillani and Swanson (4).

When the flow is abruptly stopped, the sinus vortex begins to move in the upstream direction and sweeps out of the sinus into the aorta near the upstream corner. This result was clearly seen in the cinefilms showing the motion of the point vortices. This is precisely the effect predicted at the end of section 1; it has also been observed in the physical experiments of van Steenhoven et al. (10).

3. SUMMARY, CONCLUSIONS, AND FURTHER WORK

In this paper the aortic sinus vortex has been studied

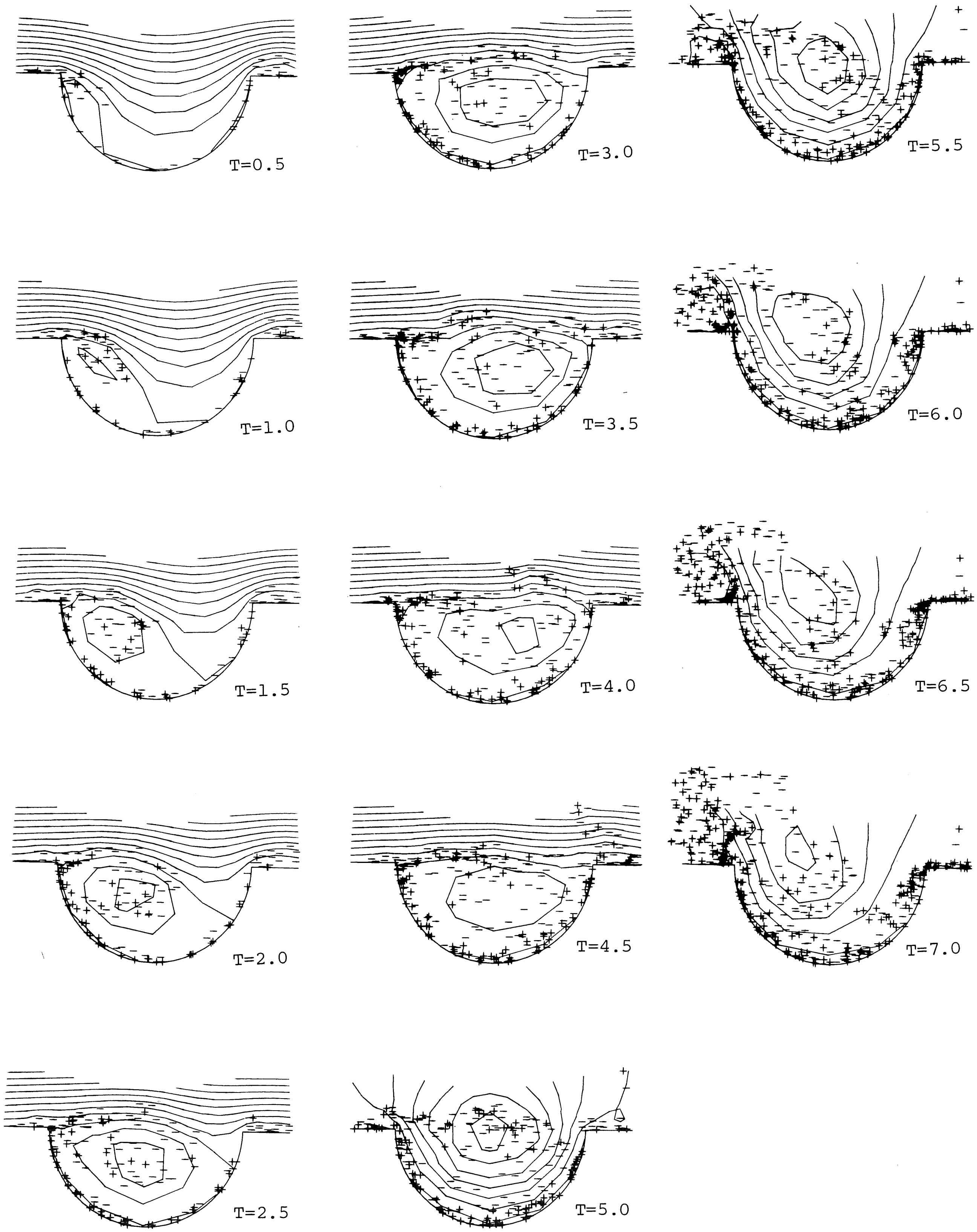


Figure 3. Vortex dynamics in the aortic sinus with fluid viscosity. The parameters are $\alpha = 2/3$, $R = 1{,}000$, $\Delta t = 0.1$, $h = \rho_0 = 0.1$. The fluid is initially at rest. Then $U(t) = 1$ for $0 < t < 5.0$ and $U(t) = 0$ for $t > 5.0$. Vortex positions are indicated by + or − according to the sign of K. Vortex formation occurs during $0 < t < 3.0$. The flow pattern is fairly steady during $3.0 < t < 5.0$ with a flat separating streamline that marks the position that would be assumed by the open valve leaflet. When the main flow is abruptly shut off at $t = 5.0$ the streamlines change so that the valve would be driven towards closure. During $t > 5.0$, the vortex migrates upstream and out of the sinus.

within the framework of point vortex dynamics. An isolated point vortex is a useful model of the aortic sinus vortex in several respects. We have used it to compute the equilibrium configuration of the sinus vortex with a free stream, to study the stability of this equilibrium configuration, and to make predictions about the mechanism of aortic valve closure. The point vortex model has severe limitations, however. It fails to predict that the separating streamline is essentially flat and that the sinus vortex is shifted downstream away from the midline of the sinus. Moreover, the model is inherently incapable of describing the process of formation of the aortic sinus vortex. All of these difficulties are related to the absence of viscosity in the point vortex model.

Fortunately, a numerical method is available which makes it possible to include viscosity in the study of vortex dynamics. This is the vortex method of A. J. Chorin (2). In this paper, we present a new variant of this method in which an explicit formula for the vortex velocities is derived by conformal mapping. This avoids the numerical solution of Laplace's equation at every time step. Our variant of the method is especially useful when the domain has corners, a situation that arises in many applications. The numerical results show that the deficiencies of the point vortex model have been overcome, and they confirm the qualitative predictions of that model with respect to the dynamics of aortic valve closure.

Two major difficulties remain. First, this work has been done entirely within the context of two-dimensional fluid dynamics. This approach may be questioned on the grounds that Bellhouse and Talbot (1) have observed an extremely interesting three-dimensional flow pattern in which fluid enters the sinus along the plane of symmetry of each aortic leaflet and leaves the sinus near the commissures after being swept around the vortex. It is not yet known whether this phenomenon is fundamental to the fluid dynamics of the aortic valve. One way to find out is to see whether a two-dimensional model can predict the detailed behavior of the aortic valve in various circumstances.

The second major difficulty is that the valve itself has been omitted from the computational work described in this paper. We are currently involved in the effort of including a flexible aortic leaflet which interacts with the fluid as an immersed boundary. Briefly, the method involves the computation of the boundary forces as in Peskin (8). The curl of the singular field of boundary forces is then used as a source of vorticity along the immersed boundary. This idea has been used successfully by Mendez (6) and by McCracken and Peskin (5).

With the valve in place, we should be able to comment on the role of the aortic leaflet in the formation of the sinus vortex and to define more precisely the role of the vortex in controlling the motions of the leaflet. FP

It goes without saying that the pioneering development of the vortex method by Alexandre Chorin has made the present work possible. We are also indebted to Gary Strumolo for pointing out the usefulness of Routh's stream function in the study of vortex trajectories.

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