A gradient flow approach to an evolution problem arising in superconductivity *

Luigi Ambrosio †
Scuola Normale Superiore
Pisa

Sylvia Serfaty ‡
Courant Institute
New York

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Abstract

We study an evolution equation proposed by Chapman-Rubinstein-Schatzman as a mean-field model for the evolution of the vortex-density in a superconductor. We treat the case of a bounded domain where vortices can exit or enter the domain. We show that the equation can be derived rigorously as the gradient-flow of some specific energy for the Riemannian structure induced by the Wasserstein distance on probability measures. This leads us to some existence and uniqueness results and energy-dissipation identities. We also exhibit some "entropies" which decrease through the flow and allow to get regularity results (solutions starting in L^p $(p > 1)$ remain in L^p).

1 Introduction

1.1 Presentation of the problem

We are interested in studying the following "mean-field model" (also called hydrodynamic limit) for superconductivity which was derived formally by Chapman, Rubinstein and Schatzman in [CRS] (see also E [E]):

$$\frac{d}{dt}\mu(t) - \mathrm{div}\ (\nabla h_{\mu(t)}|\mu(t)|) = 0 \quad \text{in } \Omega, \tag{1}$$

where for all times μ and h_μ are coupled through the relation

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

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†l.ambrosio@sns.it

‡serfaty@nyu.edu, supported by NSF CAREER award #DMS 0239121 and a Sloan Foundation Fellowship

Type-II superconductors, submitted to an external field (here normalized to be of intensity 1), have a very particular response: they "repel" the applied field, which only penetrates through "vortices". In the equations above, μ represents the suitably normalized density of vortices: a priori it should be a signed measure, but here we restrict ourselves to positive measures. The function h_μ represents the intensity of the induced magnetic field in the sample. This function can be seen as a potential generated by the vortices through the "London equation" (2). Finally, the domain Ω is a smooth bounded domain of $\mathbb{R}^2$, corresponding to a section of the superconducting material.

Several problems, which we will address later, appear in the formulation (1). First, the equation does not always have a meaning since, when μ is only a measure, the function ∇h_μ is not continuous in general and therefore the product $\mu\nabla h_\mu$ is not well defined. Second, the conditions that should be imposed on $\partial\Omega$, and whether or not the total mass in Ω should be conserved, are not very clear.

1.2 The equation as a gradient-flow

In this paper we are interested in deriving this equation as a gradient-flow of the energies

$$\Phi_\lambda(\mu) = \frac{\lambda}{2}\mu(\Omega) + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2, \tag{3}$$

where λ is a nonnegative parameter. More precisely, we observe that (1) is, at least formally, the gradient-flow of Φ_λ, with respect to the Riemannian structure on probability measures induced by the Wasserstein distance. The introduction of this Riemannian structure goes back to the seminal paper of Otto [O1], and later on it was extended in many different directions and made rigorous by Ambrosio-Gigli-Savaré in [AGS] (see also [CMV]), see §1.5 below for a short presentation of these ideas.

The motivation for looking at this specific class of energies is twofold. First, Φ_λ are directly connected to the full Ginzburg-Landau model of superconductivity: they were derived by Sandier-Serfaty [SS1] (see also [SS3], Chapter 7) as the Γ-limit of the Ginzburg-Landau energy

$$G_\varepsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |\mathrm{curl}\, A - h_{\mathrm{ex}}|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}, \tag{4}$$

as $\varepsilon \to 0$. More precisely, the Γ-limit that was derived was

$$\frac{\lambda}{2}|\mu|(\Omega) + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2$$

which reduces to Φ_λ for positive measures. Here λ equals $\lim_{\varepsilon\to 0} |\log\varepsilon|/h_{\mathrm{ex}}$, the parameter h_{ex} being the intensity of the applied magnetic field, and μ is the limit of the suitably normalized vortex-density of the complex function u. Another link to the Ginzburg-Landau equations was established for steady-states: it was shown in [SS2] (see also [SS3], Chapter

13) that critical points of G_ε have vortex-densities whose limits μ as $\varepsilon \to 0$ are (a weak version of) stationary solutions of (1), formally satisfying $\nabla h_\mu \mu = 0$.

Therefore, understanding the evolution equation (1) in the gradient flow framework provides a good basis for deriving it from the full time-dependent Ginzburg-Landau equation, gradient flow of G_ε. The gradient flow approach also gives, as a byproduct, many additional results that do not seem immediate at a purely PDE level: they concern, for instance, energy dissipation identities and entropies associated to the evolution problem. Furthermore, there might be many solutions to the PDE (1), corresponding to different possible choices of boundary conditions. Our gradient flow approach selects a steepest-descent solution and leads to a natural choice of boundary conditions, arising from the limit of the implicit Euler scheme.

The second main motivation is that this problem provides an interesting application of the framework developed in [AGS] for gradient flows in the Wasserstein spaces. In particular, this problem is technically harder than most of the ones already studied (see for instance [JKO], [O1], [O2]) for two reasons: first, because the restriction to the case of absolutely continuous (with respect to the Lebesgue measure) measures is not natural (since the natural energy space for the problem is the Sobolev space H^{-1} dual of $H^1_0(\Omega)$, and since mass might concentrate on $\partial\Omega$); second, because even with this restriction the energy Φ_λ is *not* α-displacement convex in the sense of McCann (see [MC] and the discussion in Section 1.5). Therefore, all the results developed in [AGS] and [CMV] for α-displacement convex functionals do not apply to our case, although we will borrow many technical tools from [AGS] to study the gradient flow of Φ_λ.

1.3 Previous studies of the equation

Following the work of Chapman, Rubinstein and Schatzman and E, there were quite many papers about the equation (1). Notably Schätzle and Styles [SSt] proved the existence of solutions with a Dirichlet boundary data $\mu = 0$ on $\partial\Omega$, via a vanishing viscosity method. Also Styles gave a numerical scheme for the approximation of these solutions. Later, Lin-Zhang [LZ] studied a very analogous equation but in the whole plane $\mathbb{R}^2$:

$$\frac{d}{dt}\mu(t) + \mathrm{div}\ (\mu(t)\nabla\Delta^{-1}\mu(t)) = 0 \quad \text{in } \mathcal{D}'((0,+\infty)\times\mathbb{R}^2). \tag{5}$$

Observe that this equation (as well as (1)) has a clear analogy with the vorticity formulation of the incompressible Euler equation: here the velocity field along which the measure is transported is $\nabla\Delta^{-1}\mu$ instead of the perpendicular one for Euler $\nabla^\perp\Delta^{-1}\mu$. As a result (5) is dissipative, instead of being conservative. Lin and Zhang proved, using a vortex-blob method and pseudo-differential operators, the existence of weak solutions, and existence and uniqueness in the class of L^∞ solutions. Their result was generalized by Du-Zhang [DZ] to velocity fields that are combinations of $\nabla\Delta^{-1}\mu$ and $\nabla^\perp\Delta^{-1}\mu$.

Poupaud studied in [Po] nonnegative solutions to the PDE

$$\frac{d}{dt}\mu(t) + \mathrm{div}\ (v(t)\mu(t)) = 0 \quad \text{in } \mathcal{D}'((0,+\infty)\times\mathbb{R}^2)$$

with $v(t) := -\nabla\Delta^{-1}\mu(t)$. Notice the change of sign with respect to (5): formally, since

$$\frac{d}{dt}\mu(t) + v(t)\cdot\nabla\mu(t) = -\mu(t)\mathrm{div}\; v(t) = \mu^2(t), \tag{6}$$

this PDE leads to blow-up in finite time and concentration phenomena, contrarily to (5). For analogous reasons, in our case, for nonnegative solutions, blow-up is not expected.

More recently, the paper of Masmoudi and Zhang [MZ] studied the appropriate analogue of (5), but with signed measures:

$$\frac{d}{dt}\mu(t) + \mathrm{div}\;(\nabla\Delta^{-1}\mu(t)|\mu(t)|) = 0 \qquad \text{in } \mathcal{D}'((0,+\infty)\times\mathbb{R}^2). \tag{7}$$

They show that solutions can blow up in finite time, and thus that distributional solutions cannot always be found. In turn, they prove the existence of global renormalized solutions in the sense of Lions.

Observe that working on the whole plane prevents the use of the energy approach and also suppresses the problems related to the boundary and the possibility of mass exiting or entering the domain, all issues that we would like to address.

1.4 Weak formulation

In order to make sense of the product $\mu\nabla h_\mu$ appearing in (1), we use the "stress-energy tensor"

$$\mathbf{T}\mu := \frac{1}{2}\begin{pmatrix} (\partial_1 h_\mu)^2 - (\partial_2 h_\mu)^2 - h_\mu^2 & 2\partial_1 h_\mu \partial_2 h_\mu \\ 2\partial_1 h_\mu\partial_2 h_\mu & (\partial_2 h_\mu)^2 - (\partial_1 h_\mu)^2 - h_\mu^2 \end{pmatrix}$$

Then we observe that, defining div $\mathbf{T}\mu$ as the vector $\left(\sum_{i=1}^2 \partial_i(\mathbf{T}\mu)_{i1}, \sum_{i=1}^2 \partial_i(\mathbf{T}\mu)_{i2}\right)$, we have, provided $\mu \in L^{4/3}(\Omega)$, that

$$\mathrm{div}\; \mathbf{T}\mu = -\mu\nabla h_\mu.$$

On the other hand, div $\mathbf{T}\mu$ is well defined in the sense of distributions for every measure μ such that $\Phi_\lambda(\mu) < \infty$ (i.e. every measure belonging to H^{-1}). So, an appropriate weak formulation of (1) will be

$$\frac{d}{dt}\mu(t) + \mathrm{div}\,(\mathrm{div}\; \mathbf{T}\mu(t)) = 0 \tag{8}$$

(this is also the weak formulation used in [LZ, DZ], with a slightly different tensor, due to the fact that the velocity is $\nabla\Delta^{-1}\mu$ in their case). The apparition of the stress-energy tensor is also quite natural from the Ginzburg-Landau viewpoint: in [SS2], the authors passed to the limit in the stress-energy tensor associated to G_ε to get div $\mathbf{T}\mu = 0$ for limiting vorticity measures μ of critical points of G_ε.

Moreover, thanks to Delort's theorem [De] (a theorem established in order to pass to the limit in weak solutions to Euler's equation), although the entries of $\mathbf{T}\mu$ are nonlinear functions of μ, the tensor $\mathbf{T}\mu$ is stable under weak convergence of positive measures, which will allow us to pass to the limit in approximation schemes even when dealing with data in H^{-1}.

1.5 The Wasserstein space of probability measures

The formal Riemannian structure introduced by Otto is very appropriate to study a large class of evolution PDE's, including Fokker-Planck equations and porous medium equations, see [AGS], [Vi] and the references therein. Let us recall briefly the main features of this structure.

First, in order to deal with possibly varying mass in Ω, we will consider measures over $\overline{\Omega}$. We may also assume without loss of generality that the measures are probabilities over $\overline{\Omega}$. We shall denote by $P(\overline{\Omega})$ these probability measures.

Given μ, $\nu \in P(\overline{\Omega})$ we define the set of admissible plans $\Gamma(\mu,\nu)$ as the set of probability measures γ in $\overline{\Omega} \times \overline{\Omega}$ whose marginals are μ and ν. The 2-Wasserstein distance is then defined by

$$W_2(\mu,\nu) := \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2 \, d\gamma(x,y) \right)^{1/2}.$$

The minimum is achieved for *optimal plans*. It is also well-known since the work of Brenier, that if μ is absolutely continuous with respect to the Lebesgue measure, there exists a unique optimal plan, induced by a map $\mathbf{t}$ (formally, $\gamma(x,y) = \mu(x)\delta_{\mathbf{t}(x)}(y)$). The map $\mathbf{t}$ solves the optimal transport problem in Monge's original formulation:

$$\inf \left\{ \left(\int_{\overline{\Omega}} |\mathbf{s}(x) - x|^2 \, d\mu(x) \right)^{1/2} \; : \; \mathbf{s}_{\#}\mu = \nu \right\},$$

where $\mathbf{s}_{\#}\mu = \nu$ denotes the push-forward of μ by $\mathbf{s}$ i.e. $\mu(\mathbf{s}^{-1}(A)) = \nu(A)$ for all Borel sets A.

Let us now describe why our equation is formally the gradient flow of (3) for the Wasserstein structure on probability measures.

Assume here for simplicity that the mass $\mu(\Omega)$ is constant along the flow and equal to 1. This way, we can reduce ourselves to probability measures in Ω. Following [AGS], absolutely continuous curves $\mu(t) : [0,T] \to P(\Omega)$ are characterized by the continuity equation

$$\partial_t \mu(t) + \operatorname{div}(v(t)\mu(t)) = 0 \tag{9}$$

and the integrability condition $\|v(t)\|_{L^2(\mu(t))} \in L^1(0,T)$. Different PDE's arise, choosing different relations between velocity $v(t)$ at time t and density $\mu(t)$; in the case of the gradient flow of a functional $\Phi : P(\Omega) \to \mathbb{R} \cup \{+\infty\}$, the coupling is $v(t) = -\nabla^W \Phi(\mu(t))$, where the "Wasserstein gradient" (strictly speaking, a subdifferential) $\nabla^W\Phi(\mu)$ is the vector $\xi \in L^2(\mu;\mathbb{R}^2)$ defined by the subdifferential relation

$$\Phi(\mathbf{s}_{\#}\mu) - \Phi(\mu) \geq \int \xi \cdot (\mathbf{s} - I)\, d\mu + o(\|\mathbf{s} - I\|^2_{L^2(\mu)}).$$

So, the basic difference with respect to the conventional theory of gradient flows is that the gradient of the functional (or, better, its subdifferential) has to be computed by differentiating Φ along transport maps.

Let us particularize this to our functional (3). Using the representation (see Proposition 2.1)

$$\Phi_\lambda(\mu) = \frac{1}{2}(\lambda\mu(\Omega) + |\Omega|) + \sup_{h-1\in H^1_0(\Omega)} \left\{ \int_\Omega (h-1)\, d\mu - \frac{1}{2}\int_\Omega |\nabla h|^2 + |h|^2 \right\} \tag{10}$$

it is easy to get

$$\begin{aligned} \Phi_\lambda(\mathbf{s}_{\#}\mu) - \Phi_\lambda(\mu) &\geq \int_\Omega (h_\mu - 1)(\mathbf{s}_{\#}\mu - \mu) \\ &= \int_\Omega (h_\mu(\mathbf{s}(x)) - h_\mu(x))\, d\mu \sim \int_\Omega \nabla h_\mu(x) \cdot (\mathbf{s}(x) - x)\, d\mu \end{aligned}$$

as $\|\mathbf{s} - I\|_{L^2(\mu)} \to 0$. Hence, the formal Wasserstein gradient of Φ_λ at μ is ∇h_μ, and the associated gradient flow is

$$\partial_t \mu(t) - \operatorname{div}(\nabla h_{\mu(t)}\mu(t)) = 0,$$

which is our equation.

The representation (10) also tells us how far Φ_λ is from being α-displacement convex in McCann's sense [MC]. Indeed, the functionals $\mu \mapsto \int V\, d\mu$ have this property only if $D^2V \geq \alpha I$ and in our case, having $V = h_\mu - 1$, we would need an L^∞ bound on second derivatives of h_μ. It is well known that even though $\mu \in L^\infty$, only L^p bounds with $p < \infty$ are available on $D^2 h_\mu$ from (2). Therefore we cannot expect α-convexity of Φ_λ. However, this property barely fails, and we can still adapt in Theorem 3.2 some tools from [AGS] to show uniqueness of L^∞ solutions inside Ω, valid whenever no mass reaches $\partial\Omega$.

1.6 The time-discretization scheme and sketch of results

We establish existence of solutions to (1) via a classical time-discretization scheme (see [JKO] for its first use in a Wasserstein framework): the method consists in taking the initial data $\mu^0_\tau = \bar{\mu} \in P(\overline{\Omega}) \cap H^{-1}(\Omega)$, and minimizing recursively

$$\min_{\nu \in P(\overline{\Omega})} \Phi_\lambda(\nu) + \frac{1}{2\tau} W_2^2(\mu^k_\tau, \nu). \tag{11}$$

The first difficulty arising is that μ does not have enough regularity to write down an Euler equation for this minimization problem. To remedy this, we regularize the problem by adding a $\delta \int_\Omega \nu^4$ term to the energy to be minimized. We then find an Euler-Lagrange equation in which we can pass to the limit thanks to the weak continuity of the stress-energy tensor (Delort's theorem). We also find several qualitative properties of the minimizers μ_τ^k: passing from μ_τ^k to μ_τ^{k+1}, no mass is brought in from $\partial\Omega$ to Ω (for $\lambda > 0$), and all the mass that leaves Ω does so along the shortest path connecting to $\partial\Omega$. Moreover, we exhibit a family of "entropy functions" φ which are such that

$$\int_\Omega \varphi(\mu_\tau^{k+1}) \leq \int_\Omega \varphi(\mu_\tau^k), \tag{12}$$

see Proposition 5.4. The entropies can be chosen to grow like x^p, and this allows to show that if μ is initially in $L^p(\Omega)$, with $1 < p \leq \infty$, then μ_τ^k and their limits as $\tau \downarrow 0$ stay in the same class.

Once the sequence μ_τ^k is built, we interpolate in time by $\mu_\tau(t) = \mu_\tau^k$ in $[k\tau, (k+1)\tau]$. In a final step we take the limit $\tau \to 0$ to find an extracted limit $\mu(t)$ which formally corresponds to a solution of the gradient flow of Φ_λ. A general result from [AGS] (see Theorem 6.1 here) ensures that we can derive a continuity equation at the limit:

$$\frac{d}{dt}\mu(t) + \operatorname{div}(v(t)\mu(t)) = 0 \quad \text{in } \mathcal{D}'((0,+\infty) \times \mathbb{R}^2) \tag{13}$$

where the limit velocity field $v(t)$ can be computed taking suitable limits of the discrete velocities appearing in the Euler-Lagrange equation of (11). In our case, for general H^{-1} initial conditions, we prove in Theorem 7.1 that

$$v(t)\mu(t) = \begin{cases} \operatorname{div} \mathbf{T}\mu(t) & \text{in } \Omega \\ [\operatorname{div} \mathbf{T}\mu(t)]^{\tan} & \text{on } \partial\Omega \end{cases}$$

In addition, the discrete scheme yields suitable energy-dissipation inequalities, and decrease of the entropies along the flow.

We present here our existence result in the case when the initial datum $\bar\mu$ belongs to L^p, $p \geq 4/3$: in this case we can really obtain a solution of (1), without using the stress-energy tensor. In the statement of this theorem and in the sequel we use the following notation: given a measure μ in $\overline{\Omega}$ (or in $\mathbb{R}^2$, supported in $\overline{\Omega}$), we denote by $\hat\mu$ its "internal" part (i.e. $\hat\mu = \chi_\Omega \mu$) and by $\tilde\mu = \mu - \hat\mu$ its "boundary" part. Sometimes, with a slight abuse of notation, we shall identify $\hat\mu$ with its density with respect to the Lebesgue measure whenever $\hat\mu$ is absolutely continuous.

Theorem 1.1 (Initial condition in L^p, $p \geq 4/3$, and $\lambda \geq 0$). *Assume that $\hat{\bar\mu} \in L^p$ for some $p \geq 4/3$ and $\lambda \geq 0$. Then there exists a weakly continuous map $\mu(t) : [0,+\infty) \to P(\overline{\Omega})$ such that:*

(a) $\|\hat\mu(t)\|_p \leq C$, with C depending only on $\hat{\bar\mu}$ (and not on t, λ);

(b) $\mu(0) = \bar\mu$ and the PDE

$$\frac{d}{dt}\mu(t) - \operatorname{div}(\nabla h_{\mu(t)}\hat\mu(t)) = 0 \qquad \text{in } \mathcal{D}'((0,+\infty) \times \mathbb{R}^2) \tag{14}$$

holds;

(c) $t \mapsto \tilde{\mu}(t)$ *is nondecreasing as a measure-valued map and the energy-dissipation inequality*

$$\Phi_\lambda(\mu(t)) + \int_s^t \int_\Omega |\nabla h_{\mu(\tau)}|^2 \, d\mu(\tau)\, d\tau \le \Phi_\lambda(\mu(s)) \qquad t \ge s \ge 0 \tag{15}$$

holds;

(d) if $p = \infty$ *then* $\tilde{\mu}(t) - \tilde{\mu}(0) = f(t,\cdot)\sigma_{\partial\Omega}$ *with* $\|f(t,\cdot)\|_\infty \le Ct$ *(here* $\sigma_{\partial\Omega}$ *is the length measure on* $\partial\Omega$*).*

Furthermore, the inequality (c) is an equality if $\lambda = 0$ *and* $p \ge 3/2$.

The paper is organized as follows: in §2 we introduce our main notation and some preliminary results, and in §3 we present a short-time existence result for our problem, in the case when the initial condition $\bar{\mu}$ satisfies $\hat{\bar{\mu}} \in L^\infty$: even though Theorem 1.1 provides a much stronger result, the proof of short-time existence (based on an explicit, rather than implicit, Euler scheme and on a Lagrangian formulation) is much simpler. In the same section we also present a short-time uniqueness result, based on an adaptation, in the Wasserstein framework, of Yudovitch's [Yu1, Yu2] uniqueness result for $2d$-incompressible Euler equations: notice that uniqueness of solutions $\mu(t)$ in Theorem 1.1 is an open problem, even for $p = \infty$.

In §4 we study the properties of the stress-energy tensor $\mathbf{T}\mu$ and the weak formulation of (1), while §5 contains a detailed analysis of the Euler-Lagrange equation associated to (11) and a proof that the entropies decrease along the discrete scheme as in (12).

In §6 and §7 we pass to the limit as $\tau \downarrow 0$ to obtain existence of a solution $\mu(t)$ for the most general case of initial data $\mu(0) \in H^{-1}(\Omega)$ (see Theorem 7.1). Finally, in §8 we discuss some open problems and exhibit a "stationary point" $\mu \in H^{-1}$ of Φ_λ which seems to be a good candidate for nonuniqueness.

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2 Notation and preliminary results

2.1 Wasserstein distance

Throughout this paper, Ω denotes a bounded open set in $\mathbb{R}^2$ with smooth boundary, whose unit outer normal will be denoted by $\mathbf{n}_\Omega$ and unit tangent (with suitable orientation) by τ_Ω. We denote by $\mathcal{M}_+(\Omega)$ the space of finite and nonnegative Radon measures in Ω. When talking of convergence in $\mathcal{M}_+(\Omega)$ we shall always mean convergence in the duality with continuous compactly supported functions in Ω. Similarly, we denote by $\mathcal{M}_+(\overline{\Omega})$

(resp. $P(\overline{\Omega})$) the space of finite and nonnegative Radon measures in $\overline{\Omega}$ (resp. probability measures) in $\overline{\Omega}$, sometimes thought of as probability measures in $\mathbb{R}^2$ concentrated in $\overline{\Omega}$, and when talking of convergence in $\mathcal{M}_+(\overline{\Omega})$ we shall always mean convergence in the duality with bounded continuous functions in $\overline{\Omega}$.

It is well known ([AGS], [Vi]) that convergence in $P(\overline{\Omega})$ is induced by the 2-Wasserstein distance W_2, defined as follows: for given μ, $\nu \in P(\overline{\Omega})$, let us denote by $\Gamma(\mu,\nu)$ the set of probability measures γ in $\overline{\Omega} \times \overline{\Omega}$ whose marginals are μ and ν, i.e.

$$\gamma(A \times \overline{\Omega}) = \mu(A), \qquad \gamma(\overline{\Omega} \times B) = \nu(B)$$

for any choice of Borel sets A, $B \subset \overline{\Omega}$. The elements of $\Gamma(\mu,\nu)$ are called *admissible plans.* Then, the Wasserstein distance is defined by

$$W_2(\mu,\nu) := \inf \left\{ \left(\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2 \, d\gamma \right)^{1/2} \ : \ \gamma \in \Gamma(\mu,\nu) \right\}.$$

It is also well-known that the infimum in the definition above is a minimum, and minimizing γ's are called *optimal plans.* We shall denote by $\Gamma_0(\mu,\nu)$ the class of optimal plans. Recall that when μ is absolutely continuous with respect to the Lebesgue measure, $\Gamma_0(\mu,\nu)$ contains only an element γ induced by a map $\mathbf{t}$: this map satisfies the transport condition $\mathbf{t}_\#\mu = \nu$ (i.e. $\mu(\mathbf{t}^{-1}(A)) = \nu(A)$ for all Borel sets A) and is a solution of the optimal transport problem in Monge's original formulation:

$$\inf \left\{ \left(\int_{\overline{\Omega}} |\mathbf{s}(x) - x|^2 \, d\mu(x) \right)^{1/2} \ : \ \mathbf{s}_\#\mu = \nu \right\}.$$

We extend this notion naturally to the case where μ and ν are just positive measures with same mass.

We will also use the following stability property of optimal transport plans, see for instance Proposition 7.1.3 in [AGS]: if $\mu_n \to \mu$ and $\nu_n \to \nu$ in $P(\overline{\Omega})$, then

$$\gamma_n \in \Gamma_0(\mu_n,\nu_n), \ \gamma_n \to \gamma \text{ in } P(\overline{\Omega}\times\overline{\Omega}) \qquad \Longrightarrow \qquad \gamma \in \Gamma_0(\mu,\nu). \tag{16}$$

Notice that $W_2^2(\mu,\cdot)$ is convex in $P(\overline{\Omega})$: indeed, let ν_1 and ν_2 be in $P(\overline{\Omega})$, and let $\gamma_1 \in \Gamma_0(\mu,\nu_1)$ and $\gamma_2 \in \Gamma_0(\mu,\nu_2)$. It is easy to check that $(1-t)\gamma_1 + t\gamma_2 \in \Gamma(\mu,(1-t)\nu_1 + t\nu_2)$ for all $t \in [0,1]$, so that

$$\begin{aligned} W_2^2(\mu,(1-t)\nu_1 + t\nu_2) &\leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2 \, d((1-t)\gamma_1 + t\gamma_2) \\ &= (1-t)W_2^2(\mu,\nu_1) + tW_2^2(\mu,\nu_2). \end{aligned}$$

2.2 First properties of the energy

Definition 2.1 (The energy functional). Let $\lambda \geq 0$. We consider the functional

$$\Phi_\lambda(\mu) := \frac{\lambda}{2}\mu(\Omega) + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2, \tag{17}$$

where h_μ is the solution of the elliptic PDE

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

So the natural domain of Φ_λ is $\mathcal{M}_+(\Omega) \cap H^{-1}(\Omega)$, and we may think of extending Φ_λ by the value $+\infty$ to $\mathcal{M}_+(\Omega) \setminus H^{-1}(\Omega)$. Unless otherwise stated, we always think of h_μ as an H^1_{loc} function on the whole of $\mathbb{R}^2$, extended with the value 1 outside of Ω.

Lemma 2.1 (Convexity and lower semicontinuity of Φ_λ). *We have*

$$\Phi_\lambda(\mu) = \frac{\lambda}{2}\mu(\Omega) + \frac{1}{2}\int_\Omega |\nabla w_\mu|^2 + w_\mu^2 + \int_\Omega (h_0 - 1)\, d\mu + C_0,$$

where C_0 is an explicit constant, h_0 a smooth function, and w_μ is the solution to

$$\begin{cases} -\Delta w_\mu + w_\mu = \mu & \text{in } \Omega \\ w_\mu = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

Consequently, Φ_λ is strictly convex and lower semicontinuous along bounded sequences converging in $\mathcal{M}_+(\Omega)$.

Proof. Introducing h_0 as the solution to (18) for $\mu = 0$, we have $h_\mu = w_\mu + h_0$, where w_μ is the solution of (19). Hence, we have

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2 &= \frac{1}{2}\int_\Omega |\nabla h_0|^2 + |h_0 - 1|^2 + \frac{1}{2}\int_\Omega |\nabla w_\mu|^2 + w_\mu^2 \\ &\quad + \int_\Omega \nabla w_\mu \cdot \nabla(h_0 - 1) + w_\mu(h_0 - 1) \\ &= C_0 + \frac{1}{2}\int_\Omega |\nabla w_\mu|^2 + w_\mu^2 + \int_\Omega (h_0 - 1)(-\Delta w_\mu + w_\mu), \end{aligned}$$

from which the first assertion follows. The strict convexity of Φ_λ directly follows from this representation. Finally, lower semicontinuity is easy to prove: if $\mu_n \to \mu$ in $\mathcal{M}_+(\Omega)$ and both $\mu_n(\Omega)$ and $\Phi_\lambda(\mu_n)$ are bounded sequences, then we can assume with no loss of generality that w_{μ_n} weakly converge in $H^1_0(\Omega)$ to some function w. A simple truncation argument, based on the fact that $\mu_n(\Omega)$ are equi-bounded, shows that

$$\lim_{n\to\infty} \int_\Omega f\, d\mu_n = \int_\Omega f\, d\mu. \tag{20}$$

for any $f \in C(\overline{\Omega})$ vanishing on $\partial\Omega$ so in particular for $f = h_0 - 1$.

The linearity of (19) gives $w = w_\mu$, so lower semicontinuity of Φ_λ reduces to the lower semicontinuity of the H^1 norm and of $\mu \mapsto \mu(\Omega)$. $\square$

Next, we want to analyze the convexity properties of Φ_λ with respect to the Wasserstein structure.

Proposition 2.1. *For all $\mu \in \mathcal{M}_+(\Omega)$ we have*

$$\Phi_0(\mu) - \frac{1}{2}|\Omega| = \sup_{h-1 \in H_0^1(\Omega)} \left\{ \int_\Omega (h-1)\, d\mu - \frac{1}{2} \int_\Omega |\nabla h|^2 + |h|^2 \right\}. \tag{21}$$

Proof. Standard direct methods imply that the supremum is attained, and that the maximizer h is unique. The first variation then gives

$$\int_\Omega \varphi\, d\mu - \int_\Omega \nabla h \cdot \nabla \varphi + h\varphi = 0 \qquad \forall \varphi \in C_c^\infty(\Omega),$$

and therefore $h = h_\mu$. Finally, we can use the identity

$$\int_\Omega (h_\mu - 1)\, d\mu = \int_\Omega (h_\mu - 1)(-\Delta h_\mu + h_\mu) = \int_\Omega h_\mu^2 - h_\mu + |\nabla h_\mu|^2$$

to obtain (21). □

Finally, let us point out a useful "monotonicity" property of Φ_λ.

Proposition 2.2. *For all μ, $\nu \in \mathcal{M}_+(\Omega)$ we have*

$$\Phi_\lambda(\mu) - \frac{\lambda}{2}\mu(\Omega) \geq \Phi_\lambda(\nu) - \frac{\lambda}{2}\nu(\Omega) + \int_\Omega (h_\nu - 1)\, d(\mu - \nu).$$

In particular, if $\nu \leq \mu$ we have

$$\Phi_\lambda(\nu) \leq \Phi_\lambda(\mu) - \int_\Omega (h_0 - 1 + \frac{\lambda}{2})\, d(\mu - \nu). \tag{22}$$

Proof. The first stated inequality follows combining the equality

$$\Phi_\lambda(\nu) - \frac{\lambda}{2}\nu(\Omega) - \frac{1}{2}|\Omega| = \int_\Omega (h_\nu - 1)\, d\nu - \frac{1}{2}\int_\Omega |\nabla h_\nu|^2 + |h_\nu|^2$$

and the inequality

$$\Phi_\lambda(\mu) - \frac{\lambda}{2}\mu(\Omega) - \frac{1}{2}|\Omega| \geq \int_\Omega (h_\nu - 1)\, d\mu - \frac{1}{2}\int_\Omega |\nabla h_\nu|^2 + |h_\nu|^2,$$

that both follow from Proposition 2.1 and its proof. The second stated inequality follows from the first one, simply using the fact that $h_\nu = h_0 + w_\nu$, with w_ν as in (19), and w_ν is nonnegative. □

3 Some existence and uniqueness results on L^p solutions

In this section we present some results on L^p solutions to our evolution problem which are independent of the Wasserstein gradient flow approach. Notice that if $\hat\mu \in L^p$ for some $p \in (1,2)$, by standard elliptic regularity theory and Sobolev embedding we get $\nabla h_\mu \in L^{2p/(2-p)}$, with (thanks to the linear dependence of h_μ on $\mu - 1$)

$$\|\nabla h_\mu\|_{2p/(2-p)} \le C\|\hat\mu - 1\|_p. \tag{23}$$

In particular, for $p \in [4/3, 2)$, we have

$$\|\nabla h_\mu \hat\mu\|_{2p/(4-p)} \le C\|\hat\mu - 1\|_p \|\hat\mu\|_p. \tag{24}$$

Motivated by the heuristics in Section 1.5, we give the following definition:

Definition 3.1 (Gradient flow in L^p). *Let $T > 0$ and $\mu(t) \in P(\overline{\Omega})$ with $\|\hat\mu(t)\|_{3/2} \in L^q(0,T)$ for some $q \ge 3$. We say that $\mu(t)$ is a gradient flow of Φ_λ if*

$$\frac{d}{dt}\mu(t) - \operatorname{div}\left(\nabla h_{\mu(t)}\hat\mu(t)\right) = 0 \qquad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^2) \tag{25}$$

and $t \mapsto \tilde\mu(t)$ is nondecreasing as a measure-valued map (where we recall $\tilde\mu$ is the boundary part of μ).

Observe that in this definition the velocity field is $\nabla h_{\mu(t)}\chi_\Omega$, hence is 0 on $\partial\Omega$. This fact, and that $t \mapsto \tilde\mu(t)$ is monotone are motivated, at least when $\lambda > 0$, by Proposition 2.2: indeed, if $\mu = \nu + \sigma$ with $\sigma \ge 0$ supported in a small neighbourhood (depending on λ) of $\partial\Omega$, then (22) gives $\Phi_\lambda(\nu) \le \Phi_\lambda(\nu+\sigma)$, with equality only if $\sigma = 0$. Therefore, while some leaking of mass to $\partial\Omega$ might be energetically favourable, the entrance of mass (at least in a continuous way) is not. Our formulation of the PDE corresponds to the idea that mass can reach the boundary from the inside, and then stop.

From (25) and (24) with $p = 3/2$ we get a useful formula for the increments of $t \mapsto \int \phi \, d\mu(t)$:

$$\begin{aligned} \left|\int_{\overline{\Omega}} \phi \, d(\mu(s) - \mu(t))\right| &= \left|\int_s^t \int_\Omega \nabla h_{\mu(\tau)} \cdot \nabla\phi \, d\mu(\tau) d\tau\right| \\ &\le C\|\nabla\phi\|_6 \left|\int_s^t \|\hat\mu(\tau)\|_{3/2}\|\hat\mu(\tau) - 1\|_{3/2} \, d\tau\right|. \end{aligned} \tag{26}$$

Our terminology is also justified by the following result, showing that $t \mapsto \Phi_0(\mu(t))$ is nonincreasing along the flow, and an explicit formula for its derivative. Since $t \mapsto \lambda\mu(\Omega)$ is also nonincreasing along the flow (by the monotonicity of $t \mapsto \tilde\mu(t)$) we also obtain that the whole energy $\Phi_\lambda(\mu(t))$ is nonincreasing.

Proposition 3.1 (Energy dissipation identity). *Let $\mu(t) : [0,T] \to P(\overline{\Omega})$ be as in Definition 3.1. Then $\Phi_0(\mu(t)) \in W^{1,q/3}(0,T)$, and*

$$\frac{d}{dt}\Phi_0(\mu(t)) = -\int_\Omega |\nabla h_{\mu(t)}|^2\, d\mu(t) \qquad \text{for a.e. } t. \tag{27}$$

Proof. We will use the following criterion for absolute continuity: if $f : [0,T] \to \mathbb{R}$ is a continuous map satisfying

$$|f(s) - f(t)| \leq (g(s) + g(t)) \left| \int_s^t \rho(\tau)\, d\tau \right|$$

for some Borel functions g, ρ with $g\rho \in L^r$, then $f \in W^{1,r}(0,T)$. This is proved, in the case when ρ is constant and $r = 1$, in of Lemma 1.2.6 of [AGS], and the proof in the more general case requires only minor variants.

We are going to apply this criterion with $f(t) = \Phi_0(\mu(t))$, $g(t) = \|\nabla h_{\mu(t)}\|_6$, and $\rho(t) = \|\nabla h_{\mu(t)}\hat{\mu}(t)\|_{5/6}$ (notice that with the assumption $\|\hat{\mu}(t)\| \in L^q(0,T)$, $f\rho \in L^{q/3}$ thanks to (24)): integrating by parts the difference $|\nabla h_\mu|^2 - |\nabla h_\nu|^2$ as $\nabla(h_\mu + h_\nu - 2) \cdot \nabla(h_\mu - h_\nu)$, we obtain the identity

$$\begin{aligned}
\Phi_0(\mu) - \Phi_0(\nu) &= \frac{1}{2}\int_\Omega (h_\mu + h_\nu - 2)(\Delta h_\nu - \Delta h_\mu) + \frac{1}{2}\int_\Omega (h_\mu - 1)^2 - (h_\nu - 1)^2 \\
&= \frac{1}{2}\int_\Omega (h_\mu + h_\nu - 2)(\mu - \nu - h_\mu + h_\nu) + \frac{1}{2}\int_\Omega (h_\mu - 1)^2 - (h_\nu - 1)^2 \\
&= \nu(\Omega) - \mu(\Omega) + \frac{1}{2}\int_\Omega (h_\mu + h_\nu)(\mu - \nu) \\
&= \frac{1}{2}\int_{\overline{\Omega}} (h_\mu + h_\nu)(\mu - \nu).
\end{aligned} \tag{28}$$

Using this identity with $\mu = \mu(t)$ and $\nu = \mu(s)$, and (26) with $\phi = h_{\mu(t)}$ and $\phi = h_{\mu(s)}$, we get

$$|\Phi_0(\mu(s)) - \Phi_0(\mu(t))| \leq C(g(s) + g(t)) \left| \int_s^t \rho(\tau)\, d\tau \right|.$$

This proves that $\Phi_0(\mu(t)) \in W^{1,q/3}(0,T)$. At any differentiability point t, (27) can be proved by applying (28) with $\mu = \mu(t)$, $\nu = \mu(t+s)$, and letting $s \to 0$. □

The following result provides some control on the mass dissipated on $\partial\Omega$ when $p > 2$, showing that it is absolutely continuous with respect to $\sigma_{\partial\Omega}$.

Proposition 3.2 (Boundary mass dissipation). *Let $\mu(t) : [0,T] \to P(\overline{\Omega})$ be as in Definition 3.1, and assume that $\|\hat{\mu}(t)\|_\infty \in L^\infty(0,T)$. Then $\tilde{\mu}(t) - \tilde{\mu}(0) = f(t,\cdot)\sigma_{\partial\Omega}$ with $\|f(t,\cdot)\|_\infty \leq Ct$.*

Proof. Let C be the least upper bound in $[0,T]$ of $\|\nabla h_{\mu(t)}\hat{\mu}(t)\|_\infty$. Let $\Omega_\varepsilon \subset \Omega$ be the set of all points with distance from $\partial\Omega$ greater than ε, with ε small enough; inserting into the PDE a test function $\chi\phi$, where $\chi \in C^1_c(\mathbb{R}^2)$ and ϕ is a smooth approximation of $1 - \chi_{\Omega_\varepsilon}$, we find

$$\frac{d}{dt}\int_{\overline{\Omega}\setminus\Omega_\varepsilon} \chi\, d\mu(t) = -\int_{\Omega\setminus\Omega_\varepsilon} \nabla h_{\mu(t)} \cdot \nabla\chi\, d\mu(t) - \int_{\partial\Omega_\varepsilon} \chi \frac{\partial h_{\mu(t)}}{\partial\nu}\hat{\mu}(t)$$

for a.e. $\varepsilon > 0$. Now we can bound the last integral with $C\int_{\partial\Omega_\varepsilon}|\chi|$ and pass to the limit as $\varepsilon \downarrow 0$ to obtain

$$\left|\frac{d}{dt}\int_{\partial\Omega}\chi\, d\mu(t)\right| \le C\int_{\partial\Omega}|\chi|\, d\sigma_{\partial\Omega}.$$

By integration we get

$$\left|\int_{\partial\Omega}\chi\, d\mu(t) - \int_{\partial\Omega}\chi\, d\mu(0)\right| \le Ct\int_{\partial\Omega}|\chi|\, d\sigma_{\partial\Omega}.$$

Since χ is arbitrary, the statement is proved. □

Although we will obtain later on more general global existence results for the PDE, via implicit time discretization and a-priori estimates, we sketch here the proof of a weaker short-time existence result based on a Lagrangian formulation, in the case $p = \infty$. We are going to use in particular the DiPerna-Lions theory of flows associated to Sobolev vector fields, see [DPL].

Theorem 3.1 (Short-time existence). *Assume that $\bar{\mu} \in P(\overline{\Omega})$ and that $\hat{\bar{\mu}} \in L^\infty(\Omega)$. Then for $T > 0$ sufficiently small there exists a solution $\mu(t)$ of* (25) *with $\mu(0) = \bar{\mu}$ and $\hat{\mu}(t) \in L^\infty(0,T)$.*

Proof. We use an explicit Euler scheme. To begin with, we fix an auxiliary domain $\tilde{\Omega}$ with $\overline{\Omega} \subset \tilde{\Omega}$. Then, we fix an initial velocity $v(0) \in W^{1,2}(\tilde{\Omega};\mathbb{R}^2)$ equal to $-\nabla h_{\bar{\mu}}$ in Ω, equal to 0 in a neighbourhood of $\partial\tilde{\Omega}$ and satisfying

$$\Big(\|\nabla v(0)\|_{L^2(\tilde{\Omega})} + \|v(0)\|_\infty\Big) \le C\|\bar{\mu}\|_\infty$$

for some constant C. This is possible, thanks to a reflection argument near $\partial\Omega$ and multiplication with a cut-off function near $\partial\tilde{\Omega}$, because h_μ can be estimated both in $C^1(\overline{\Omega})$ and $W^{2,2}(\Omega)$ with $\|\bar{\mu}\|_\infty$. Furthermore, as

$$-\Delta h_{\bar{\mu}} = \bar{\mu} - h_{\bar{\mu}} \ge -h_{\bar{\mu}}$$

we can, by the same argument, assume that

$$\operatorname{div} v(0) \ge -C\|\bar{\mu}\|_\infty. \tag{29}$$

Then, we denote by $X(t,x) : \mathbb{R}^+ \times \tilde{\Omega} \to \tilde{\Omega}$ the DiPerna-Lions flow associated to $v(0)$ and we modify it with a stopping time argument as follows: for $x \in \overline{\Omega}$ fixed, we define

$\tilde{X}(t,x) = X(t,x)$ until $X(t,x)$ reaches $\partial\Omega$; if this happens at some minimal time $t(x)$ ($t(x) = 0$ if $x \in \partial\Omega$), we define $\tilde{X}(t,x) = X(t(x),x)$ for all $t \geq t(x)$. Then, we build an approximate solution in $[0,\tau]$, setting

$$\mu(t) := \tilde{X}(t,\cdot)_{\#}\bar{\mu} \qquad t \in [0,\tau].$$

It is immediate to check that $\mu(t) \in P(\overline{\Omega})$ and that the continuity equation

$$\frac{d}{dt}\mu(t) + \nabla \cdot (v(0)\hat{\mu}(t)) = 0 \qquad \text{in } \mathcal{D}'((0,\tau) \times \mathbb{R}^2)$$

is satisfied. By construction, we also have that $t \mapsto \tilde{\mu}(t)$ is nondecreasing. From the DiPerna-Lions theory we have that $X(t,\cdot)_{\#}\hat{\bar{\mu}}$ have a bounded density in $\tilde{\Omega}$, with

$$\|X(t,\cdot)_{\#}\hat{\bar{\mu}}\|_\infty \leq e^{t\|[\mathrm{div}\ v(0)]^-\|_\infty}\|\hat{\bar{\mu}}\|_\infty \leq \left(1 + 2C\tau\|\hat{\bar{\mu}}\|_\infty\right)\|\hat{\bar{\mu}}\|_\infty$$

for τ sufficiently small. Since $\hat{\mu}(t) \leq X(t,\cdot)_{\#}\hat{\bar{\mu}}$ we obtain also

$$\|\hat{\mu}(t)\|_\infty \leq (1 + 2C\tau\|\hat{\bar{\mu}}\|_\infty)\|\hat{\bar{\mu}}\|_\infty. \tag{30}$$

Therefore, we can iterate this construction, taking τ as an initial time, using a suitable extension of $-\nabla h_{\mu(\tau)}$ as an initial velocity and then a new stopping time argument. This way we extend the solution to $[\tau, 2\tau]$, $[2\tau, 3\tau]$ and so on, obtaining a discrete solution μ^τ of the delayed equation

$$\frac{d}{dt}\mu^\tau(t) + \nabla \cdot (v^\tau(t)\hat{\mu}^\tau(t)) = 0 \qquad \text{in } \mathcal{D}'((0,+\infty) \times \mathbb{R}^2) \tag{31}$$

with (here $[s]$ denotes the integer part of s)

$$v^\tau(t) = -\nabla h_{\mu^\tau(\tau[t/\tau])} \quad \text{in } \Omega. \tag{32}$$

Using (30) one can check that in a sufficiently small time interval $[0,T]$ (depending only on $\|\hat{\bar{\mu}}\|_\infty$) the L^∞ norm of $\hat{\mu}^\tau$ remains bounded. Notice also that, by construction, $t \mapsto \tilde{\mu}^\tau(t)$ is nondecreasing. By a routine argument (see for instance [AG]) we can pass to the limit in (31), (32), at least along subsequences, as $\tau \to 0$ to obtain a solution $\mu(t)$ of the gradient flow. $\square$

The reason why existence results are global when the coupling between velocity v and density μ is $v = -\nabla\Delta^{-1}\mu$, and $\mu \geq 0$, is due to the fact that (29) in this case is replaced by the nonnegativity of the divergence: this yields a global estimate in time of the L^∞ norm of μ. In any case, the Lagrangian approach does not seem to be applicable in the case when $p < \infty$ or μ is a signed measure.

Next, we are going to present, still in the case $p = \infty$, a partial result concerning uniqueness of gradient flows. To this aim, we will first prove a sub-differentiability property of Φ_λ at measures $\mu \in L^\infty$, involving a logarithmic modulus of continuity depending on the L^∞ norm of μ.

Proposition 3.3. *Let $M \geq 1$. There exists a constant $C = C(\Omega)$ such that, setting*

$$\omega(t) = CM^2 t |\ln t|,$$

the functional Φ_λ satisfies

$$\Phi_\lambda(\nu) \geq \Phi_\lambda(\mu) + \int_\Omega \nabla h_\mu \cdot (\mathbf{t} - I)\, d\mu - \omega(W_2^2(\mu,\nu))$$

for all μ, $\nu \in L^\infty$ with $\mu(\Omega) = \nu(\Omega)$, $\|\hat{\mu}\|_\infty \leq M$, $\|\hat{\nu}\|_\infty \leq M$ and $W_2^2(\mu,\nu) \leq e^{-4}$, where $\mathbf{t}$ is the optimal transport map between μ and ν.

Proof. Let us use Proposition 2.2 to obtain

$$\Phi_\lambda(\nu) \geq \Phi_\lambda(\mu) + \int_\Omega h_\mu \, d(\nu - \mu).$$

Next we use the fact that $\nu = \mathbf{t}_\# \mu$ to obtain

$$\Phi_\lambda(\nu) \geq \Phi_\lambda(\mu) + \int_\Omega (h_\mu \circ \mathbf{t} - h_\mu)\, d\mu.$$

In this proof we shall extend h_μ to a ball B containing Ω in such a way that

$$\|\nabla^2 h_\mu\|_{L^p(B)} \leq c(\Omega) \|\nabla^2 h_\mu\|_{L^p(\Omega)}, \tag{33}$$

with $c(\Omega)$ independent of p. This can be achieved in a canonical way, locally straightening $\partial\Omega$ and using a reflection argument. Now we can do a first-order Taylor expansion of h_μ, writing the remainder in integral form, to obtain

$$\Phi_\lambda(\nu) \geq \Phi_\lambda(\mu) + \int_\Omega \nabla h_\mu \cdot (\mathbf{t} - I)\, d\mu + \frac{1}{2}\int_0^1 \left(\int_\Omega \nabla^2 h_\mu((1-\theta)I + \theta\mathbf{t})(\mathbf{t} - I)\cdot(\mathbf{t} - I) \right) d\mu\, d\theta.$$

So, we have to bound the remainder term by $\omega(W_2^2(\mu,\nu))$.

Setting $\mathbf{t}_\theta := (1-\theta)I + \theta\mathbf{t}$ and $\mu_\theta := \mathbf{t}_{\theta\#}\mu$, our assumptions on μ and ν give that still $\mu_\theta \in L^\infty$ and $\|\mu_\theta\|_\infty \leq M$ (see [Ag], [O1]). Changing variables, for $p > 1$ we obtain

$$\begin{aligned}
\int_\Omega \left| \nabla^2 h_\mu((1-\theta)I + \theta\mathbf{t})(\mathbf{t} - I)\cdot(\mathbf{t} - I) \right| d\mu &\leq \frac{1}{\theta^2}\int_B |\nabla^2 h_\mu| |I - \mathbf{t}_\theta^{-1}|^2 \, d\mu_\theta \\
&\leq \frac{1}{\theta^2} \|\nabla^2 h_\mu\|_{L^p(B,\mu_\theta)} \| |I - \mathbf{t}_\theta^{-1}|^2 \|_{L^{p'}(\mu_\theta)} \\
&\leq \frac{M^{1/p}[\mathrm{diam}\,(\Omega)]^{2-2/p'}}{\theta^2} \|\nabla^2 h_\mu\|_{L^p(B)} W_2^{2/p'}(\mu_\theta, \mu).
\end{aligned}$$

Now, $M^{1/p} \leq M$, $[\mathrm{diam}\,(\Omega)]^{2-2/p'}$ can be bounded uniformly in p, say by C_1, and it is well known that $W_2(\mu_\theta,\mu) = \theta W_2(\nu,\mu)$, therefore we get that for $p > 2$ the remainder term can be estimated by

$$C_1 M W_2^{2/p'}(\mu,\nu) \|\nabla^2 h_\mu\|_{L^p(B)} \int_0^1 \theta^{2/p'-2}\, d\theta = \frac{C_1 M \|\nabla^2 h_\mu\|_{L^p(B)}}{1 - 2/p} W_2^{2/p'}(\mu,\nu).$$

Furthermore, we know by elliptic regularity theory and (33) that $\|\nabla^2 h_\mu\|_{L^p(B)} \le C_2 p\|\mu\|_\infty$, with C_2 depending only on Ω, so eventually we can estimate for $p \ge 4$ the remainder term by $2C_1C_2pM^2W_2^{2/p'}(\mu,\nu)$.

Now, as in Yudovitch proof [Yu1, Yu2] of uniqueness of L^∞ solutions to 2-d Euler equations, we can minimize with respect to $p \ge 4$ the function $p \mapsto pW_2^{2/p'}(\mu,\nu)$ (the minimum is attained at $p = -\ln W_2^2(\mu,\nu)$) to obtain the logarithmic bound stated in the theorem, with $C = 2eC_1C_2$. □

Now we combine ideas coming from the theory of optimal transportation with Yudovitch proof for uniqueness of 2-d solutions of incompressible Euler equations with bounded vorticity to obtain the following uniqueness result, valid until the solution has a bounded density in Ω and no mass is present on $\partial\Omega$ (see also [Lo] for a uniqueness result based on optimal transportation tools). In particular, in the case when the velocity is bounded, it gives uniqueness for short time when the initial datum $\mu(0)$ has compact support in Ω.

Theorem 3.2 (Uniqueness). *Let μ^i, $i = 1,2$ be solutions of (25) with $\|\hat{\mu}^i(t)\|_\infty \in L^\infty(0,T)$ and $\tilde{\mu}^i(t) = 0$. Then $\mu^1(0) = \mu^2(0)$ implies $\mu^1(t) = \mu^2(t)$ for all $t \in [0,T]$.*

Proof. We first check that any solution $\mu(t) \in P(\overline{\Omega})$ of the PDE with $\tilde{\mu}(t) = 0$ satisfies the family of evolution inequalities (analogous to Benilan's formulation [Be] of evolution PDE's in Banach spaces)

$$\frac{d}{dt}\frac{1}{2}W_2^2(\mu(t),\nu) - \omega(W_2^2(\mu(t),\nu)) \le \Phi_\lambda(\nu) - \Phi_\lambda(\mu(t)) \tag{34}$$

for any $\nu \in P(\Omega)$ with $\nu(\Omega) = 1$, where $\omega(t) = Lt|\ln t|$ for t sufficiently small (here L depends only on Ω and the L^∞ bounds on $\mu(t)$ and ν). To obtain this we first apply Theorem 8.4.7 of [AGS], that gives the explicit formula for the derivative of $W_2^2(\cdot,\nu)$ along a solution $\mu(t)$ of the continuity equation with velocity field $v_t = -\nabla h_{\mu(t)}$:

$$\frac{d}{dt}\frac{1}{2}W_2^2(\mu(t),\nu) = \int_\Omega \nabla h_{\mu(t)} \cdot (\mathbf{t}_t - I)\, d\mu(t).$$

Here $\mathbf{t}_t$ is the optimal transport map between $\mu(t)$ and ν. Combining this with Proposition 3.3 we obtain (34).

Let $\mu^1(t)$ and $\mu^2(t)$ be as in the statement of the theorem. By applying (34) first to $\mu = \mu^1$, with $\nu = \mu^2(s)$, and then reversing the roles of μ^1 and μ^2 (this is reminiscent of Kruzhkov's doubling of variables argument, see for instance Lemma 4.3.4 of [AGS] for details) we get

$$\frac{d}{dt}W_2^2(\mu^1(t),\mu^2(t)) \le 4\omega(W_2^2(\mu^1(t),\mu^2(t)))$$

for almost every t. Since $\int_0^1 1/\omega(s)\,ds = \infty$ we can apply Gronwall's lemma and obtain that $\mu^1(t) = \mu^2(t)$ for all $t \ge 0$. □

4 The stress-energy tensor and the weak formulation of the equation

We introduce the stress-energy tensor $\mathbf{T}\mu : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ associated to μ (or, better, to h_μ), as the symmetric 2×2 tensor defined by

$$\mathbf{T}\mu := \frac{1}{2}\begin{pmatrix} (\partial_1 h_\mu)^2 - (\partial_2 h_\mu)^2 - h_\mu^2 & 2\partial_1 h_\mu \partial_2 h_\mu \\ 2\partial_1 h_\mu \partial_2 h_\mu & (\partial_2 h_\mu)^2 - (\partial_1 h_\mu)^2 - h_\mu^2 \end{pmatrix} \tag{35}$$

(see [SS2] for example). Recall that h_μ is thought as a function on the whole of $\mathbb{R}^2$. This is the tensor associated to inner variations of a modification of Φ_0 seen as a function of h, i.e.

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla(h(x+t\phi(x)))|^2 + |h(x+t\phi(x))|^2\Big|_{t=0} = \int_\Omega \sum_{i,j} \partial_i \phi_j \mathbf{T}_{ij}.$$

Notice that, due to the fact that h_μ has zero tangential derivative on $\partial\Omega$ and $h_\mu = 1$ on $\partial\Omega$; when $h_\mu \in C^1(\overline{\Omega})$ the tensor $\mathbf{T}\mu$ has a special structure on $\partial\Omega$, namely

$$\mathbf{T}\mu = \frac{1}{2}\left(|\nabla h_\mu|^2 \mathbf{n}_\Omega \otimes \mathbf{n}_\Omega - |\nabla h_\mu|^2 \tau_\Omega \otimes \tau_\Omega - \mathbf{I}\right) \qquad \text{on } \partial\Omega. \tag{36}$$

In what follows we denote by div $\mathbf{T}\mu$ the vector

$$\left(\sum_{i=1}^2 \partial_i(\mathbf{T}\mu)_{i1}, \sum_{i=1}^2 \partial_i(\mathbf{T}\mu)_{i2}\right).$$

Lemma 4.1. *If $\hat\mu \in L^{4/3}(\Omega)$ then*

$$\operatorname{div} \mathbf{T}\mu = -\hat\mu\nabla h_\mu - \frac{1}{2}(|\nabla h_\mu|^2 - 1)\mathbf{n}_\Omega \sigma_{\partial\Omega} \qquad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{37}$$

where $\sigma_{\partial\Omega}$ is the surface measure on $\partial\Omega$.

Proof. If μ is smooth, this follows by a direct calculation and (36), using the chain rule. If $\hat\mu \in L^{4/3}$ we obtain from (24) that $\nabla h_\mu \mu \in L^1$. Moreover, since the trace of a $W^{1,p}$ function belongs to $L^{p/(2-p)}$, by applying this property to ∇h_μ with $p = 4/3$ we find that $|\nabla h_\mu|^2$ is well defined and integrable on $\partial\Omega$. The relation can thus be deduced by a density argument, using the strong continuity of the trace operator. □

The previous lemma plays an important role in the paper, as it allows to define, at least as a distribution, the product $\mu\nabla h$ even when μ is only in H^{-1}. This definition is consistent in the case when μ is sufficiently regular. Furthermore, Delort's theorem shows that the map $\mu \mapsto \mathbf{T}\mu$ is sequentially weakly continuous even though $\mathbf{T}\mu$ depends on μ in a nonlinear way.

Proposition 4.1 (Delort's Theorem). *If $\mu_n \in \mathcal{M}_+(\Omega)$ converge to $\mu \in \mathcal{M}_+(\Omega)$ and (μ_n) is bounded in H^{-1}, then $\mathbf{T}\mu_n \rightharpoonup \mathbf{T}\mu$ in $\mathcal{D}'(\Omega)$.*

Proof. The proof of Delort's theorem can be found in [De], or in a simplified form in [Ch, BM]. For our particular situation one may apply the version stated in [Ch]: since $\Delta h_{\mu_n} = h_n - \mu_n$ we have that Δh_{μ_n} is the sum of a part which is a negative measure, bounded as measures, and a part which is bounded in L^p for some $p > 1$. Then Chemin's result applies and gives us the weak convergence of the quadratic functions appearing in the stress-energy tensor. $\square$

We conclude the section showing that Φ_0 is continuous whenever we have bounds on the total variation in $\mathbb{R}^2$ of div $\mathbf{T}\mu$.

Proposition 4.2 (Continuity of Φ_0). *Let μ_n be a sequence converging to μ in $\mathcal{M}_+(\Omega)$ such that μ_n is bounded in H^{-1} and $|\mathrm{div}\ \mathbf{T}\mu_n|$ is bounded in $\mathcal{M}_+(\mathbb{R}^2)$. Then $h_{\mu_n} \to h_\mu$ strongly in $W^{1,2}(\Omega)$ and, in particular, $\Phi_0(\mu_n) \to \Phi_0(\mu)$ as $n \to \infty$.*

Proof. We denote by $X_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ the vector field

$$\left((\partial_1 h_\mu)^2 - (\partial_2 h_\mu)^2, 2\partial_1 h_\mu \partial_2 h_\mu\right). \tag{38}$$

It is easy to check that, because of our assumption on div $\mathbf{T}\mu_n$, we can bound uniformly as measures both the divergence and the curl of X_{μ_n}.

Let $\tilde{\Omega}$ be a bounded smooth domain strictly containing Ω. Let us define φ_n as the solution to

$$\begin{cases} -\Delta\varphi_n = \mathrm{div}\ X_{\mu_n} & \text{in } \tilde{\Omega} \\ \varphi_n = 0 & \text{on } \partial\tilde{\Omega}, \end{cases} \tag{39}$$

and ψ_n as the solution to

$$\begin{cases} -\Delta\psi_n = \mathrm{curl}\, X_{\mu_n} & \text{in } \tilde{\Omega} \\ \psi_n = 0 & \text{on } \partial\tilde{\Omega}. \end{cases} \tag{40}$$

Since div X_{μ_n} and curl X_{μ_n} remain bounded in $\mathcal{M}(\tilde{\Omega})$, we have by elliptic regularity that $\nabla\varphi_n$ and $\nabla\psi_n$ are compact in $L^p(\tilde{\Omega})$ for every $p < 2$.

In particular $Y_n = X_{\mu_n} + \nabla\varphi_n + \nabla^\perp\psi_n$ is bounded in $L^1(\tilde{\Omega})$. Moreover, Y_n is both divergence-free and curl-free in $\tilde{\Omega}$, thus is harmonic in $\tilde{\Omega}$. We deduce that interior estimates hold, i.e. for every compact subset K of $\tilde{\Omega}$ and for every k, p we have

$$\|Y_n\|_{W^{k,p}(K)} \le C_{k,p,K}\|Y_n\|_{L^1(\tilde{\Omega})}.$$

Consequently, Y_n is compact in $L^p(K)$ for any p, and from the compactness of $\nabla\varphi_n$ and $\nabla\psi_n$, we deduce that X_{μ_n} is compact in $L^p_{loc}(\tilde{\Omega})$ for every $p < 2$. This implies directly that X_{μ_n} is compact in $L^p(\Omega)$, $p < 2$. On the other hand, from Proposition 4.1, we know that $X_{\mu_n} \rightharpoonup X_\mu$ in the sense of distributions. We deduce that $X_{\mu_n} \to X_\mu$ in $L^p(\Omega)$ for all $p < 2$. Since $2|X_{\mu_n}| = |\nabla h_{\mu_n}|^2$ we find $|\nabla h_{\mu_n}|^2 \to |\nabla h_\mu|^2$ in $L^1(\Omega)$. Since weak convergence plus convergence of the norms implies strong convergence, we find $\nabla h_{\mu_n} \to \nabla h_\mu$ in $L^2(\Omega)$, and the result follows. $\square$

5 The one step minimization

We recall that we consider measures in $\mathcal{M}_+(\Omega)$ which have masses different from 1 (due to leaking of mass through the boundary, or entrance of mass through the boundary). Observe that Φ (originally defined on $\mathcal{M}_+(\Omega)$) can be extended as a functional on $P(\overline{\Omega})$ by setting $\Phi(\mu) := \Phi(\hat{\mu})$.

Given $\tau > 0$ and $\mu \in P(\overline{\Omega})$, we denote by μ_τ a minimizer of

$$\min_{\nu \in P(\overline{\Omega})} \nu \mapsto \Phi_\lambda(\hat{\nu}) + \frac{1}{2\tau} W_2^2(\nu, \mu). \tag{41}$$

Since $P(\overline{\Omega})$ is compact for the weak convergence, existence is an easy consequence of the lower semicontinuity of Φ in $P(\overline{\Omega})$, proved in Lemma 2.1, and of the continuity of $W_2^2(\cdot, \mu)$. Uniqueness of minimizers, on the other hand, is not completely clear, since the strictly convex part of the energy, namely Φ_0, depends only on the mass of $\hat{\mu}$, while the other part of the energy need not be strictly convex. However, this argument suffices to prove that $\hat{\mu}_\tau$ is unique.

5.1 Euler-Lagrange equation

We first prove some simple property of all minimizers μ_τ that can be achieved by non-differential variations: no mass is moved within $\partial\Omega$, and if some mass in Ω is moved to $\partial\Omega$, in passing from μ to μ_τ, it has to be moved along a shortest connection to $\partial\Omega$. Furthermore, if $\lambda > 0$, for τ sufficiently small no mass moved on $\partial\Omega$ returns to Ω, i.e. no new mass enters Ω.

Lemma 5.1. *For any minimizer μ_τ of (41) and any $\eta \in \Gamma_0(\mu, \mu_\tau)$ we have*

$$|y - x| = \mathrm{dist}(x, \partial\Omega) \qquad \textit{for } \eta\textit{-a.e. } (x, y) \in \overline{\Omega} \times \partial\Omega. \tag{42}$$

In addition, if $\lambda > 0$ and $\tau > 0$ is sufficiently small (depending only on Ω and λ), we have that $\eta(\partial\Omega \times \Omega) = 0$.

Proof. Let χ be the characteristic function of the set of points (x, y) where (42) fails, i.e. $x \in \overline{\Omega}$, $y \in \partial\Omega$ and $|y - x| > \mathrm{dist}(x, \partial\Omega)$. Let us choose (in a Borel way) for any $x \in \overline{\Omega}$ a point $s(x) \in \partial\Omega$ with shortest distance, set

$$\sigma := (1 - \chi)\eta$$

and let ν and ρ be respectively its first and second marginals; notice that $\nu \leq \mu$ and that $\hat{\rho} = \hat{\mu}_\tau$, because $\chi(x, y)$ can be nonzero only when $y \in \partial\Omega$. Defining

$$\eta' := \sigma + (I, s)_\#(\mu - \nu)$$

and μ'_τ as the second marginal of η', we have that η' still has μ as first marginal, and therefore $\eta' \in \Gamma(\mu, \mu'_\tau)$. Taking into account that $\mu - \nu$ is the first marginal of $\chi\eta$, we also have

$$
\begin{aligned}
(43) \qquad W_2^2(\mu, \mu'_\tau) &\leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2(1-\chi)\,d\eta + \int_{\overline{\Omega}} |s(x)-x|^2\,d(\mu-\nu) \\
&= W_2^2(\mu, \mu_\tau) + \int_{\overline{\Omega}\times\partial\Omega} (|s(x)-x|^2 - |y-x|^2)\chi\,\eta \\
&\leq W_2^2(\mu, \mu_\tau),
\end{aligned}
$$

with strict inequality whenever $\int \chi\,d\eta > 0$. On the other hand, since the second marginal of $(I, s)_\#(\mu - \nu)$ is concentrated on $\partial\Omega$ (because s is $\partial\Omega$-valued) we have

$$
\hat{\mu}'_\tau = \hat{\rho} = \hat{\mu}_\tau,
$$

and therefore $\Phi_\lambda(\hat{\mu}'_\tau) = \Phi_\lambda(\hat{\mu}_\tau)$. The minimality of μ_τ then gives that (42) holds.

Let us prove that $\eta(\partial\Omega \times \Omega) = 0$, for τ sufficiently small, in an informal way (the argument can be easily made rigorous using the same formalism used to show (42)). First, we can find $r > 0$ such that $h_0 - 1 + \lambda/2 \geq \lambda/4$ for all $x \in \Omega$ with $\mathrm{dist}(x, \partial\Omega) < r$. As a consequence, Proposition 2.2 shows that $\Phi_\lambda(\nu) - \Phi_\lambda(\mu) \leq -\lambda(\hat{\mu}(\Omega) - \hat{\nu}(\Omega))/4$ whenever $\hat{\nu} \leq \hat{\mu}$. This shows that, for any $\tau > 0$, if an amount δ of mass in $\partial\Omega$ is sent by η inside Ω, it has to be sent at a distance larger than r (the energy would decrease leaving steady on $\partial\Omega$ the mass sent at a smaller distance). Then, let us compare the energy of μ_τ with the energy of the new measure μ'_τ obtained by leaving this mass at rest: the term $W_2^2(\mu_\tau, \mu)$ decreases at least by δr^2, while (still by Proposition 2.2), the term $\Phi_\lambda(\mu_\tau)$ increases at most by $C\delta$, with $C = \sup\,(h_0 - 1 + \lambda/2)^-$. By the inequality $\delta r^2/(2\tau) \leq C\delta$ it follows that $\delta = 0$ for $\tau < r^2/(2C)$. □

In order to derive the Euler-Lagrange equation associated to (41), we consider a family of approximating variational problems in $P(\overline{\Omega})$.

Lemma 5.2. *Let $\delta > 0$, let $\Phi_\lambda^\delta : P(\overline{\Omega}) \to [0, +\infty]$ be defined by*

$$
(44) \qquad \Phi_\lambda^\delta(\nu) = \Phi_\lambda(\hat{\nu}) + \delta \int_\Omega \hat{\nu}^4,
$$

with the convention $\Phi_\delta(\nu) = +\infty$ if $\hat{\nu}$ is not absolutely continuous with respect to the Lebesgue measure, and let us consider the minimization problem

$$
(45) \qquad \min_{\nu \in P(\overline{\Omega})} \Phi_\lambda^\delta(\nu) + \frac{1}{2\tau} W_2^2(\mu, \nu).
$$

Then this problem has a solution μ_τ^δ, the family μ_τ^δ has limit points both for the strong H^{-1} topology and the $P(\overline{\Omega})$ topology, $\int_\Omega (\hat{\mu}_\tau^\delta)^4 \to 0$ as $\delta \to 0$, and any limit point μ_τ as $\delta \to 0$ solves (41).

Proof. The existence of μ_τ^δ follows by the same argument used to prove the existence of μ_τ. Let us prove the convergence of μ_τ^δ. Let M_δ be the minimum in (45) and let M be the minimum of the functional in (41). It is clear that $M_\delta \geq M$; on the other hand, since $\Phi_\lambda^\delta \to \Phi_\lambda$ pointwise on

$$\left\{\nu \in P(\overline{\Omega}) :\ \hat{\nu} \in L^4(\Omega)\right\},$$

we have

$$\limsup_{\delta\downarrow 0} M_\delta \leq \Phi_\lambda(\nu) + \frac{1}{2\tau} W_2^2(\nu,\mu)$$

for all ν in this subspace. By a density argument we obtain that $\limsup_\delta M_\delta \leq M$, and therefore $M_\delta \to M$ as $\delta \to 0$.

If μ_τ is a limit point, in the weak $P(\overline{\Omega})$ topology, of μ_τ^δ along some sequence $\delta_i \to 0$, the lower semicontinuity of Φ_λ gives

$$\Phi_\lambda(\mu_\tau) + \frac{1}{2\tau} W_2^2(\mu_\tau,\mu) \leq \liminf_{i\to\infty} \Phi_\lambda(\mu_\tau^{\delta_i}) + \frac{1}{2\tau} W_2^2(\mu_\tau^{\delta_i},\mu) \leq \liminf_{i\to\infty} M_{\delta_i} = M,$$

therefore μ_τ is a solution of (41). Finally, taking into account the continuity of $\nu \mapsto W_2^2(\nu,\mu)$, the convergence of M_δ to M gives

$$\limsup_{i\to\infty} \Phi_\lambda(\hat{\mu}_\tau^{\delta_i}) + \delta_i \int_\Omega (\hat{\mu}_\tau^{\delta_i})^4 \leq \Phi_\lambda(\mu_\tau).$$

By the lower semicontinuity of Φ_λ it follows that $\Phi_\lambda(\mu_\tau^{\delta_i}) \to \Phi_\lambda(\mu_\tau)$ and $\delta_i \int_\Omega (\hat{\mu}_\tau^{\delta_i})^4 \to 0$. Now, taking into account that $\Phi_\lambda(\nu)$ is the sum of two lower semicontinuous terms, namely $\Phi_0(\nu)$ and $\lambda\nu(\Omega)/2$, we obtain that

$$\text{(46)}\quad \lim_{i\to\infty} \lambda\mu^{\delta_i}(\Omega) = \lambda\mu_\tau(\Omega) \quad\text{and}\quad \lim_{i\to\infty} \int_\Omega |\nabla h_{\mu_\tau^{\delta_i}}|^2 + (h_{\mu_\tau^{\delta_i}} - 1)^2 = \int_\Omega |\nabla h_\nu|^2 + (h_\nu - 1)^2.$$

In particular $\hat{\mu}_\tau^{\delta_i} \to \hat{\nu}$ strongly in $H^{-1}(\Omega)$. □

Since μ_τ^δ has L^4 regularity inside Ω, we can write down an Euler-Lagrange equation for (45). Here and in the sequel we extend $\hat{\mu}_\tau^\delta$ with the 0 value 0 outside Ω: this is natural, thinking that the measure μ_τ^δ is supported in $\overline{\Omega}$.

Proposition 5.1. *Any minimizer μ_τ^δ of (45) satisfies*

$$\text{(47)}\qquad -3\delta\nabla((\hat{\mu}_\tau^\delta)^4) - \nabla h_{\mu_\tau^\delta}\hat{\mu}_\tau^\delta = \frac{1}{\tau}\pi_{x\#}\left(\chi_\Omega(x)(x-y)\gamma\right) \quad \text{in } \mathcal{D}'(\mathbb{R}^2),$$

where γ is any optimal plan between μ_τ^δ and μ and $\pi_x : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ denotes the projection on the first factor.

Proof. We perform variations around μ_τ^δ in (45). To this aim, it is useful to notice that we can equivalently minimize in (45) among all probability measures ν in $\mathbb{R}^2$ with finite second moment, due to the fact that, for any such ν, there exists ν' with support contained in $\overline{\Omega}$ with $\hat{\nu} = \hat{\nu}'$ and $W_2^2(\mu,\nu') \leq W_2^2(\mu,\nu)$. Indeed, given any optimal plan η between μ and ν, it suffices to consider the map $\Phi(x,y) = (x,y')$, where y' is equal to y if $y \in \Omega$, and is equal to the first point on the segment from x to y hitting $\partial\Omega$ otherwise; then, the second marginal of $\Phi_{\#}\eta$ gives us ν'. Also recall that the result of Lemma 5.1 also holds for (45).

For simplicity, let us denote $\nu = \mu_\tau^\delta$. Then, we write

$$\nu_\varepsilon = \tilde{\nu} + (I + \varepsilon\xi(x))_{\#}\hat{\nu},$$

where ξ is a smooth vector field compactly supported in $\mathbb{R}^2$ (so, we move only the mass of ν inside Ω). We set

$$\Omega_\varepsilon := \{x \in \Omega : \ x + \varepsilon\xi(x) \in \Omega\}.$$

By minimality of ν and the previous remark, we have

$$\Phi_\lambda^\delta(\nu_\varepsilon) + \frac{1}{2\tau}W_2^2(\mu,\nu_\varepsilon) \geq \Phi_\lambda^\delta(\nu) + \frac{1}{2\tau}W_2^2(\mu,\nu). \tag{48}$$

Let us evaluate

$$\frac{1}{2\tau\varepsilon}\left(W_2^2(\mu,\nu_\varepsilon) - W_2^2(\mu,\nu)\right).$$

Since $\nu_\varepsilon = \tilde{\nu} + (I + \varepsilon\xi)_{\#}\hat{\nu}$, defining

$$\gamma_\varepsilon = (I + \varepsilon\xi, I)_{\#}(\chi_\Omega(x)\gamma) + \chi_{\partial\Omega}(x)\gamma,$$

we have that $\gamma_\varepsilon \in \Gamma(\nu_\varepsilon,\mu)$. Therefore, by the definition of W_2, we have

$$\begin{aligned} W_2^2(\nu_\varepsilon,\mu) &\leq \int_{\mathbb{R}^2\times\overline{\Omega}} |y-x|^2\,d\gamma_\varepsilon(x,y) \\ &= \int_{\Omega\times\overline{\Omega}} |x + \varepsilon\xi(x) - y|^2\,d\gamma(x,y) + \int_{\partial\Omega\times\overline{\Omega}} |x-y|^2\,d\gamma(x,y). \end{aligned} \tag{49}$$

A Taylor expansion then gives

$$\frac{1}{2\varepsilon}\left(W_2^2(\nu_\varepsilon,\mu) - W_2^2(\nu,\mu)\right) \leq \int_{\Omega\times\overline{\Omega}} (x-y)\cdot\xi(x)\,d\gamma(x,y) + O(\varepsilon). \tag{50}$$

Secondly, let us evaluate $\Phi_\delta(\nu_\varepsilon) - \Phi_\delta(\nu)$; since $\hat{\nu}_\varepsilon = (I + \varepsilon\xi)_{\#}(\chi_{\Omega_\varepsilon}\hat{\nu})$, we have that

$$\hat{\nu}_\varepsilon(\Omega) = \hat{\nu}(\Omega_\varepsilon) = \nu(\Omega_\varepsilon).$$

Furthermore, the change of variables formula gives

$$\hat{\nu}_\varepsilon(x) = \frac{\hat{\nu}}{\det(I + \varepsilon D\xi)} \circ \left[(I + \varepsilon D\xi)\big|_{\Omega_\varepsilon}\right]^{-1}, \tag{51}$$

so that

$$\int_\Omega \hat{\nu}_\varepsilon^p = \int_{\Omega_\varepsilon} \frac{\hat{\nu}^p}{\det^{p-1}(I + \varepsilon D\xi)} \qquad \forall p > 0. \tag{52}$$

In particular $\hat{\nu_\varepsilon} \to \hat\nu$ in $L^4(\Omega)$ as $\varepsilon \to 0$, hence $h_{\nu_\varepsilon} \to h_\nu$ in $W^{2,4}(\Omega)$, and in particular in $C^1(\overline{\Omega})$. This convergence holds also in a small neighbourhood $\tilde\Omega$ of $\overline{\Omega}$, considering suitable extensions of these functions. By (28) we obtain that $\Phi_\lambda(\nu_\varepsilon) - \Phi_\lambda(\nu)$ equals

$$
\begin{aligned}
& \frac{2-\lambda}{2}\nu(\Omega\setminus\Omega_\varepsilon) + \frac{1}{2}\int_\Omega (h_{\nu_\varepsilon} + h_\nu)(\nu_\varepsilon - \nu) \\
= \; & \frac{2-\lambda}{2}\nu(\Omega\setminus\Omega_\varepsilon) + \frac{1}{2}\int_{\Omega_\varepsilon} (h_{\nu_\varepsilon}\circ(I+\varepsilon\xi) + h_\nu\circ(I+\varepsilon\xi))\,d\nu - \int_\Omega (h_{\nu_\varepsilon} + h_\nu)\,d\nu \\
= \; & \frac{2-\lambda}{2}\nu(\Omega\setminus\Omega_\varepsilon) + \frac{1}{2}\int_\Omega (h_{\nu_\varepsilon}\circ(I+\varepsilon\xi) - h_{\nu_\varepsilon} + h_\nu\circ(I+\varepsilon\xi) - h_\nu)\,d\nu \\
- \; & \frac{1}{2}\int_{\Omega\setminus\Omega_\varepsilon} (h_{\nu_\varepsilon}\circ(I+\varepsilon\xi) + h_\nu\circ(I+\varepsilon\xi))\,d\nu.
\end{aligned}
$$

Now, by C^1 regularity and the fact that $h_\nu = 1$ on $\partial\Omega$, we have

$$
\int_{\Omega\setminus\Omega_\varepsilon} (h_{\nu_\varepsilon}\circ(I+\varepsilon\xi) + h_\nu\circ(I+\varepsilon\xi))\;d\nu = 2\nu(\Omega\setminus\Omega_\varepsilon) + O(\varepsilon)\nu(\Omega\setminus\Omega_\varepsilon) = 2\nu(\Omega\setminus\Omega_\varepsilon) + o(\varepsilon).
$$

We deduce that $\Phi_\lambda(\nu_\varepsilon) - \Phi_\lambda(\nu) = -\frac{\lambda}{2}\nu(\Omega\backslash\Omega_\varepsilon) + \frac{\varepsilon}{2}\int_\Omega \nabla h_\nu\cdot\xi\,d\nu + o(\varepsilon)$, taking into account the strong $C^1(\tilde\Omega)$ convergence of h_{ν_ε} to h_ν. Therefore, neglecting the term $-\frac{\lambda}{2}\nu(\Omega\setminus\Omega_\varepsilon)$ we get

$$
\limsup_{\varepsilon\to 0^+} \frac{\Phi_\lambda(\nu_\varepsilon) - \Phi_\lambda(\nu)}{\varepsilon} \le \int_\Omega \nabla h_\nu\cdot\xi\,d\nu.
$$

On the other hand,

$$
\frac{\delta}{\varepsilon}\left[\int_\Omega \hat\nu_\varepsilon^4 - \int_\Omega \hat\nu^4\right] = \frac{\delta}{\varepsilon}\left[\int_{\Omega_\varepsilon} \frac{\hat\nu^4}{\det^3(I+\varepsilon D\xi)} - \int_\Omega \hat\nu^4\right] \le -3\delta\int_\Omega \hat\nu^4 \mathrm{div}\,\xi + o(1), \tag{53}
$$

as $\varepsilon \to 0^+$. We deduce

$$
\limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left(\Phi_\lambda^\delta(\nu_\varepsilon) - \Phi_\lambda^\delta(\nu)\right) \le \int_\Omega \nabla h_\nu\cdot\xi\,d\hat\nu - 3\delta\int_\Omega \hat\nu^4\mathrm{div}\,\xi. \tag{54}
$$

Inserting this and (50) into (48), and passing to the limit as $\varepsilon \to 0$, we find

$$
\frac{1}{\tau}\int_{\Omega\times\overline{\Omega}} (x-y)\cdot\xi(x)\,d\gamma(x,y) + \int_\Omega \nabla h_\nu\cdot\xi\,d\hat\nu - 3\delta\int_\Omega \hat\nu^4\mathrm{div}\,\xi \ge 0,
$$

and changing ξ to $-\xi$, we have

$$
\frac{1}{\tau}\int_{\Omega\times\overline{\Omega}} (x-y)\cdot\xi(x)\,d\gamma(x,y) + \int_\Omega \nabla h_\nu\cdot\xi\,d\hat\nu - 3\delta\int_\Omega \hat\nu^4\mathrm{div}\,\xi = 0. \tag{55}
$$

This proves (47). □

Passing to the limit as $\delta \to 0$ in the previous result, and using the stress-energy tensor, we are led to:

Proposition 5.2. *Assume that $\lambda > 0$. Then there exist a minimizer μ_τ of (41) and $\gamma \in \Gamma_0(\mu_\tau, \mu)$ satisfying*

$$\operatorname{div} \mathbf{T}\mu_\tau = -\frac{1}{2} Z(\mu_\tau)\mathbf{n}_\Omega + \frac{1}{\tau}(\pi_x)_\#(\chi_\Omega(x)(x-y)\gamma) \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{56}$$

with $Z(\mu_\tau) + \sigma_{\partial\Omega}$ nonnegative and concentrated on $\partial\Omega$.

Proof. We consider a limit point (in the weak $P(\overline{\Omega})$ topology) $\mu_\tau = \lim_i \mu_\tau^{\delta_i}$ as in Lemma 5.2, and consider plans $\gamma_i \in \Gamma_0(\mu_\tau^{\delta_i}, \mu)$. Recall that $\delta_i \int_\Omega (\hat{\mu}_\tau^{\delta_i})^4 \to 0$, that $\hat{\mu}_\tau^{\delta_i}$ converge to $\hat{\mu}_\tau$ in $H^{-1}(\Omega)$ and that $\hat{\mu}_\tau^{\delta_i}(\Omega)$ converges to $\hat{\mu}_\tau(\Omega)$ (this follows from (46)).

Returning to (55) and taking into account Lemma 4.1 we obtain

$$\begin{aligned} \frac{1}{\tau}\int_{\Omega\times\overline{\Omega}} (x-y)\cdot\xi(x)\, d\gamma_i(x,y) \quad &+ \quad \int_\Omega \mathbf{T}\mu_\tau^{\delta_i}\cdot D\xi\, d\hat{\mu}_\tau^{\delta_i} \\ &- \quad 3\delta_i \int_\Omega (\hat{\mu}_\tau^{\delta_i})^4 \operatorname{div}\xi = \frac{1}{2}\int_{\partial\Omega} (|\nabla h_i|^2 - 1)\xi\cdot\mathbf{n}_\Omega \end{aligned} \tag{57}$$

for all smooth ξ, with $h_i := h_{\mu_\tau^{\delta_i}}$. In particular, choosing ξ equal to $\mathbf{n}_\Omega$ on $\partial\Omega$ with $|\xi| \le 1$, we obtain that $(|\nabla h_i|^2 - 1)$ are uniformly bounded in $L^1(\sigma_{\partial\Omega})$, so we can assume that they converge, as measures, to some measure $Z(\mu_\tau)$ supported on $\partial\Omega$, with $Z(\mu_\tau) + \sigma_{\partial\Omega}$ obviously nonnegative.

Next, we pass to the limit in (57) as $i \to \infty$, possibly along a subsequence: we know that $\delta_i \int_\Omega (\hat{\mu}_\tau^{\delta_i})^4 \operatorname{div}\xi \to 0$ and that, by strong H^{-1} convergence, $\int_\Omega \mathbf{T}\mu_\tau^{\delta_i}\cdot D\xi \to \int_\Omega \mathbf{T}\mu_\tau \cdot D\xi$. Let us see how the first term in the left hand side can be handled: we can assume, by the stability properties of optimal plans, that γ_i weakly converges in $P(\overline{\Omega}\times\overline{\Omega})$ to some $\gamma \in \Gamma_0(\mu_\tau, \mu)$, and the fact that $\mu_\tau^{\delta_i}(\Omega)$ converges to $\mu_\tau(\Omega)$ tells us that there is no concentration of mass near $\partial\Omega$; using this fact we easily obtain that $\chi_\Omega(x)\gamma_i$ still weakly converge to $\chi_\Omega(x)\gamma$, so that also the first term in the left-hand side goes to the limit. □

We conclude this analysis of the one-step minimization noticing that the second measure appearing in the right hand side of (56) is concentrated on Ω and absolutely continuous with respect to μ_τ, with an L^2 density. This directly follows from the next lemma.

Lemma 5.3. *Let $\gamma \in \Gamma(\mu, \nu)$ and set*

$$\sigma := (\pi_x)_\# \left((x-y)\gamma\right).$$

Then $\sigma \ll \mu$ and the density $f \in L^2(\mu; \mathbb{R}^2)$ of σ with respect to μ satisfies

$$\|f\|^2_{L^2(\mu)} \le \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2\, d\gamma(x,y). \tag{58}$$

Proof. It is immediate to check that $|\sigma| \le (\operatorname{diam}\Omega)\mu$, therefore $\sigma \ll \mu$. Denoting by $f \in L^\infty(\mu;\mathbb{R}^2)$ the density and by $\|\cdot\|$ the $L^2(\mu)$ norm, we have

$$\begin{aligned}
\|f\|_{L^2(\mu)} &= \sup_{\xi\in C_c,\ \|\xi\|=1} \int_{\overline{\Omega}} f\cdot\xi\, d\mu = \sup_{\xi\in C_c,\ \|\xi\|=1} \int_{\overline{\Omega}\times\overline{\Omega}} (x-y)\cdot\xi\, d\gamma(x,y) \\
&\le \sup_{\xi\in C_c,\ \|\xi\|=1} \left(\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2\, d\gamma(x,y)\right)^{\frac{1}{2}} \left(\int_{\overline{\Omega}\times\overline{\Omega}} |\xi(x)|^2\, d\gamma(x,y)\right)^{\frac{1}{2}} \\
&= \left(\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2\, d\gamma(x,y)\right)^{\frac{1}{2}}.
\end{aligned}$$

□

Choosing a smooth vector field $\xi : \overline{\Omega} \to \mathbb{R}^2$ with $|\xi| \le 1$ and $\xi = \mathbf{n}_\Omega$ on $\partial\Omega$, we can combine (56) and (58) to obtain

$$\frac{1}{2} Z(\mu_\tau)(\partial\Omega) \le C(\Omega)\int_\Omega |\mathbf{T}\mu_\tau|\, dx + \frac{W_2(\mu,\mu_\tau)}{\tau}, \tag{59}$$

with $C(\Omega) = \|\nabla\xi\|_\infty$.

5.2 A class of entropies for the discrete scheme

We prove here that $\int_\Omega \varphi(\hat\mu)$ decreases along the discrete flow, for a suitable family of "entropies" φ. The idea of the proof was suggested to us by G. Savaré for the case of absolutely continuous measures. Our measures are not a priori absolutely continuous, but fortunately it turns out that the δ-approximant μ_τ^δ provides a good approximation which is stable for the inequality we wish to prove.

Let φ be a C^1 function from $[0,+\infty)$ to $\mathbb{R}$. We say φ is an "entropy" for the flow, if:

(a) φ is convex nondecreasing, C^1, and $x\varphi'(x) = \varphi(x)$ for all $x \in [0,1]$;

(b) the following inequality holds:

$$2x\varphi''(x) \ge x\varphi'(x) - \varphi(x). \tag{60}$$

Notice that since $\varphi(0) = 0$ the right-hand side in (60) is nonnegative, so that the inequality can be considered as a stronger convexity requirement on φ. It corresponds indeed to the convexity of the map $s \mapsto s^2\varphi(s^{-2})$, yielding the so-called displacement convexity of the map $\mu \mapsto \int_\Omega \varphi(\mu)\, dx$ [MC].

Proposition 5.3. *Let φ be an entropy and let $\mu \in P(\overline{\Omega})$ be such that $\int_\Omega \varphi(\mu) < \infty$. Then, for any minimizer μ_τ^δ of (45), we have*

$$\int_\Omega \varphi(\hat\mu_\tau^\delta) \le \int_\Omega \varphi(\mu).$$

Proof. We are going to consider a function ψ such that

$$\psi'(x) = x\varphi'(x) - \varphi(x). \tag{61}$$

Notice that $\psi''(x) = x\varphi''(x)$, so that ψ is also convex and $\psi'(x) \equiv 0$ on $[0,1]$.

- *Step 1:* The first step consists in deriving some regularity for $\rho := \hat{\mu}_\tau^\delta$ from (47). More precisely, we show that the function ρ^4 belongs to $W^{1,4}(\mathbb{R}^2)$ and that $\frac{\nabla\rho^4}{\rho}$ coincides ρ-a.e. with a $BV \cap L^\infty$ vector field G on Ω.

For simplicity we write h for $h_{\mu_\tau^\delta}$. Recall that $\rho \in L^4$, hence by Sobolev embedding, we have $h \in W^{2,4}$ hence $h \in C^{1,1/2}$. Therefore $\rho\nabla h \in L^4$.

On the other hand, since $\hat{\mu}_\tau^\delta$ is absolutely continuous, the optimal transport plan γ from μ_τ^δ to μ is induced, inside Ω, by the gradient r of a Lipschitz convex function defined on $\mathbb{R}^2$ (see for instance §6.2.3 in [AGS]); in particular $r \in BV_{loc}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Thus

$$(\pi_x)_\# \left(\chi_\Omega(x)(x-y)\gamma\right) = (I - r)\rho \in BV(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).$$

Inserting all this information into (47) we find that $\nabla\rho^4 \in L^4(\mathbb{R}^2)$. Therefore $\rho^4 \in W^{1,4}(\mathbb{R}^2)$ and ρ is thus continuous. Observe that by definition $\rho = 0$ outside Ω, hence $\hat{\mu}_\tau^\delta$ is a continuous function which vanishes on $\partial\Omega$.

Returning to (47), dividing by ρ we may write

$$3\delta\frac{\nabla\rho^4}{\rho} + \nabla h = \frac{1}{\tau}(r - I) \quad \rho\text{-a.e. in } \Omega. \tag{62}$$

We deduce that $\frac{\nabla\rho^4}{\rho}$ can be extended to the whole of Ω as a $BV \cap L^\infty$ vector-field G.

- *Step 2:* Let now φ be an entropy and let ψ be a convex function such that ψ' satisfies (61). We show that

$$\int_\Omega \psi(h) + \int_\Omega \psi''(h)|\nabla h|^2 \le \int_\Omega \psi(\rho). \tag{63}$$

Indeed, recall

$$\begin{cases} -\Delta h + h = \rho & \text{in } \Omega \\ h = 1 & \text{on } \partial\Omega, \end{cases}$$

hence multiplying this equation by $\psi'(h)$, which belongs to $H_0^1(\Omega)$ since ψ' is continuous and $h = 1$ on $\partial\Omega$, we find, after an integration by parts,

$$\int_\Omega \psi''(h)|\nabla h|^2 + \psi'(h)(h - \rho) = 0 \tag{64}$$

But the convexity of ψ gives $\psi(\rho) \ge \psi(h) + \psi'(h)(\rho - h)$, thus inserting this into (64) we obtain (63).

- *Step 3:* Let us now evaluate $\int_\Omega \varphi(\hat{\mu}) - \int_\Omega \varphi(\hat{\mu}_\tau^\delta) = \int_\Omega \varphi(\hat{\mu}) - \varphi(\rho)$. Since $\gamma(\Omega \times \partial\Omega) = 0$, $r_\# \rho$ is concentrated on Ω and therefore $r_\# \rho \le \hat{\mu}$. Using that φ is nondecreasing, we find that

$$\int_\Omega \varphi(\hat{\mu}) - \varphi(\rho) \ge \int_\Omega \varphi(r_\# \rho) - \int_\Omega \varphi(\rho).$$

By (60), φ is displacement convex, therefore (see Lemma 10.4.4 in [AGS])

$$\int_\Omega \varphi(r_\# \rho) - \int_\Omega \varphi(\rho) \ge - \int_\Omega \psi'(\rho) \mathrm{trace} \nabla (r - I).$$

Here the gradient ∇r is understood in the pointwise sense (i.e. the absolutely continuous part of the distributional derivative of r). Moreover, since r is the gradient of a convex function, we can bound from above the trace in the pointwise sense by div $(r - I)$, the divergence being in the distributional sense.

Recalling that from (62) we have

$$\mathrm{div}\ (r - I) = \tau \Delta h + 3\delta\tau \mathrm{div}\ G,$$

we are thus led to the inequality

$$\int_\Omega \varphi(\hat{\mu}) - \int_\Omega \varphi(\hat{\mu}_\tau^\delta) \ge -\tau \int_\Omega \psi'(\rho)\, d\left(\Delta h + 3\delta \mathrm{div}\, G\right). \tag{65}$$

Notice that $\psi'(\rho)$ is continuous and compactly supported in Ω, because $\psi'(t) = 0$ for $t \in [0,1]$ and $\rho = 0$ on $\partial\Omega$. Moreover, writing $g(t) = \psi'(t^{1/4})$, we have that $\psi'(\rho) = g(\rho^4) \in W^{1,1}(\Omega)$. Therefore, integrating by parts, we get

$$\int_\Omega \psi'(\rho)\, d\mathrm{div}\ G = - \int_\Omega G \cdot \nabla \left(g(\rho^4)\right) = - \int_\Omega \frac{|\nabla \rho^4|^2}{\rho} g'(\rho^4) \le 0 \tag{66}$$

because $4g'(t) = t^{-3/4}\psi''(t^{1/4}) \ge 0$. On the other hand

$$\int_\Omega \psi'(\rho) \Delta h = \int_\Omega \psi'(\rho)(h - \rho) \le \int_\Omega \psi(h) - \psi(\rho) \tag{67}$$

by convexity of ψ, and the right-hand side is less than $-\int_\Omega \psi''(h)|\nabla h|^2$ by (63). Combining (65)-(66)-(67), we conclude that

$$\int_\Omega \varphi(\hat{\mu}) - \int_\Omega \varphi(\hat{\mu}_\tau^\delta) \ge \tau \int_\Omega \psi''(h)|\nabla h|^2 \tag{68}$$

Using again the fact that $\psi'' \ge 0$, the proposition follows. $\square$

Let us now draw consequences of this result. We may choose as particular φ the functions

$$\begin{cases} \varphi(x) = x & \text{for } x \le 1 \\ \varphi(x) = x^p + (p-1)(1-x) & \text{for } x \ge 1, \end{cases} \tag{69}$$

where $p > 1$. By straighforward calculations one may check that φ satisfies all the conditions to be an "entropy". The following result does not really require that φ takes specifically the form (69), but just is an entropy with sufficient growth at infinity.

Proposition 5.4. *Let $p \in (1,\infty]$, $\mu \in P(\overline{\Omega})$ with $\hat{\mu} \in L^p(\Omega)$. Then there exists a minimizer μ_τ of (41) satisfying Proposition 5.2 and*

(i) If $p < \infty$ and φ is defined as in (69)*, then*

$$\int_\Omega \varphi(\hat{\mu}_\tau) \leq \int_\Omega \varphi(\hat{\mu}) < \infty, \tag{70}$$

hence $\hat{\mu}_\tau \in L^p(\Omega)$.

(ii) If $p = \infty$ and $M = \max\{1, \|\hat{\mu}\|_\infty\}$, we have

$$\|\hat{\mu}_\tau\|_\infty \leq M, \qquad 0 \leq h_{\mu_\tau} \leq M.$$

Proof. Applying Proposition 5.3 to the particular φ given in (69), we find

$$\int_\Omega \varphi(\hat{\mu}_\tau^\delta) \leq \int_\Omega \varphi(\hat{\mu}) \tag{71}$$

for any minimizer μ_τ^δ of (45). In view of the growth of φ in (69), we deduce that $\hat{\mu}_\tau^\delta$ are bounded in L^p uniformly with respect to δ. So, for any weak limit μ_τ of $\mu_\tau^{\delta_i}$ we have that $\hat{\mu}_\tau^{\delta_i}$ weakly converge in L^p to $\hat{\mu}$, hence (70) follows by the weak lower semicontinuity of $\mu \mapsto \int_\Omega \varphi(\mu)$ in L^p.

For the L^∞ case, let $M \geq 1$ be such that $|\hat{\mu}| \leq M$ and let us consider instead of (69) a sequence of entropies φ_n converging monotonically to

$$\varphi(x) := \begin{cases} x & \text{for } x \leq M \\ +\infty & \text{for } x > M \end{cases}$$

(this is possible because $M \geq 1$). We may also assume that $\psi_n''(x) = x\varphi_n''(x)$ converge monotonically to $+\infty$ if $x > M$.

Using now (68) instead of Proposition 5.3, we find

$$\int_\Omega \varphi_n(\hat{\mu}_\tau^\delta) + \tau \int_\Omega \psi_n''(h_{\mu_\tau^\delta})|\nabla h_{\mu_\tau^\delta}|^2 \leq \int_\Omega \varphi_n(\hat{\mu}).$$

Passing to the limit $\delta_i \to 0$ as above we find

$$\int_\Omega \varphi_n(\hat{\mu}_\tau) + \tau \int_\Omega \psi_n''(h_{\mu_\tau})|\nabla h_{\mu_\tau}|^2 \leq \int_\Omega \varphi(\hat{\mu}).$$

Passing to the limit $n \to +\infty$ we find by monotone convergence theorem

$$\int_\Omega \varphi(\hat{\mu}_\tau) + \tau \int_{\{h_{\mu_\tau} > M\}} |\nabla h_{\mu_\tau}|^2 \leq \int_\Omega \varphi(\hat{\mu}) < \infty.$$

We deduce that $|\hat{\mu}_\tau| \leq M$ a.e. by definition of φ, and that $\nabla h_{\mu_\tau} \equiv 0$ on the open set $\{h_{\mu_\tau} > M\}$. It follows that this set is empty and $h_{\mu_\tau} \leq M$. The inequality $h_{\mu_\tau} \geq 0$ is just a simple consequence of the maximum principle and (18). □

6 Convergence of the implicit time discretization scheme

In this section we consider the convergence of the implicit time discretization scheme associated to Φ_λ. Let us first describe this problem in a more general context: given a lower semicontinuous functional $\Psi : P(\overline{\Omega}) \to \mathbb{R} \cup \{+\infty\}$, an initial condition $\bar{\mu}$ and a time step $\tau > 0$, for $k \geq 0$ integer we define μ_τ^k in such a way that $\mu_\tau^0 = \bar{\mu}$ and, for any k, μ_τ^{k+1} is a minimizer of

$$\nu \mapsto \Psi(\nu) + \frac{1}{2\tau} W_2^2(\nu, \mu_\tau^k) \tag{72}$$

in $P(\overline{\Omega})$. The minimality of μ_τ^{k+1} gives the inequality

$$\Psi(\mu_\tau^{k+1}) + \frac{1}{2\tau} W_2^2(\mu_\tau^{k+1}, \mu_\tau^k) \leq \Psi(\mu_\tau^k),$$

whence

$$\sum_{k=0}^{N} W_2^2(\mu_\tau^{k+1}, \mu_\tau^k) + 2\tau\Psi(\mu_\tau^{N+1}) \leq 2\tau\Psi(\bar{\mu}) \qquad \forall N \geq 0 \tag{73}$$

and, in particular, the energy bound

$$\Psi(\mu_\tau^k) \leq \Psi(\bar{\mu}). \tag{74}$$

We can define piecewise constant continuous trajectories and rescale in time, setting

$$\bar{\mu}_\tau(t) := \mu_\tau^{k+1} \qquad \text{if } t \in (k\tau, (k+1)\tau], \tag{75}$$

and investigate the behaviour of $\bar{\mu}_\tau(t)$ as $\tau \downarrow 0$.

If $\Psi(\bar{\mu}) < +\infty$ and $m := \inf \Psi > -\infty$, using (73) it is immediate to obtain, from the triangle inequality, the uniform discrete $C^{0,1/2}$ estimate

$$W_2(\bar{\mu}_\tau(t), \bar{\mu}_\tau(s)) \leq \sqrt{2(\Psi(\bar{\mu}) - m)}\sqrt{|t-s| + \tau} \qquad \forall s,\, t \in [0, \infty). \tag{76}$$

Since for fixed t, the family $\bar{\mu}_\tau(t)$ is sequentially compact in $P(\overline{\Omega})$, a standard diagonal argument based on (76) ensures the existence of limit points of $\bar{\mu}_\tau(t)$ as $\tau \downarrow 0$ in this sense: given any sequence $(\tau_n) \downarrow 0$ we can extract a subsequence (that we don't relabel, just for notational simplicity) such that

$$\lim_{n\to\infty} \bar{\mu}_{\tau_n}(t) = \mu(t) \qquad \forall t \geq 0. \tag{77}$$

Furthermore, the energy bound (74) ensures that $\Psi(\mu(t)) < \infty$ for all $t \geq 0$.

If we choose $\gamma_\tau^{k+1} \in \Gamma_0(\mu_\tau^{k+1}, \mu_\tau^k)$, we may think that $(x-y)/\tau$, for $(x,y) \in \operatorname{supp} \gamma_\tau^{k+1}$, gives us a sort of discrete velocity for the scheme (72). This motivates the following definition.

Definition 6.1 (Limiting velocity of the scheme). Let (τ_n), $\mu(t)$ be as in (77). We say that $w \in L^2(\mu(t);\mathbb{R}^2)$ is a *limiting velocity* at $\mu(t)$ if there exists a subsequence $\tau_{n(l)} \to 0$ such that, denoting by $k(l)$ the integer part of $t/\tau_{n(l)}$, we have

(a) $\liminf_{l\to\infty} W_2(\mu_{\tau_{n(l)}}^{k(l)+1}, \mu_{\tau_{n(l)}}^{k(l)})/\tau_{n(l)} < \infty$;

(b) $\frac{1}{\tau_{n(l)}}(\pi_x)_\#((x-y)\gamma_{\tau_{n(l)}}^{k(l)+1})$ weakly converge to $w\mu(t)$.

We shall denote by $V_l\Psi(\mu(t)) \subset L^2(\mu(t);\mathbb{R}^2)$ the limiting velocities.

Notice that the definition of $k(l)$ and the equi-continuity estimate (76) give

$$\lim_{l\to\infty} \mu_{\tau_{n(l)}}^{k(l)+1} = \lim_{l\to\infty} \mu_{\tau_{n(l)}}^{k(l)} = \mu(t) \qquad \text{in } P(\overline{\Omega}). \tag{78}$$

We also observe that, according to Lemma 5.3, $\frac{1}{\tau_{n(l)}}(\pi_x)_\#((y-x)\gamma_{\tau_{n(l)}}^{k(l)+1})$ have an L^2 density with respect to $\mu_{\tau_{n(l)}}^{k(l)+1}$ whose norm is bounded by $\frac{1}{\tau_{n(l)}}W_2(\mu_{\tau_{n(l)}}^{k(l)+1}, \mu_{\tau_{n(l)}}^{k(l)})$. Therefore, a general lower semicontinuity argument, illustrated in Lemma 6.1 below, gives

$$\|w\|_{L^2(\mu(t))} \le \liminf_{l\to\infty} \frac{1}{\tau_{n(l)}} W_2(\mu_{\tau_{n(l)}}^{k(l)+1}, \mu_{\tau_{n(l)}}^{k(l)}) \tag{79}$$

for all w as in (b).

Lemma 6.1. *Let μ_n be converging to μ in $P(\overline{\Omega})$, and let $w_n\mu_n$ be converging to σ, with $w_n \in L^2(\mu_n;\mathbb{R}^2)$ and $\|w_n\|_{L^2(\mu_n)}$ bounded. Then $\sigma = w\mu$ for some $w \in L^2(\mu;\mathbb{R}^2)$ and*

$$\int |w|^2\, d\mu \le \liminf_{n\to\infty} \int |w_n|^2\, d\mu_n.$$

Proof. Let $C = \liminf_n \|w_n\|_{L^2(\mu_n)}$. For every $\eta > 0$ and for every smooth test-vector field ξ compactly supported in Ω,

$$\left|\int_\Omega w_n \cdot \xi\, d\mu_n\right| \le (C+\eta)\left(\int_\Omega |\xi|^2\, d\mu_n\right)^{\frac{1}{2}}, \tag{80}$$

for infinitely many n. We can pass to the limit first as $n \to \infty$ and then as $\eta \downarrow 0$ to get

$$|\langle \xi, \sigma\rangle| \le C\left(\int_\Omega |\xi|^2\, d\mu\right)^{\frac{1}{2}}.$$

Therefore, by the density of smooth functions, the linear map $\xi \mapsto \langle \xi, \sigma\rangle$ can be uniquely extended to a linear map on $L^2(\mu;\mathbb{R}^2)$ with norm less than C. By the Riesz representation theorem we can represent this linear map by some $w \in L^2(\mu;\mathbb{R}^2)$ with $\|w\|_{L^2(\mu;\mathbb{R}^2)} \le C$ and we conclude. $\square$

The following result, borrowed from Theorem 11.1.6 of [AGS], provides us with a general scheme, totally independent of the specific functional Ψ under consideration, for the construction of a velocity field $v(t)$ associated to $\mu(t)$ and its relation with the implicit scheme (72).

Theorem 6.1 (Limiting velocity field [AGS]). *Assume that $\Psi : P(\overline{\Omega}) \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and bounded from below, and let $\mu_\tau^k \in P(\overline{\Omega})$ be minimizers obtained from the recursive minimization of* (72) *starting from $\bar{\mu}$, with $\Psi(\bar{\mu}) < \infty$. Let $\bar{\mu}_{\tau_n}(t)$, $\mu(t)$ be as in* (75)–(77)*. Then:*

(i) there exist $v(t) \in L^2(\mu(t); \mathbb{R}^2)$ such that $\|v(t)\|_{L^2(\mu(t))} \in L^2(0, +\infty)$ and

$$\frac{d}{dt}\mu(t) + \mathrm{div}\,(v(t)\mu(t)) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0, +\infty)); \tag{81}$$

(ii) The following energy dissipation inequality at time 0 *holds:*

$$\Psi(\mu(t)) + \int_0^t \|v(r)\|^2_{L^2(\mu(r))}\,dr \le \Psi(\bar{\mu}) \qquad \forall t \ge 0; \tag{82}$$

(iii) for a.e. $t > 0$, $v(t)$ belongs to the closed convex hull in $L^2(\mu(t); \mathbb{R}^2)$ of $V_l\Psi(\mu(t))$.

Finally, if for some $s \ge 0$ the implication

$$\lim_{l\to\infty} \frac{W_2(\mu_{\tau_{n(l)}}^{k(l)+1}, \mu_{\tau_{n(l)}}^{k(l)})}{\tau_{n(l)}} < \infty \qquad \Longrightarrow \qquad \lim_{l\to\infty} \Psi(\mu_{\tau_{n(l)}}^{k(l)+1}) = \Psi(\mu(s)) \tag{83}$$

is fulfilled for any subsequence $n(l)$, with $k(l)$ equal to the integer part of $s/\tau_{n(l)}$, then the energy dissipation inequality at time s holds:

$$\Psi(\mu(t)) + \int_s^t \|v(r)\|^2_{L^2(\mu(r))}\,dr \le \Psi(\mu(s)) \qquad \forall t \ge s. \tag{84}$$

So, in specific cases one has to look for the properties of the closed convex hull of the limiting subdifferential. In the case when $\Psi = \Phi_\lambda$, this task can be achieved by choosing in the discrete scheme particular minimizers and optimal plans, whose existence is ensured by Proposition 5.2, and passing to the limit in (56). We use in the sequel this notation: given some $\mathbb{R}^2$-valued measure σ in $\partial\Omega$, we may write it as $\sigma = f|\sigma|$ for some $\mathbf{S}^1$-valued Borel map f (the so-called *polar decomposition* of σ); then, we write

$$\sigma^{\tan} := f \cdot \mathbf{n}_\Omega^\perp |\sigma|, \qquad \sigma^\perp := f \cdot \mathbf{n}_\Omega |\sigma|,$$

so that

$$\int g\,d\sigma^{\tan} = \int g \cdot \mathbf{n}_\Omega^\perp\,d\sigma, \qquad \int g\,d\sigma^\perp = \int g \cdot \mathbf{n}_\Omega\,d\sigma.$$

The following theorem provides an almost complete characterization of the limiting velocities, up to some term concentrated on $\partial\Omega$ and normal to $\partial\Omega$ (the characterization is indeed complete if $\bar{\mu} \in L^p$, $p \ge 4/3$).

Proposition 6.1 (Characterization of limiting velocities). *Let $\mu_\tau^k \in P(\overline{\Omega})$, $\gamma_\tau^{k+1} \in \Gamma_0(\mu_\tau^{k+1}, \mu_\tau^k)$ be minimizers obtained from Proposition 5.2, starting from $\bar\mu$, with $\Phi_\lambda(\bar\mu) < \infty$ and $\lambda > 0$. Let $\bar\mu_{\tau_n}(t)$, $\mu(t)$ as in (75)–(77). Then, for a.e. t and for all $w \in V_l\Psi_\lambda(\mu(t))$ we have that* div $\mathbf{T}\mu(t)$ *is a finite measure in $\mathbb{R}^2$ and*

$$w\mu(t) = \operatorname{div} \mathbf{T}\mu(t) + \frac{1}{2}Z(w)\mathbf{n}_\Omega, \tag{85}$$

with $Z(w) \geq -\sigma_{\partial\Omega}$ concentrated on $\partial\Omega$. We also have

$$Z(w) = \frac{1}{2}(|\nabla h_{\mu(t)}|^2 - 1)\sigma_{\partial\Omega} \tag{86}$$

if $\hat{\bar\mu} \in L^p$ for some $p \geq 4/3$.

Proof. We first recall that, by Lemma 5.1, $\gamma_\tau^{k+1}(\Omega \times \partial\Omega) = 0$ for τ small enough, hence

$$\begin{aligned} \mu_\tau^{k+1}(\partial\Omega) &= \gamma_\tau^{k+1}(\partial\Omega \times \overline{\Omega}) = \gamma_\tau^{k+1}(\partial\Omega \times \Omega) + \gamma_\tau^{k+1}(\partial\Omega \times \partial\Omega) \\ &= \gamma_\tau^{k+1}(\partial\Omega \times \Omega) + \gamma_\tau^{k+1}(\overline{\Omega} \times \partial\Omega) \\ &= \gamma_\tau^{k+1}(\partial\Omega \times \Omega) + \mu_\tau^k(\partial\Omega). \end{aligned} \tag{87}$$

By the monotonicity of $k \mapsto \mu_\tau^k(\partial\Omega)$ we can assume, thanks to Helly's compactness theorem, that $\mu_{\tau_{n(l)}}^{[t/\tau_{n(l)}]}(\partial\Omega)$ converge as $l \to \infty$ for all $t \geq 0$ to some nondecreasing function f.

Let us fix t such that f is continuous at t and let us rewrite (56) as follows:

$$-\int_\Omega \mathbf{T}\mu_\tau \cdot D\xi = -\frac{1}{2}\int_{\partial\Omega} \xi \cdot \mathbf{n}_\Omega \, dZ(\mu_\tau) + \frac{1}{\tau}\int_{\overline{\Omega}\times\overline{\Omega}} \xi(x)\cdot(x-y)d\gamma - \frac{1}{\tau}\int_{\partial\Omega\times\overline{\Omega}} \xi(x)\cdot(x-y)d\gamma.$$

Set now $\tau = \tau_{n(l)}$ and $\mu_\tau = \mu_{\tau_{n(l)}}^{k(l)+1}$, in accordance with Definition 6.1 of a limiting velocity, and assume that conditions (a) and (b) are fulfilled for some $w \in L^2(\mu(t);\mathbb{R}^2)$. Denoting

$$\gamma_l := \gamma_{\tau_{n(l)}}^{k(l)+1} \in \Gamma_0(\mu_{\tau_{n(l)}}^{k(l)+1}, \mu_{\tau_{n(l)}}^{k(l)}), \qquad Z_l := Z(\mu_{\tau_{n(l)}}^{k(l)+1}),$$

we obtain

$$-\int_\Omega \mathbf{T}\mu_{\tau_{n(l)}}^{k(l)+1}\cdot D\xi = -\frac{1}{2}\int_{\partial\Omega} \xi\cdot\mathbf{n}_\Omega \, dZ_l + \frac{1}{\tau_{n(l)}}\int_{\overline{\Omega}\times\overline{\Omega}} \xi(x)\cdot(x-y)d\gamma_l - \frac{1}{\tau_{n(l)}}\int_{\partial\Omega\times\overline{\Omega}} \xi(x)\cdot(x-y)d\gamma_l. \tag{88}$$

By the continuity of f at t, (87) with $\tau = \tau_{n(l)}$ and $k = k(l)$ give

$$\lim_{l\to\infty} \gamma_l(\partial\Omega \times \Omega) = 0. \tag{89}$$

Recall also that (59) gives

$$\frac{1}{2}Z_l(\partial\Omega) \leq C(\Omega)\int_\Omega |\mathbf{T}\mu_{\tau_{n(l)}}^{k(l)+1}| + \frac{W_2(\mu_{\tau_{n(l)}}^{k(l)+1}, \mu_{\tau_{n(l)}}^{k(l)})}{\tau_{n(l)}}, \tag{90}$$

so that the energy bound (74) ensures that $Z_l(\partial\Omega)$ is bounded (the energy controls $\int_\Omega |\mathbf{T}\mu|$). As a consequence (88) gives that $|\mathrm{div}\ \mathbf{T}\mu^{k(l)+1}_{\tau_{n(l)}}|(\mathbb{R}^2)$ is bounded.

Let us take limits as $l \to \infty$ in (88): the convergence of the left hand side follows by the strong $H^1(\Omega)$ convergence of $h_{\mu^{k(l)+1}_{\tau_{n(l)}}}$ to h_{μ_t}, ensured by Proposition 4.2. By compactness, taking again into account (90) and the lower bound $Z_l \geq -\sigma_{\partial\Omega}$, we can assume that the first term in the right hand side converges to $-\frac{1}{2}\int_{\partial\Omega} \xi \cdot \mathbf{n}_\Omega \, dZ$ for some signed measure $Z(w)$, still concentrated on $\partial\Omega$ and such that $Z(w) \geq -\sigma_{\partial\Omega}$ and

$$\tag{91} \frac{1}{2} Z(w)(\partial\Omega) \leq C(\Omega) \int_\Omega |\mathbf{T}\mu(t)| + \liminf_{l\to\infty} \frac{W_2(\mu^{k(l)+1}_{\tau_{n(l)}}, \mu^{k(l)}_{\tau_{n(l)}})}{\tau_{n(l)}}.$$

By assumption (b) the second term converges to $\int_{\overline{\Omega}} \xi \cdot w \, d\mu(t)$.

It remains to analyze the behaviour of the third term in the right hand side in (88). Using Lemma 5.1 we see that the integration can also be done on $\partial\Omega \times \Omega$ only; then, using Hölder's inequality and (89) we immediately see that this term gives no contribution to the limit.

Summing up, upon passing to the limit as $l \to \infty$ in (88) we get

$$\tag{92} -\int_\Omega \mathbf{T}\mu(t) \cdot D\xi = -\frac{1}{2}\int_{\partial\Omega} \xi \cdot \mathbf{n}_\Omega \, dZ(w) + \int_{\overline{\Omega}} \xi \cdot w \, d\mu(t).$$

□

7 Proof of the main existence results

In this section we are going to show our main global existence result for the PDE

$$\tag{93} \frac{d}{dt}\mu(t) + \mathrm{div}(v(t)\mu(t)) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0, +\infty)),$$

where $v(t) = -\nabla h_{\mu(t)}$ if $\mu(t)$ is sufficiently regular inside Ω (otherwise the coupling will be based on div $\mathbf{T}\mu(t)$, see (95)), with the initial condition $\mu(0) = \bar{\mu}$.

We are also going to show that our solution also satisfies suitable energy dissipation inequalities. To this aim, as in the statement of Theorem 6.1, we say that $\mu(t)$ satisfies the energy dissipation inequality at time s if

$$\tag{94} \Phi_\lambda(\mu(t)) + \int_s^t \int_{\overline{\Omega}} |v(\tau)|^2 \, d\mu(\tau) \, d\tau \leq \Phi_\lambda(\mu(s)) \qquad \forall t \geq s.$$

Theorem 7.1 (Initial condition in H^{-1} and $\lambda > 0$). *Assume that $\Phi_\lambda(\bar{\mu}) < \infty$ and $\lambda > 0$. Then there exists a weakly continuous map $\mu(t) : [0, +\infty) \to P(\overline{\Omega})$ such that:*

(a) for a.e. t, div $\mathbf{T}\mu(t)$ *is a finite measure in $\mathbb{R}^2$;*

(b) $\mu(0) = \bar{\mu}$ *and the PDE* (93) *holds for a velocity field* $v(t) \in L^2(\mu(t); \mathbb{R}^2)$ *satisfying*

$$v(t)\mu(t) = \begin{cases} \mathrm{div}\ \mathbf{T}\mu(t) & \text{in } \Omega \\ [\mathrm{div}\ \mathbf{T}\mu(t)]^{\tan} & \text{on } \partial\Omega \end{cases} \qquad \text{for a.e. } t \geq 0; \tag{95}$$

(c) the energy dissipation inequality at time 0 *holds.*

Furthermore, if $\hat{\bar{\mu}} \in L^p(\Omega)$ *for some* $p > 1$ *we have that* $\|\hat{\mu}(t)\|_p \in L^\infty(0, +\infty)$, $t \mapsto \tilde{\mu}(t)$ *is nondecreasing, and the energy dissipation inequality holds at a.e. time* $s \geq 0$.

Proof. Let $\mu_\tau^k \in P(\overline{\Omega})$, $\gamma_\tau^{k+1} \in \Gamma_0(\mu_\tau^{k+1}, \mu_\tau^k)$ be minimizers obtained from Proposition 5.2, starting from $\bar{\mu}$. We interpolate in time between the discrete solutions, to build maps $\mu_\tau(t)$ and find a subsequence (τ_n) with $\tau_n \downarrow 0$ and $\mu_{\tau_n}(t) \to \mu(t)$ weakly in $P(\overline{\Omega})$ for all $t \geq 0$.

Then, Theorem 6.1 provides us with a velocity field $v(t)$ for which the continuity equation (81) holds, which belongs for a.e. t to the closed convex hull of limiting velocities. Now, Proposition 6.1 gives that div $\mathbf{T}\mu(t)$ is a finite measure in $\mathbb{R}^2$ for a.e. t, and the characterization (85) of limiting velocities shows that

$$v(t)\mu(t) = \mathrm{div}\ \mathbf{T}\mu(t) + Z\mathbf{n}_\Omega,$$

for some measure Z concentrated on $\partial\Omega$ with $Z \geq -\sigma_{\partial\Omega}$. Eventually we use the fact that the velocity field $v(t)$ preserves the domain to show (see Lemma 7.1 below) that $Z\mathbf{n}_\Omega = -\,[\mathrm{div}\ \mathbf{T}\mu(t)]^\perp$: this leads to (95).

Finally, the energy dissipation inequality (94) at time 0 follows from (82) and statement (iii) in Theorem 6.1.

Now, let us assume that $\hat{\bar{\mu}} \in L^p$ for some $p > 1$. Choosing an entropy with p growth as in (69) we obtain that $\hat{\mu}_\tau^k$ is uniformly bounded in L^p. This, together with the monotonicity of $k \mapsto \tilde{\mu}_\tau^k$ yields the monotonicity of $t \mapsto \tilde{\mu}(t)$, because the uniform L^p bound provides a separate convergence of $\hat{\mu}_{\tau_n}^{[t/\tau_n]}$ to $\hat{\mu}(t)$ (weakly in L^p) and of $\tilde{\mu}_{\tau_n}^{[t/\tau_n]}$ to $\tilde{\mu}(t)$ (weakly in $P(\overline{\Omega})$).

In order to prove the energy dissipation inequality at a.e. $s \geq 0$, we have to check condition (83) with $\Psi = \Phi_\lambda$. The continuity of the term $\mu \mapsto \lambda\mu(\Omega)$ is trivial, for the reasons illustrated above. The continuity of Φ_0 follows directly from Proposition 4.2, taking into account the formula (56) for div $\mathbf{T}\mu_\tau$ and the bound (59). □

Proof of Theorem 1.1. In the case $\lambda > 0$ the proof follows the same scheme as that of Theorem 7.1, but using the information coming from the uniform L^p bounds on μ_τ^k provided by Proposition 5.4: the characterization of $v(t)$ is in this case simpler, because (86) shows that $V_l\Psi_\lambda(\mu(t))$ is a singleton. Moreover, under the $L^{4/3}$ integrability assumption we know from Lemma 4.1 that div $\mathbf{T}\mu(t)$ has no tangential component on $\partial\Omega$, so that (95) simply gives that $v(t) = 0$ $\mu(t)$-a.e. on $\partial\Omega$: this leads to (14). The proof of the energy dissipation inequality (15), now for all times s, is also similar. Statement (d) follows by Proposition 3.2.

In the case $\lambda = 0$ the proof can be achieved by a standard approximation argument, taking into account the uniform L^p bounds on the solutions $\mu^\lambda(t)$ built with $\lambda > 0$. Notice that the inequalities $\tilde{\mu}^\lambda(t) \geq \tilde{\mu}^\lambda(s)$ for $t \geq s$ are retained in the limit, because the uniform L^p bounds force a separate weak convergence of the interior and boundary parts. Finally, when $p \geq 3/2$ the inequality (15) becomes an equality thanks to Proposition 3.1. $\square$

Lemma 7.1 (Tangential velocity). *Let $\mu(t) : [0, T] \to P(\overline{\Omega})$ be solving the continuity equation*

$$\frac{d}{dt}\mu(t) + \nabla \cdot (v(t)\mu(t)) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0, +\infty))$$

for some velocity field $v(t)$ with $\|v(t)\|^2_{L^2(\mu(t))} \in L^1(0, T)$. Then, for a.e. t, we have that the normal component of $v(t)$ vanishes $\mu(t)$-a.e. on $\partial\Omega$.

Proof. Assume first that $\mu(t) = \delta_{\gamma(t)}$, with $\gamma : [0, T] \to \overline{\Omega}$ absolutely continuous, so that $v(t)\mu(t) = \gamma'(t)\mu(t)$ for a.e. t. In this case the statement is trivial, as for any point t of differentiability of γ such that $\gamma(t) \in \partial\Omega$ the normal component of $\gamma'(t)$ has to vanish.

In the general case, it is proved in Theorem 8.2.1 [AGS] that we can represent μ_t as

$$\mu(t) = (e_t)_\# \eta \qquad \forall t \in [0, T], \tag{96}$$

where $e_t : C([0, T]; \mathbb{R}^2) \to \mathbb{R}^2$ is the evaluation map at time t (i.e. $e_t(\gamma) = \gamma(t)$), for a suitable positive and finite measure η in $C([0, T]; \mathbb{R}^2)$, concentrated on the class of absolutely continuous maps γ solutions to the ODE $\gamma' = v(t, \gamma)$ for a.e. t (here we use the notation $v(t, x) = v(t)(x)$). By Fubini's theorem we obtain that, for a.e. t, there exists a Borel set $E_t \subset C([0, T]; \mathbb{R}^2)$ where η is concentrated and γ is differentiable, with $\gamma'(t) = v(t, \gamma(t))$ for all $\gamma \in E_t$. Fix a time t with this property, and use the previous remark to obtain that

$$v(t, \gamma(t)) \cdot \mathbf{n}_\Omega(\gamma(t)) = \gamma'(t) \cdot \mathbf{n}_\Omega(\gamma(t)) = 0$$

for all $\gamma \in E_t$ with $\gamma(t) \in \partial\Omega$. By (96) we infer that $v(t, x)) \cdot \mathbf{n}_\Omega = 0$ for $\mu(t)$-a.e. $x \in \partial\Omega$. $\square$

8 Open problems and an example

We conclude this paper by pointing out some open problems, comments and possible extensions.

[**The case of signed measures**] The extension of our results to the case of measures of varying sign presents several difficulties: first, in this case there is no reason to rule out the possibility of mass entering the domain; second, many estimates (even in the short-time existence result) seem to be of difficult extension; third, the Wasserstein framework should be adapted to this more general situation. However, the restriction to nonnegative measure is not so unrealistic, in view of the fact that minimizers of Φ_λ, even among signed measures, are known to be unique, nonnegative and compactly supported [SS1].

[**The role of energy dissipation inequalities**] Notice that, although Φ_λ depends on λ, the PDE (1) does not. This is not surprising, since the λ term appears only as a multiplier of the "null Lagrangian" $\mu \mapsto \mu(\Omega)$. However, the time-discrete version of our problem does depend on λ, and λ appears in the energy dissipation inequalities (15) and (94), which should be considered as a nontrivial part of the existence result. At least for L^p solutions with $p \geq 3/2$, for which the energy identity

$$\Phi_0(\mu(t)) + \int_s^t \int_\Omega |\nabla h_{\mu(\tau)}|^2 \, d\mu(\tau) \, d\tau = \Phi_0(\mu(s)) \qquad t \geq s \geq 0$$

is available (see Theorem 3.1), it would be interesting to do a more refined analysis of the mechanism of mass dissipation through $\partial\Omega$, possibly adding a new term to the energy dissipation rate $\int_\Omega |\nabla h_{\mu(t)}|^2 \, d\mu(t)$.

[**Uniqueness and asymptotic behaviour of solutions**] The results presented in Theorem 7.1 and Theorem 1.1 are only concerned with existence of solutions. It would be nice to extend Theorem 3.2, valid only until some mass reaches the boundary, to more general situations. In this connection, notice that for any η and λ, the minimizer μ_η of Φ_η is also a stationary point of Φ_λ (being compactly supported, and because the two functionals differ by a null-lagrangian). So, given an initial compactly supported measure $\bar{\mu}$, it is natural to conjecture that the solution $\mu(t)$ starting from $\bar{\mu}$ should remain compactly supported (and therefore unique) and converge as $t \to \infty$ to some μ_η, where η is chosen in such a way that $\mu_\eta(\Omega) = \bar{\mu}(\Omega)$.

On the other hand, if $\bar{\mu}$ is not compactly supported then the mass dissipation rate through the boundary should probably depend on λ, causing nonuniqueness for solutions to the PDE (which again is independent of λ) as soon as some mass reaches the boundary. In other words, in Theorem 3.2 the restriction to measures μ satisfying $\tilde{\mu} = 0$ might be necessary.

[**Structure of the measure** div $\mathbf{T}\mu$] The main difference between the PDE's in Theorem 1.1 and Theorem 7.1 is that in the former the velocity is 0 on $\partial\Omega$ (while, of course, the normal trace of the velocity from inside might well be nonzero) and in the latter the velocity is $[\mathrm{div}\, \mathbf{T}\mu(t)]^{\tan}$. However, having in mind the identity (37), we conjecture that div $\mathbf{T}\mu$ has *no* tangential component on $\partial\Omega$ whenever $\mu \in H^{-1}$ and div $\mathbf{T}\mu$ is a finite measure in $\mathbb{R}^2$. Were this result true, the velocity would be 0 also in the general H^{-1} case.

[**Slope and stationary points**] In the variational theory of gradient flows (see [AGS]) a key role is played by De Giorgi's (descending) metric slope:

$$|\partial\Psi|(\mu) := \limsup_{\nu \to \mu} \frac{(\Psi(\mu) - \Psi(\nu))^+}{W_2(\mu, \nu)}. \tag{97}$$

Indeed, points μ where the metric slope vanishes correspond somehow to critical points, i.e. points where the gradient flow is allowed to stop. The slope can also be computed (see Lemma 3.1.5 in [AGS]) by

$$|\partial\Psi|(\mu) = \lim_{n \to \infty} \frac{W_2(\mu_{\tau_n}, \mu)}{\tau_n},$$

for a suitable sequence $(\tau_n) \downarrow 0$, where μ_τ are minimizers of $\nu \mapsto \Psi(\nu) + W_2^2(\nu, \mu)/(2\tau)$. In the case when $\Psi = \Phi_\lambda$, using Lemma 6.1 and (56) one can obtain

$$|\partial\Phi_\lambda|(\mu) \geq \left\| \frac{\mathrm{div}\ \mathbf{T}\mu}{\hat{\mu}} \right\|_{L^2(\hat{\mu})}. \tag{98}$$

This relation shows somehow that, being a critical point from the metric viewpoint is a stronger property, compared to having a divergence-free stress-energy tensor. We give here an example (mentioned in [SS2]) of a measure μ supported on a curve in Ω (hence not belonging to L^p) such that the metric slope at μ is not zero but div $\mathbf{T}\mu$ vanishes. Taking $\bar{\mu} = \mu$ as initial condition, this example should probably lead to nonuniqueness, as we may reasonably expect that the solution built by the implicit Euler scheme should not be constant but rather have decreasing energy.

The example mentioned in [SS2] is built as follows. Let $c \in \mathbb{R}$, $0 < R_1 < R_2$, and let us solve

$$\begin{cases} -\Delta h_1 + h_1 = 0 & \text{in } B_{R_1}(0) \\ h_1 = c & \text{on } \partial B_{R_1}(0), \end{cases} \qquad \begin{cases} -\Delta h_2 + h_2 = 0 & \text{in } B_{R_2}(0) \setminus \overline{B}_{R_1}(0) \\ h_2 = c & \text{on } \partial B_{R_1}(0) \\ h_2 = 1 & \text{on } \partial B_{R_2}(0). \end{cases}$$

Both functions are radial, and we can adjust c, R_1 and R_2 in such a way that

$$\frac{\partial h_1}{\partial r}(R_1) = -\frac{\partial h_2}{\partial r}(R_1) = \frac{1}{4\pi R_1}.$$

A rigorous treatment of this using modified Bessel functions can be found in [Ay] Chapter 8.

Now, we can take $\Omega = B_{R_2}(0)$ and

$$h := \begin{cases} h_1 & \text{on } B_{R_1}(0) \\ h_2 & \text{on } \Omega \setminus B_{R_1}(0), \end{cases}$$

so that $h \in H^1(\Omega)$, ∇h is discontinuous on $\partial B_{R_1}(0)$, while $|\nabla h|$ remains continuous. This ensures that, letting $\mu = -\Delta h + h$, which is a measure supported on the circle $\partial B_{R_1}(0)$, we have div $\mathbf{T}\mu = 0$ in Ω and $\mu(\Omega) = 1$.

Proposition 8.1. *For the measure μ constructed above, we have $|\partial\Phi_\lambda|(\mu) > 0$ while* div $\mathbf{T}\mu = 0$*, hence there is strict inequality in (98).*

Proof. Since h is radial, we can identify it with a function of r only which solves

$$-h'' - \frac{h'}{r} + h = 2\alpha\delta_{R_1} \tag{99}$$

in $(0, R_2)$, where

$$\alpha = h'_-(R_1) = -h'_+(R_1) = \frac{1}{4\pi R_1}, \tag{100}$$

see again [Ay]. We introduce a perturbation of h as follows. First let $\varepsilon > 0$ be small and let h_ε be the function h truncated in a neighborhood of $\partial B_{R_1}(0)$ at the level $h(R_1 - \varepsilon)$. More precisely, let $R_\varepsilon > R_1$ be the infimum of all $r > R_1$ such that $h(r) = h(R_1 - \varepsilon)$ and set $h_\varepsilon(r) = h(R_1 - \varepsilon)$ for $R_1 - \varepsilon \le r \le R_\varepsilon$ and $h_\varepsilon = h$ elsewhere.

From (99), we have

$$\lim_{r \to R_1^-} h''(r) = c - \frac{\alpha}{R_1}, \qquad \lim_{r \to R_1^+} h''(r) = c + \frac{\alpha}{R_1},$$

so that

$$h'(R_1 - \varepsilon) = \alpha - \varepsilon(c - \frac{\alpha}{R_1}) + O(\varepsilon^2), \qquad h'(R_\varepsilon) = -\alpha + \varepsilon(c + \frac{\alpha}{R_1}) + O(\varepsilon^2). \tag{101}$$

Also, since (100) holds and h is smooth in $(0, R_1) \cup (R_1, R_2)$, we must have

$$R_\varepsilon = R_1 + \varepsilon + O(\varepsilon^2), \qquad h(R_1 - \varepsilon) = c - \alpha\varepsilon + O(\varepsilon^2). \tag{102}$$

By construction, $-\Delta h_\varepsilon + h_\varepsilon$ is a radial measure which is supported only in $\overline{B}_{R_\varepsilon}(0) \setminus B_{R_1 - \varepsilon}(0)$. It is equal to the positive constant $h(R_1 - \varepsilon)$ in the interior of that annulus, and it has a singular part on the boundary of the annulus which can easily seen to be positive. In total $\mu_\varepsilon := -\Delta h_\varepsilon + h_\varepsilon \ge 0$. Let us now evaluate $m_\varepsilon := \int_\Omega -\Delta h_\varepsilon + h_\varepsilon$; clearly $m_0 = 1$, and let us prove that $m_\varepsilon = 1 + O(\varepsilon^2)$. Indeed, the first order terms in ε in the identity

$$m_\varepsilon = h_1(R_1 - \varepsilon)\pi(R_\varepsilon^2 - (R_1 - \varepsilon)^2) + 2\pi(R_1 - \varepsilon)h'(R_1 - \varepsilon) - 2\pi R_\varepsilon h'(R_\varepsilon)$$

are given by (taking into account (101) and (102)) $4\pi R_1 c$ for the first summand, $-2\pi R_1 \alpha - 2\pi R_1(c - \alpha/R_1)$ for the second, and $2\pi R_1 \alpha - 2\pi R_1(c + \alpha/R_1)$ for the third, and their sum is 0.

Now, we define h_0 to be the solution of (2) for $\mu = 0$, hence $-\Delta h_0 + h_0 = 0$ and we set

$$g_\varepsilon := \frac{h_\varepsilon}{m_\varepsilon} + \left(1 - \frac{1}{m_\varepsilon}\right) h_0, \qquad \tilde{\mu_\varepsilon} := -\Delta g_\varepsilon + g_\varepsilon,$$

so that $g_\varepsilon = 1$ on $\partial\Omega$, $\tilde{\mu_\varepsilon} = \frac{\mu_\varepsilon}{m_\varepsilon} \ge 0$ and $\tilde{\mu_\varepsilon}$ is a probability measure on Ω. Let us now evaluate $\Phi_\lambda(\tilde{\mu_\varepsilon})$. We easily check that

$$\int_\Omega |\nabla g_\varepsilon|^2 + |g_\varepsilon - 1|^2 = (1 + O(\varepsilon^2)) \int_\Omega |\nabla h_\varepsilon|^2 + |h_\varepsilon - 1|^2 + O(\varepsilon^2).$$

But

$$\begin{aligned} &\int_\Omega |\nabla h_\varepsilon|^2 + |h_\varepsilon - 1|^2 - |\nabla h|^2 - |h - 1|^2 \\ &\qquad = \int_{B_{R_\varepsilon}(0) \setminus B_{R_1 - \varepsilon}(0)} -|\nabla h|^2 - |h - 1|^2 + |h_\varepsilon - 1|^2 = -4\pi R_1 \alpha^2 \varepsilon + o(\varepsilon). \end{aligned}$$

We conclude that $\Phi_\lambda(\tilde{\mu_\varepsilon}) - \Phi_\lambda(\mu) = -\frac{\alpha}{2}\varepsilon + o(\varepsilon)$. Moreover, by construction $\tilde{\mu_\varepsilon} = \mu_\varepsilon / m_\varepsilon$ is supported on $\overline{B}_{R_\varepsilon}(0) \setminus B_{R_1 - \varepsilon}(0)$, hence it is clear that $W_2(\tilde{\mu_\varepsilon}, \mu) \le \varepsilon + o(\varepsilon)$. Recalling the definition (97), this proves that $|\partial\Phi_\lambda|(\mu) \ge \alpha/2 > 0$. $\square$

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