Stability in 2D Ginzburg-Landau Passes to the Limit

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Abstract

We prove that if we consider a family of *stable* solutions to the Ginzburg-Landau equation, then their vortices converge to a *stable* critical point of the "renormalized energy". Moreover in the case of instability, the number of "directions of descent" is bounded below by the number of directions of descent for the renormalized energy. A consequence is a result of nonexistence of stable nonconstant solutions to Ginzburg-Landau with homogeneous Neumann boundary condition.

I Introduction and main results

We are interested in characterizing the vortices of solutions of the Ginzburg-Landau equation

$$-\Delta u = \frac{1}{\varepsilon^2} u(1-|u|^2) \quad \text{in } \Omega, \tag{I.1}$$

which are also the critical points of the Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2. \tag{I.2}$$

Here Ω is a smooth bounded two-dimensional simply connected domain and u is a complex-valued function. Around each of its zeroes, the map u can have a nonzero *winding number*, or *degree* of the zero. Such points are called *vortices*, and when their number is bounded, they converge as $\varepsilon \to 0$ to a limiting finite set of points or limiting vortices $\{a_i\}$.

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The main result established by Bethuel, Brezis and Hélein in [BBH] is that vortices of critical points of the Ginzburg-Landau energy (I.2) with a fixed Dirichlet boundary condition converge, as $\varepsilon \to 0$, to critical points of the *renormalized energy* W, a function they introduced (see below for its expression), which depends only on the limiting vortices $\{a_i\}$ (and on the degrees). They also established that E_ε Γ-converges to W in some sense, and thus minimizers of E_ε converge to minimizers of the renormalized energy. The converse issue of proving that to each critical point $\{a_i\}$ of W can be associated a sequence of critical points of (I.2) whose vortices converge to the $\{a_i\}$ has been answered positively (at least in the case of degrees ± 1), first in [Li, LL] through a construction for local minimizers and minmax solutions, then in more details and all generality in [PR] through local inversion (see also [AB] for a construction of solutions using the topological landscape of W). So essentially, through all these works, the relation of E_ε to W has been understood at order of the first derivative.

The main question that we address here is the following: if u_ε is a family of stable/unstable critical points of E_ε, then is the limiting set of points $\{a_i\}$ a stable/unstable critical point of W? In other words, we know that the criticality and minimality of u_ε translates into that of $\{a_i\}$, and want to know whether the second order property of stability translates into that of $\{a_i\}$ for the limiting energy W. This will in turn give some information about solutions of (I.1), for example we will see in Theorem 3 that with the Neumann boundary condition, W has no stable critical point, and thus deduce that u_ε has no nonconstant stable critical point, for ε small enough.

In order to answer this question, we place it in a more general framework. We established in [SS4] a scheme to prove that if a family of functionals E_ε Γ-converges to a limiting functional F, then there are conditions that ensure that "well-prepared" (meaning having no excess energy) solutions of the gradient-flow of E_ε converge to solutions of the gradient-flow of F. This corresponds to a sort of C^1 notion of Γ-convergence, that guarantees that the C^1 structure of the energy landscape is preserved through the limiting process. In particular well-prepared critical points of E_ε converged to critical points of F. Here we show that the same kind of reasoning can be applied to the second order, or in other words, we present some criteria that ensure that the second order structure of the energy landscape is preserved, thus corresponding to a sort of C^2 notion of Γ-convergence. Since we only deal with critical points and static situations, the criteria are quite simpler. The main criterion involves exactly as in [SS4], performing some construction (variation of the original family u_ε), similar to that of [SS4], but pushed to the second order. Then, we will just need to show that it is possible to do this construction for Ginzburg-Landau, in order to obtain the result on stability/instability of vortices.

I.1 Abstract result

Here is thus the abstract result, providing the general scheme. The setting is similar to [SS4]. E_ε (resp. F) are C^2 functionals defined over $\mathcal{M}$ (resp. $\mathcal{N}$), open set of an affine

space associated to a Banach space $\mathcal{B}$ (resp. $\mathcal{B}'$). Since E_ε is assumed to be C^2, let us denote by $D^2E_\varepsilon(u)$ its Hessian at u and $Q_\varepsilon(u)$ the associated quadratic function, associated to the bilinear continuous function $B_\varepsilon(u)(.,.)$ defined over $\mathcal{B}$ (resp. $D^2F(u)$, $Q(u)$ and $B(u)$ for F). We will say that a critical point is stable if the Hessian is nonnegative, unstable otherwise, purely unstable if the Hessian is nonpositive.
We assume that there is a sense of convergence S such that if $E_\varepsilon(u_\varepsilon) \le C$ there exists a subsequence $u_\varepsilon \rightharpoonup^S u \in \mathcal{N}$ (i.e. we have compactness of families of solutions in that sense). As in [SS4], this sense is to be specified each time. It can be a strong or weak convergence of u_ε, it can also be a convergence of a nonlinear function of u_ε.

Theorem 1 *Let u_ε be a family of critical points of E_ε, and assume $u_\varepsilon \rightharpoonup^S u \in \mathcal{N}$. Assume also that the following hold: for any $V \in \mathcal{B}'$, we can find $v_\varepsilon(t) \in \mathcal{M}$ defined in a neighborhood of $t = 0$, such that $\partial_t v_\varepsilon(0)$ depends on V in a linear and one-to-one manner, and*

$$v_\varepsilon(0) = u_\varepsilon \tag{I.3}$$

$$\lim_{\varepsilon\to 0} \frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = \frac{d}{dt}_{|t=0} F(u+tV) = dF(u).V \tag{I.4}$$

$$\lim_{\varepsilon\to 0} \frac{d^2}{dt^2}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = \frac{d^2}{dt^2}_{|t=0} F(u+tV) = Q(u)(V). \tag{I.5}$$

Then
- if (I.3)-(I.4) are satisfied, u is a critical point of F
- if (I.3)-(I.4)-(I.5) are satisfied, then if u_ε are stable (resp. purely unstable) critical points of E_ε, u is a stable (resp. purely unstable) critical point of F. More generally, denoting by n_ε^+ the dimension (possibly infinite) of the space spanned by eigenvectors of $D^2E_\varepsilon(u_\varepsilon)$ associated to positive eigenvalues, and n^+ the dimension of the space spanned by eigenvectors of $D^2F(u)$ associated to positive eigenvalues (resp. n_ε^- and n^- for negative eigenvalues); for ε small enough we have

$$n_\varepsilon^+ \ge n^+ \qquad n_\varepsilon^- \ge n^-.$$

Let us observe that here we do not specifically require the Γ-convergence of E_ε to F, the relation between E_ε and F is specified in (I.4)-(I.5).

I.2 Application to Ginzburg-Landau

We will be interested in the situation of families such that $E_\varepsilon(u_\varepsilon) \le C|\log \varepsilon|$. This bound guarantees that the number of vortices remains bounded as $\varepsilon \to 0$. We will consider two types of boundary conditions, and emphasize that our results are valid for both. Either as in [BBH] a Dirichlet boundary condition $u_\varepsilon = g$, where g is a fixed map with $|g| = 1$ and $\deg(g, \partial\Omega) = d > 0$, or a homogeneous Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

The limiting vortices a_i and their degrees d_i being given, we define Φ_0 by

$$
\text{(I.6)} \qquad \begin{cases} \Delta \Phi_0 = 2\pi \sum_{i=1}^{k} d_i \delta_{a_i} & \text{in } \Omega \\ \dfrac{\partial \Phi_0}{\partial n} = \left(ig, \dfrac{\partial g}{\partial \tau} \right) & \text{on } \partial\Omega \text{ (resp. } \Phi_0 = 0 \text{ on } \partial\Omega \text{ for Neumann),} \end{cases}
$$

where δ denotes the Dirac mass, and $(.,.)$ denotes the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$. The renormalized energy is given by

$$
\text{(I.7)} \qquad W(a_1, \cdots, a_k) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| - \pi \sum_{i=1}^{k} d_i R(a_i) + \frac{1}{2} \int_{\partial\Omega} \Phi_0 (ig, \partial_\tau g),
$$

where

$$
\text{(I.8)} \qquad R(x) = \Phi_0(x) - \sum_{i=1}^{k} d_i \log |x - a_i|.
$$

In (I.7), the last boundary term is taken to be 0 in the case of the Neumann boundary condition. By critical point of W, we mean critical points with the d_i's being fixed.

We prove that conditions (I.3)—(I.5) are satisfied for solutions of Ginzburg-Landau, and we deduce the following theorem.

Theorem 2 *Let u_ε be a family of solutions of (I.1) such that $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$, with either Dirichlet or homogeneous Neumann boundary conditions. Then, there exists a family of points $a_1, \cdots, a_k$ and nonzero integers $d_1, \cdots, d_k$ such that, up to extraction of a subsequence,*

$$
\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^{k} d_i \delta_{a_i},
$$

where the family $\{a_i\}$ is a critical point of W. Moreover, if u_ε is a stable (resp. purely unstable) solution of (I.1) then $\{a_i\}$ is a stable (resp. purely unstable) critical point of W; and more generally, denoting by n_ε^+ the dimension of the space spanned by eigenvectors of $D^2 E_\varepsilon(u_\varepsilon)$ associated to positive eigenvalues and n^+ the dimension of the space spanned by eigenvectors of $D^2 W(a_i)$ associated to positive eigenvalues (resp. n_ε^- and n^- for negative eigenvalues), we have, for ε small enough,

$$
\text{(I.9)} \qquad n_\varepsilon^+ \geq n^+ \qquad n_\varepsilon^- \geq n^-.
$$

Let us mention that the result of [BBH] contained a stronger convergence of u_ε to u_*, the canonical harmonic map associated to the a_i's and d_i's, and that the compactness of $\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)$ is true in a stronger sense, but we are not focusing on that aspect here.

On the other hand, some related results can be obtained from the analysis of [PR] (restricted to degrees ± 1). For example, conversely to Theorem 2, if the degrees d_i are all equal to ± 1, then if u_ε are unstable, the a_i's are unstable too. Indeed, if they were stable, then one would be able from [Li] to construct stable solutions v_ε of (I.1) by local minimization, and these would have to be equal to the u_ε by the uniqueness result of [PR] (stating that there is a unique solution of (I.1) converging to prescribed vortices of degree ± 1). Our analysis does not use techniques of this type at all.

More generally, one can expect equality $n_\varepsilon^- = n^-$ when the degrees are ± 1, because then solutions of (I.1) are essentially minimizers with prescribed vortices (as first observed in [CM1, CM2]), and a strict inequality when some $|d_i| > 1$ due to the inner instability of vortices of higher degrees.

One of the interesting consequences of Theorem 2 is the following

Theorem 3 *Let u_ε be a family of nonconstant solutions of (I.1) with homogeneous Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, such that $E_\varepsilon(u_\varepsilon) \le C|\log\,\varepsilon|$, then for ε small enough, u_ε is unstable.*

This theorem says that there are no nonconstant stable solutions of Ginzburg-Landau with homogeneous Neumann boundary condition. In other words, there is no possibility of having stable vortex-configurations with this simple model in dimension 2. First, this is in contrast with the situation in dimension 3, because it has been proved by Montero, Sternberg and Ziemer in [MSZ] that as soon as the 3D domain satisfies some nonconvexity condition, there exist stable nontrivial solutions of the same equation, which have line vortices. In order to have stable vortices in 2 dimensions, one needs to add a confinement effect through some applied magnetic field, for example (see below).

Secondly, this result extends the same one by Jimbo and Morita [JM] which was valid in any dimension but which required the domain to be convex. Their result was then extended to the functional with magnetic field in 2 dimensions (see below) by Jimbo and Sternberg in [JiSt]. In both cases, they used a totally different method (a clever integration), not using the renormalized energy, and their result was in addition valid for all ε. Here, on the contrary, this result is an asymptotic one, valid only for small ε, and it uses the fact that the renormalized energy has no stable critical point, a fact that was observed by J. Rubinstein [Ru] for the case of Ginzburg-Landau with magnetic field. The questions of knowing whether Theorem 3 is true for all ε in simply connected domains, or for non-simply connected ones remain open.

I.3 The case of Ginzburg-Landau with magnetic field

Although we don't state here formal proofs, the results of the previous subsection could be extended with the same method to the case of the full Ginzburg-Landau energy with magnetic field:

$$J(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |h - h_{\text{ex}}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2. \tag{I.10}$$

This is the full two-dimensional model of superconductivity, introduced by Ginzburg and Landau. Here, u is coupled to a magnetic potential A, $\mathbb{R}^2$-valued vector field, with $\nabla_A = \nabla - iA$ the covariant derivative, and $h = \operatorname{curl} A$ the induced magnetic field. The parameter ε is related to a material constant, and can be assumed to be small. The parameter $h_{\rm ex}$ represents the intensity of an applied magnetic field. If $h_{\rm ex} = 0$ and the domain Ω is convex, Jimbo and Sternberg have proved in [JiSt] that all stable critical points of J are constant (for all ε). On the other hand, it is known that a nonzero applied field $h_{\rm ex}$ larger than some critical value H_{sc} of order 1 with respect to ε (the "subcooling field") stabilizes vortices (i.e. there exist stable solutions with vortices), for ε small enough. This has been proved in [DL] and in [S2]. Here for nonconvex domains and $h_{\rm ex} = 0$, one could obtain the analogue result of [JiSt] and of Theorem 3, that is there are no sequences (in $\varepsilon \to 0$) of nontrivial stable solutions. This would be done, as for Theorem 3, by examining the renormalized energy associated to J (for $h_{\rm ex} \leq O(1)$) which is

(I.11)

$$W_J(a_1, \cdots, a_n) = -\pi \sum_{i\neq j} d_i d_j \log|a_i - a_j| - \pi \sum_i d_i R(a_i) + \pi \sum_i d_i \xi(a_i) + \frac{h_{\rm ex}^2}{2}|\Omega| - \frac{h_{\rm ex}}{2} \int_{\partial\Omega} \frac{\partial \xi}{\partial n}$$

where R is defined as before, and

$$\text{(I.12)} \qquad \begin{cases} -\Delta^2 \xi + \Delta \xi = 2\pi \sum_i d_i \delta_{a_i} & \text{in } \Omega \\ \xi = 0 & \text{on } \partial\Omega \\ \Delta \xi = h_{\rm ex} & \text{on } \partial\Omega. \end{cases}$$

This renormalized energy has been written down by Rubinstein [Ru] and Du-Lin [DL] (see also [Sp] and [S2]). It was established in [Ru], that for $h_{\rm ex} = 0$, W_J has no stable critical points (for any $n \neq 0$). His argument in fact proves that W_J has no stable critical point for $h_{\rm ex}$ small, and has some for $h_{\rm ex}$ larger than a constant.

For $h_{\rm ex} \neq 0$, one expects to get the analogue of Theorem 2, that is that stable critical points of J converge to stable critical points of W_J. Also, characterizing the smallest $h_{\rm ex}$ such that W_J has nontrivial stable critical points would provide a lower bound for H_{sc}, which has to be an equality in view of the result of [DL].

On the other hand, in the regime $h_{\rm ex} \gg 1$, for a bounded number of vortices, we have established in [S1, SS1, SS2] that the renormalized (or Γ-limit) energy associated to J suitably rescaled, becomes simply ξ_0 where ξ_0 is the solution of

$$\text{(I.13)} \qquad \begin{cases} -\Delta \xi_0 + \xi_0 + 1 = 0 & \text{in } \Omega \\ \xi_0 = 0 & \text{on } \partial\Omega \end{cases}$$

We established in [SS2] that critical points converge to critical points of ξ_0. Applying the method presented here would yield that stable critical points converge to stable critical points of ξ_0.

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II Proof of Theorem 1

Proof: -first assertion : We have $\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = 0$ (because u_ε is a critical point), hence from (I.4) we must have $dF(u).V = 0$. But this is true for any $V \in \mathcal{B}'$, thus $dF(u) = 0$ and u is a critical point of F.
-second assertion: to every $V \in \mathcal{B}'$, we can associate a v_ε such that, according to (I.5),

$$\frac{d^2}{dt^2}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = Q_\varepsilon(u_\varepsilon)(\partial_t v_\varepsilon(0)) \to Q(u)(\partial_t v(0)).$$

If u_ε is a stable critical point of E_ε, we have $Q_\varepsilon(u_\varepsilon)(\partial_t v_\varepsilon(0)) \geq 0$, thus $Q(u)(V) \geq 0$. But this is true for all $V \in \mathcal{B}'$, hence $Q(u) \geq 0$ and u is a stable critical point of F.
-third assertion: we claim that, v_ε being associated to V and w_ε to W, we have

$$B_\varepsilon(u_\varepsilon)(\partial_t v_\varepsilon(0), \partial_t w_\varepsilon(0)) = B(u)(V, W) + o(1). \tag{II.1}$$

Let us prove this claim. From the hypotheses, we can associate s_ε to $V + W$ in such a way that

$$\partial_t s_\varepsilon(0) = \partial_t v_\varepsilon(0) + \partial_t w_\varepsilon(0) \tag{II.2}$$
$$\lim_{\varepsilon\to 0} \frac{d^2}{dt^2}_{|t=0} E_\varepsilon(s_\varepsilon(t)) = Q(u)(V + W) \tag{II.3}$$

For simplicity, let us write $V_\varepsilon = \partial_t v_\varepsilon(0)$, $W_\varepsilon = \partial_t w_\varepsilon(0)$, $S_\varepsilon = \partial_t s_\varepsilon(0)$. We have

$$Q_\varepsilon(u_\varepsilon)(V_\varepsilon + W_\varepsilon) = Q_\varepsilon(u_\varepsilon)(S_\varepsilon) = Q(u)(V + W) + o(1).$$

We can do the same reasoning with $V - W$ instead of $V + W$ and will find that

$$Q_\varepsilon(u_\varepsilon)(V_\varepsilon - W_\varepsilon) = Q(u)(V - W) + o(1).$$

Therefore by substracting these two relations, we are led to $B_\varepsilon(u_\varepsilon)(V_\varepsilon, W_\varepsilon) = B(u)(V, W) + o(1)$ which is (II.1).

Next, by definition of n^+, if n^+ is finite, we can find n^+ linearly independent vectors $V^1, \cdots, V^{n^+}$, such that the quadratic function $Q(u)$ restricted to the space they span is positive, i.e.

$$\min_{\sum_{i=1}^{n^+} x_i^2 = 1} Q(u)\left(\sum_{i=1}^{n^+} x_i V^i\right) > 0. \tag{II.4}$$

By the hypothesis, we can associate to them a family v_ε^i such that (I.3)—(I.5) are satisfied. Denoting $V_\varepsilon^i = \partial_t v_\varepsilon^i(0)$, since the V_ε^i depend linearly and in a one-to-one fashion on the V_i, they are also linearly independent. In view of (II.1), we have for all x_i,

$$\lim_{\varepsilon\to 0} Q_\varepsilon(u_\varepsilon)\left(\sum_{i=1}^{n^+} x_i V_\varepsilon^i\right) = Q(u)\left(\sum_{i=1}^{n^+} x_i V^i\right)$$

and the convergence is uniform with respect to (x_i) such that $\sum_i x_i^2 = 1$. Finally we deduce from (II.4) that for ε small enough

$$\text{(II.5)} \qquad \min_{\sum_{i=1}^{n^+} x_i^2 = 1} Q_\varepsilon(u_\varepsilon)\left(\sum_{i=1}^{n^+} x_i V_\varepsilon^i\right) > 0.$$

Since the V_ε^i span a space of dimension n^+, this proves that $D^2 E_\varepsilon(u_\varepsilon)$ has at least n^+ positive eigenvalues and thus that $n_\varepsilon^+ \geq n^+$. Observe that if n^+ is $+\infty$, we can apply the previous argument on subspaces of arbitrary large finite dimensions, and find that n_ε^+ is also $+\infty$ for ε small. The same arguments work for $n_\varepsilon^- \geq n^-$. $\square$

III Application to Ginzburg-Landau

For Ginzburg-Landau, we take

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2,$$

and F to be the renormalized energy W of [BBH] with Neumann or Dirichlet boundary conditions. $\mathcal{M}$ is $H_g^1(\Omega)$ (resp. $\{u \in H^1(\Omega), \partial_n u = 0 \text{ on } \partial\Omega\}$ for Neumann) where $H_g^1(\Omega)$ is the affine space of $H^1(\Omega)$ functions which are equal to g on $\partial\Omega$.

The sense of convergence S is $u_\varepsilon \rightharpoonup^S u = ((a_1, d_1), \cdots, (a_k, d_k))$ if $\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^k d_i \delta_{a_i}$ in the sense of distributions. $\mathcal{N}$ is then Ω^k minus its diagonals (the d_i's being considered fixed).

Proposition III.1 *Hypotheses (I.3) to (I.5) are verified for families of solutions of Ginzburg-Landau (with Dirichlet or Neumann boundary condition) such that $E_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$, with the above choices of E_ε and F.*

Proof :
- *Step 1: preliminaries.* Let u_ε be a family of critical points such that $E_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$. There exists a constant λ and a bounded number of balls $B(a_i^\varepsilon, \lambda\varepsilon)$ such that $|u_\varepsilon| \geq \frac{1}{2}$ in $\Omega \setminus \cup_i B(a_i^\varepsilon, \lambda\varepsilon)$ for ε small enough. In the case of the Dirichlet boundary condition and a starshaped domain, this is proved in [BBH] (it follows from the Pohozaev identity). In the case of the Neumann boundary condition, it can be proved following exactly the method of [BR], or by a ball-construction method for solutions of Ginzburg-Landau that will be exposed in [SS5], which proves that $\frac{1}{\varepsilon^2}\int_{|u|\leq\frac{1}{2}}(1-|u|^2)^2 \leq C$.

Extracting a subsequence if necessary, we can assume that a_i^ε converges to a_i and that

$$\text{(III.1)} \qquad \operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_{j=1}^k d_j \delta_{a_j},$$

in the sense of distributions, where the d_i's are integers and the a_i's are distinct points (this can be proved directly, or one can apply the results of [SS1, JeSo, SS3]). We can consider $\rho > 0$ small enough so that the balls $B_j = B(a_j, \rho)$ are disjoint and included in Ω. Now the main theorem in Chap. X of [BBH] asserts that

$$\frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \rightharpoonup \frac{\pi}{2}\sum_{j=1}^k d_j^2 \delta_{a_j}$$

in the sense of measures, for the Dirichlet boundary condition, and this result remains valid also for Neumann boundary condition. If one of the d_i's is 0, then

$$\text{(III.2)} \qquad \int_{B(a_i,\rho)} \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \to 0.$$

It is standard from [BBH] that this implies that $|u_\varepsilon| \geq \frac{1}{2}$ in $B(a_i, \rho)$ for ε small enough. Indeed, solutions of (I.1) satisfy the estimate $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}$, and thus if there is a point in $B(a_i, \rho)$ where $|u_\varepsilon(x_0)| < \frac{1}{2}$, this estimate ensures that $|u_\varepsilon| < \frac{3}{4}$ in a ball of size $\lambda\varepsilon$ centered at x_0, which finally yields a contradiction with (III.2). Therefore, $|u_\varepsilon| \geq \frac{1}{2}$ in $B(a_i, \rho)$, and the bad disc $B(a_i^\varepsilon, \lambda\varepsilon)$ is not really a bad disc and can be removed from the original list. Finally, we can thus assume that the d_i's are all nonzero, still with $|u_\varepsilon| \geq \frac{1}{2}$ in $\Omega \backslash \cup_j B(a_j, \rho)$ for ε small enough.

We will need the following additional results.

Lemma III.1 *If u_ε is a solution of (I.1) with either Dirichlet or Neumann boundary condition, and $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$, then Φ_0 being defined in (I.6), we have*

$$\text{(III.3)} \qquad \int_{\Omega\backslash\cup_i B_i} |\nabla u_\varepsilon - iu_\varepsilon \nabla^\perp \Phi_0|^2 \to 0 \quad \textit{as } \varepsilon \to 0.$$

$$\text{(III.4)} \qquad \|(iu_\varepsilon, \nabla u_\varepsilon) - \nabla^\perp \Phi_0\|_{L^p(\Omega)} \to 0 \quad \forall p < 2.$$

$$\text{(III.5)} \qquad \int_{\Omega\backslash\cup_i B_i} |\nabla |u_\varepsilon||^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \to 0 \quad \textit{as } \varepsilon \to 0.$$

We postpone the proof until the end of the section.

- *Step 2: construction of v_ε.* We now follow the construction of [SS4]. Given a family of vectors $V = (V_1, \cdots, V_k)$, we can find a C^1 one-parameter family of diffeomorphisms of Ω, $\chi_t(x) = x + tX(x)$ defined in a small interval around 0 such that $X(x)$ is C^1, $X = 0$ on $\partial\Omega$ and

$$\text{(III.6)} \qquad X(x) = V_i \quad \text{in each } B_i.$$

In other words, χ_t achieves a translation of vector V_i of each ball. χ_t is thus a C^1 family of diffeomorphisms of Ω, independent of ε, which keep $\partial\Omega$ fixed. By taking a basis of $(\mathbb{R}^2)^k$

and associating this way an $X_i(x)$ to each of its vectors, it is possible to obtain by linear combination an $X(x)$ which depends linearly on V.

Next, we let $a_i(t)$ denote $\chi_t(a_i)$. We define Φ_t by

$$\text{(III.7)} \qquad \begin{cases} \Delta\Phi_t = 2\pi \sum_i d_i \delta_{a_i(t)} & \text{in } \Omega \\ \dfrac{\partial \Phi_t}{\partial \tau} = (ig, \dfrac{\partial g}{\partial n}) & \text{on } \partial\Omega \quad (\text{resp.} \quad \Phi_t = 0 \text{ on } \partial\Omega), \end{cases}$$

then R_t by

$$\text{(III.8)} \qquad R_t(x) = \Phi_t(x) - \sum_j d_j \log|x - a_j(t)|.$$

R_t is a smooth harmonic function in Ω, and we recall that the renormalized energy W associated to $a_1(t), \cdots, a_k(t)$ is defined by

(III.9)

$$W(a_1(t), \cdots, a_k(t)) = -\pi \sum_{i \neq j} d_i d_j \log|a_i(t) - a_j(t)| + \frac{1}{2} \int_{\partial\Omega} \Phi_t(ig, \partial_\tau g) - \pi \sum_{i=1}^k d_i R_t(a_i(t)).$$

We can also consider $\tilde{R}_t$, the conjugate harmonic function of R_t. We then denote by θ_t^j the polar coordinate centered at $a_j(t)$ (defined modulo 2π), and define

$$\text{(III.10)} \qquad \psi_t = \sum_{j=1}^k d_j \theta_t^j \circ \chi_t - \sum_{j=1}^k d_j \theta_0^j + \tilde{R}_t \circ \chi_t - \tilde{R}_0.$$

One can check that $\psi_0 = 0$, ψ_t is a smooth function in Ω, the singularities at $a_i(0)$ in fact cancelling out, and that it is smooth in space-time. Also, we have

$$\text{(III.11)} \qquad \nabla^\perp \Phi_0 + \nabla \psi_t = \nabla \left(\sum_j d_j \theta_t^j \circ \chi_t + \tilde{R}_t \circ \chi_t \right)$$

We finally define $v_\varepsilon(x, t)$ as follows:

$$\text{(III.12)} \qquad v_\varepsilon(\chi_t(x), t) = u_\varepsilon(x) e^{i\psi_t} \quad \text{i.e. } v_\varepsilon(x, t) = u_\varepsilon(x - tX(x)) e^{i\psi_t(x - tX(x))}.$$

- *Step 3:* Let us check that v_ε satisfies the desired properties. First, $\psi_t = 0$ on $\partial\Omega$ (resp $\frac{\partial \psi_t}{\partial n} = 0$ on $\partial\Omega$) thus v_ε satisfies the right boundary conditions. In addition, v_ε is C^1 in time and clearly $v_\varepsilon(0) = u_\varepsilon$. Secondly,

$$\text{(III.13)} \qquad \partial_t v_\varepsilon(x, 0) = -\nabla u_\varepsilon(x) \cdot X(x) + i u_\varepsilon(x) \frac{d}{dt}_{|t=0} \psi_t(x),$$

thus, since X depends linearly on V, in order to check that $\partial_t v_\varepsilon(0)$ depends linearly on V, there remains to check that $(\partial_t \psi_t)_{|t=0}$ does. First, differentiating (III.7) with respect to

t, we find that $\Delta\partial_t\Phi_t = 2\pi\sum_i d_i \mathrm{div}\,(V_i\delta_{a_i(t)})$. Thus, $(\Delta\partial_t\Phi_t)_{|t=0}$ depends linearly on V, hence $(\partial_t\Phi_t)_{|t=0}$ too in view of its boundary condition. But, using (III.11) with (III.8), one may check that $(\nabla\partial_t\psi_t)_{|t=0} = (\nabla^\perp\partial_t\Phi_t)_{|t=0}$ hence $(\nabla\partial_t\psi_t)_{|t=0}$ depends linearly on V and finally $(\partial_t\psi_t)_{|t=0}$ too. Let us check that the dependence is one-to-one. If V is such that $\partial_t v_\varepsilon(0) = 0$, then in view of (III.13), we must have

$$\nabla u_\varepsilon \cdot X = i u_\varepsilon \frac{d}{dt}_{|t=0}\psi_t.$$

Thus, since $|u_\varepsilon| \le 1$, we must have $|\nabla u_\varepsilon \cdot X| \le C$ where C is independent of ε. But a result of [SS3] (see Corollaries 1 and 2) yields that

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|}\left(\int_\Omega |\nabla u_\varepsilon \cdot X|^2 \int_\Omega |\nabla u_\varepsilon \cdot X^\perp|^2\right)^{\frac12} \ge \sum_{i=1}^k |d_i||X(a_i)|^2.$$

Hence, we deduce that we have

$$C\|X\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 \ge C\int_\Omega |\nabla u_\varepsilon \cdot X^\perp|^2 \ge \left(\sum_{i=1}^k |d_i||X(a_i)|^2\right)^2 |\log\varepsilon|^2(1+o(1)),$$

a contradiction with the hypothesis $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$ unless $\sum_i |d_i||X(a_i)|^2 = 0$. Since we saw that the d_i's are all nonzero, this implies that for every i, $X(a_i) = 0$, but recall that $X(a_i) = V_i$. Therefore, $V = 0$, and the mapping is one-to-one.

- *Step 4:* We evaluate $\frac{d}{dt}E_\varepsilon(v_\varepsilon(t))$. In view of the definition of v_ε, with a change of variables $y = \chi_t(x)$, we have

$$\begin{aligned}
E_\varepsilon(v_\varepsilon) &= \frac12\int_\Omega \left(|(\nabla v_\varepsilon)\circ\chi_t(x)|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon(\chi_t(x))|^2)^2\right)|Jac\ \chi_t|(x)\,dx\\
&= \frac12\int_\Omega \left(|D\chi_t^{-1}\nabla(v_\varepsilon\circ\chi_t)|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)|Jac\ \chi_t|\\
&= \frac12\int_\Omega \left(|D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)|Jac\ \chi_t|. \qquad \text{(III.14)}
\end{aligned}$$

First, observing that $\frac{d}{dt}|Jac\ \chi_t|$ and $\frac{d^2}{dt^2}|Jac\ \chi_t|$ are 0 in $\cup_i B_i$, and bounded otherwise, we have, in view of (III.5),

$$\text{(III.15)}\quad \frac{d}{dt}\int_\Omega \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2|Jac\ \chi_t| \le C\frac{1}{2\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2 \le o(1) \quad \text{as } \varepsilon\to 0$$

$$\text{(III.16)}\quad \frac{d^2}{dt^2}\int_\Omega \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2|Jac\ \chi_t| \le C\frac{1}{2\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2 \le o(1) \quad \text{as } \varepsilon\to 0.$$

Secondly, we have

$$\text{(III.17)}\quad \frac{d}{dt}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t| = \frac{d}{dt}\int_\Omega |D\chi_t^{-1}(\nabla u_\varepsilon + iu_\varepsilon \nabla\psi_t)|^2 |Jac\ \chi_t|$$
$$= \int_\Omega 2(\frac{d}{dt}D\chi_t^{-1}\,(\nabla u_\varepsilon + iu_\varepsilon\nabla\psi_t) + D\chi_t^{-1}(iu_\varepsilon \frac{d}{dt}\nabla\psi_t)) \cdot \big(D\chi_t^{-1}(\nabla u_\varepsilon + iu_\varepsilon\nabla\psi_t)\big)\,|Jac\ \chi_t|$$
$$+ |D\chi_t^{-1}(\nabla u_\varepsilon + iu_\varepsilon\nabla\psi_t)|^2 \frac{d}{dt}|Jac\ \chi_t|$$

We can conclude with exactly the calculation of [SS4] that $\frac{d}{dt}_{|t=0}E_\varepsilon(v_\varepsilon) = \frac{d}{dt}_{|t=0}F(v)$. Let us rewrite it here for the sake of completeness. We have

$$\text{(III.18)}\quad \frac{d}{dt}_{|t=0}\frac{1}{2}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t|$$
$$= \int_\Omega (\frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla u_\varepsilon)\cdot\nabla u_\varepsilon + iu_\varepsilon \frac{d}{dt}_{|t=0}\nabla\psi_t\cdot\nabla u_\varepsilon + |\nabla u_\varepsilon|^2 \frac{d}{dt}_{|t=0}|Jac\ \chi_t|$$

But, from (III.4) and the fact that ψ_t is smooth and C^1 in time, we deduce that

$$\text{(III.19)}\qquad \int_\Omega \frac{d}{dt}_{|t=0}\nabla\psi_t\cdot(iu_\varepsilon, \nabla u_\varepsilon) = \int_\Omega \frac{d}{dt}_{|t=0}\nabla\psi_t\cdot\nabla^\perp\Phi_0 + o(1).$$

Meanwhile, observing that $\frac{d}{dt}_{|t=0}D\chi_t^{-1} = \frac{d}{dt}_{|t=0}|Jac\ \chi_t| = 0$ in $\cup_i B_i$, using (III.3), we have

$$\text{(III.20)}\quad \int_\Omega \frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla u_\varepsilon\cdot\nabla u_\varepsilon + \frac{1}{2}|\nabla u_\varepsilon|^2\frac{d}{dt}_{|t=0}|Jac\ \chi_t|$$
$$= \int_\Omega \frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla^\perp\Phi_0\cdot\nabla^\perp\Phi_0 + \frac{1}{2}|\nabla^\perp\Phi_0|^2\frac{d}{dt}_{|t=0}|Jac\ \chi_t| + o(1).$$

Inserting (III.19) and (III.20) into (III.18), we have

$$\text{(III.21)}\quad \frac{d}{dt}_{|t=0}\frac{1}{2}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t|$$
$$= \int_\Omega \frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla^\perp\Phi_0\cdot\nabla^\perp\Phi_0 + \frac{d}{dt}_{|t=0}\nabla\psi_t\cdot\nabla^\perp\Phi_0 + \frac{1}{2}|\nabla^\perp\Phi_0|^2\frac{d}{dt}_{|t=0}|Jac\ \chi_t| + o(1).$$

Using again the fact that $\frac{d}{dt}_{|t=0}D\chi_t^{-1} = \frac{d}{dt}_{|t=0}|Jac\ \chi_t| = 0$ in $\cup_i B_i$, we deduce that, for any $0 < r < \rho$,

$$\text{(III.22)}\quad \frac{d}{dt}_{|t=0}\frac{1}{2}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t|$$
$$= \frac{d}{dt}_{|t=0}\frac{1}{2}\int_{\Omega\setminus\cup_j B(a_j,r)} |D\chi_t^{-1}(\nabla^\perp\Phi_0 + \nabla\psi_t)|^2 |Jac\ \chi_t| + o(1) + o_r(1)$$
$$= \lim_{r\to 0}\frac{d}{dt}_{|t=0}\frac{1}{2}\int_{\Omega\setminus\cup_j B(a_j,r)} |D\chi_t^{-1}(\nabla^\perp\Phi_0 + \nabla\psi_t)|^2 |Jac\ \chi_t| + o(1).$$

Now observe that in view of (III.8),

$$\nabla^\perp \Phi_t = \nabla^\perp R_t + \nabla^\perp \sum_j d_i \log |x - a_j(t)| = \nabla \tilde{R}_t + \sum_j d_j \nabla \theta_t^j, \tag{III.23}$$

and hence in view of (III.10)

$$\nabla^\perp \Phi_0 + \nabla \psi_t = \nabla \sum_j d_j \theta_t^j \circ \chi_t + \tilde{R}_t \circ \chi_t. \tag{III.24}$$

Inserting this into (III.22) and doing a change of variables, we are led to

$$\begin{aligned} &\frac{d}{dt}_{|t=0} \frac{1}{2} \int_\Omega |D\chi_t^{-1} \nabla (u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t| \\ &= \lim_{r\to 0} \frac{d}{dt}_{|t=0} \frac{1}{2} \int_{\Omega \setminus \cup_j B(a_j(t),r)} |\nabla (\sum_j d_j \theta_t^j + \tilde{R}_t)|^2 + o(1) = \lim_{r\to 0} \frac{d}{dt}_{|t=0} \frac{1}{2} \int_{\Omega \setminus \cup_j B(a_j(t),r)} |\nabla^\perp \Phi_t|^2 + o(1), \end{aligned} \tag{III.25}$$

where we have used (III.23). We then introduce $S_t^j(x) = \Phi_t(x) - d_j \log |x - a_j(t)|$, smooth harmonic function in a neighborhood of a_j, also C^1 in time. As in [BBH], p. 22, we have $S_t^j(a_j(t)) = R_t(a_j(t)) + \sum_{k\neq j} d_k \log |a_j(t) - a_k(t)|$ and

(III.26)

$$\begin{aligned} \frac{d}{dt}_{|t=0} \int_{\Omega \setminus \cup_j B(a_j(t),r)} |\nabla \Phi_t|^2 &= \frac{d}{dt}_{|t=0} \left(\sum_j \int_{B(a_j(t),r)} |\nabla S_t^j|^2 + 2\pi d_j S_t^j(a_j(t)) + 2\pi d_j^2 \log r \right) \\ &= \frac{d}{dt}_{|t=0} \left(\sum_j \int_{B(a_j(t),r)} |\nabla S_t^j|^2 + 2\pi \sum_j d_j R_t(a_j(t)) + 2 \sum_{j\neq l} d_j d_l \log |a_j(t) - a_l(t)| \right) \\ &= \frac{d}{dt}_{|t=0} \sum_j \int_{B(a_j(t),r)} |\nabla S_t^j|^2 + 2 \frac{d}{dt}_{|t=0} W(a_1(t), \cdots, a_k(t)). \end{aligned}$$

But, $\lim_{r\to 0} \frac{d}{dt}_{|t=0} \sum_j \int_{B(a_j(t),r)} |\nabla S_t^j|^2 = 0$, because the S_j are smooth functions, C^1 in time, thus taking the limit $r \to 0$ in (III.26) and combining it with (III.25), we find

$$\frac{d}{dt}_{|t=0} \frac{1}{2} \int_\Omega |D\chi_t^{-1} \nabla (u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t| = \frac{d}{dt}_{|t=0} W(a_1(t), \cdots, a_k(t)) + o(1). \tag{III.27}$$

Combining this with (III.14) and (III.16), we conclude that

$$\lim_{\varepsilon\to 0} \frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(x,t)) = \frac{d}{dt}_{|t=0} W(a_1(t), \cdots, a_k(t)) = dW(a_i).V,$$

hence the desired condition (I.4). This implies that, when u_ε is a solution of (I.1), then $dW(a_i) = 0$.

- *Step 5:* We evaluate $\frac{d^2}{dt^2}_{|t=0} E_\varepsilon(v_\varepsilon)$. We differentiate a second time (III.17) and, observing that $\psi_0 = 0$, $|Jac\ \chi_0| = 1$ and $D\chi_0 = I$, we find

$$
\text{(III.28)} \quad \frac{d^2}{dt^2}_{|t=0} \frac{1}{2}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t|
$$
$$
= \int_\Omega \left(\frac{d^2}{dt^2}_{|t=0} D\chi_t^{-1}\nabla u_\varepsilon + 2\frac{d}{dt}_{|t=0} D\chi_t^{-1}(iu_\varepsilon \frac{d}{dt}_{|t=0}\nabla\psi_t) + (iu_\varepsilon \frac{d^2}{dt^2}_{|t=0}\nabla\psi_t)\right)\cdot\nabla u_\varepsilon
$$
$$
+\left|\frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla u_\varepsilon + iu_\varepsilon\frac{d}{dt}_{|t=0}\nabla\psi_t\right|^2 + 2\Big(\frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla u_\varepsilon + (iu_\varepsilon\frac{d}{dt}_{|t=0}\nabla\psi_t)\Big)\cdot\nabla u_\varepsilon \frac{d}{dt}_{|t=0}|Jac\,\chi_t|
$$
$$
+\frac{1}{2}|\nabla u_\varepsilon|^2\frac{d^2}{dt^2}_{|t=0}|Jac\ \chi_t|.
$$

Using (III.3), (III.4), (III.5), the fact that ψ_t is smooth and C^1 in time, and observing that $\frac{d}{dt}_{|t=0} D\chi_t^{-1} = \frac{d}{dt}_{|t=0}|Jac\ \chi_t| = \frac{d^2}{dt^2}_{|t=0} D\chi_t^{-1} = \frac{d^2}{dt^2}_{|t=0}|Jac\ \chi_t| = 0$ in $\cup_i B_i$, we see that we can replace, like for the first derivative, ∇u_ε by $\nabla^\perp\Phi_0$ with only a small error and eventually be led to

(III.29)
$$
\frac{d^2}{dt^2}_{|t=0}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2|Jac\,\chi_t| = \lim_{r\to 0}\frac{d^2}{dt^2}_{|t=0}\int_{\Omega\setminus\cup_i B(a_i,r)} |D\chi_t^{-1}(\nabla^\perp\Phi_0+\nabla\psi_t)|^2|Jac\,\chi_t| + o(1).
$$

Inserting (III.24) into (III.29) and doing a change of variables yields

(III.30)
$$
\frac{d^2}{dt^2}_{|t=0}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2|Jac\ \chi_t| = \lim_{r\to 0}\frac{d^2}{dt^2}_{|t=0}\int_{\Omega\setminus\cup_i B(a_j(t),r)} |\nabla(\sum_j d_j\theta_t^j + \tilde{R}_t)|^2 + o(1)
$$
$$
= \lim_{r\to 0}\frac{d^2}{dt^2}_{|t=0}\int_{\Omega\setminus\cup_i B(a_j(t),r)} |\nabla^\perp\Phi_t|^2 + o(1)
$$

.

where we have used (III.23). As before,

(III.31)
$$
\frac{d^2}{dt^2}_{|t=0}\int_{\Omega\setminus\cup_j B(a_j(t),r)} |\nabla\Phi_t|^2 = \frac{d^2}{dt^2}_{|t=0}\left(\sum_j\int_{B(a_j(t),r)} |\nabla S_t^j|^2 + 2\pi d_j S_t^j(a_j(t)) + 2\pi d_j^2\log r\right)
$$
$$
= \frac{d^2}{dt^2}_{|t=0}\left(\sum_j\int_{B(a_j(t),r)} |\nabla S_t^j|^2 + 2\pi\sum_j d_j R_t(a_j(t)) + 2\sum_{j\neq k} d_j d_k \log|a_j(t)-a_k(t)|\right)
$$
$$
= \frac{d^2}{dt^2}_{|t=0}\sum_j\int_{B(a_j(t),r)} |\nabla S_t^j|^2 + 2\frac{d^2}{dt^2}_{|t=0} W(a_1(t),\cdots,a_k(t)).
$$

But, $\lim_{r\to 0} \frac{d}{dt}_{|t=0} \sum_j \int_{B(a_j(t),r)} |\nabla S_t^j|^2 = 0$, because the S_j are smooth functions, C^1 in time, thus taking the limit $r \to 0$ in (III.31) and combining it with (III.30), we find

$$\frac{d^2}{dt^2}_{|t=0} \frac{1}{2} \int_\Omega |D\chi_t^{-1} \nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t| = \frac{d^2}{dt^2}_{|t=0} W(a_1(t), \cdots, a_k(t)). \tag{III.32}$$

Combining this with (III.14) and (III.16), we conclude that

$$\lim_{\varepsilon\to 0} \frac{d^2}{dt^2}_{|t=0} E_\varepsilon(v_\varepsilon(x,t)) = \frac{d^2}{dt^2}_{|t=0} W(a_1(t), \cdots, a_k(t)),$$

hence the desired result (I.5). This completes the proof. □

Combining Proposition III.1 with Theorem 1, we deduce Theorem 2.

Proof of Lemma III.1:
In the Dirichlet case, this follows directly from the result of [BBH], Theorem X.3, where $C^{1,\alpha}(K)$ convergence of u_ε to the "canonical harmonic map" is proved, for every K compact subset of $\Omega \setminus \cup_i \{a_i\}$. We give here a slightly different proof, valid for both Dirichlet and Neumann cases, but that borrows ingredients from [BBH] and other works.
- *Step 1 :* Taking the scalar product of (I.1) with iu_ε we find

$$0 = \Big(iu_\varepsilon, \Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2) \Big) = (iu_\varepsilon, \Delta u_\varepsilon)$$

but we have the identity

$$(iu_\varepsilon, \Delta u_\varepsilon) = \text{div } (iu_\varepsilon, \nabla u_\varepsilon) \tag{III.33}$$

hence we deduce div $(iu_\varepsilon, \nabla u_\varepsilon) = 0$ and since Ω is simply connected, we may write

$$(iu_\varepsilon, \nabla u_\varepsilon) = \nabla^\perp U_\varepsilon \tag{III.34}$$

with the boundary conditions

$$\frac{\partial U_\varepsilon}{\partial n} = (ig, \frac{\partial g}{\partial \tau}) \quad \text{on } \partial\Omega$$

in the case of the Dirichlet boundary condition, and

$$U_\varepsilon = 0 \quad \text{on } \partial\Omega$$

in the case of the Neumann boundary condition. We deduce from (III.34) that $|\nabla U_\varepsilon| \leq |\nabla u_\varepsilon|$ hence $\int_\Omega |\nabla U_\varepsilon|^2 \leq C|\log \varepsilon|$. From the analysis of [SS1], we also have

$$\text{curl}\, (iu_\varepsilon, \nabla u_\varepsilon) \longrightarrow 2\pi \sum_i d_i \delta_{a_i} = \text{curl}\, \nabla^\perp \Phi_0 \quad \text{in } W^{-1,p}(\Omega)\ \forall p < 2. \tag{III.35}$$

Since we also have div $(iu_\varepsilon, \nabla u_\varepsilon) = 0$, we deduce that

$$(iu_\varepsilon, \nabla u_\varepsilon) \to \nabla^\perp \Phi_0 \ + cst \quad \text{strongly in } L^p(\Omega). \tag{III.36}$$

Examining the boundary conditions, we deduce that the constant is 0 and that (III.4) is proved.

- *Step 2:* Let η be a smooth cut-off function, equal to 0 in $\cup_i B(a_i, \frac{\rho}{2})$ and to 1 in $\Omega \backslash \cup_i B(a_i, \rho)$. For ε small enough, we have $|u_\varepsilon| \geq \frac{1}{2}$ wherever $\eta > 0$, and in view of (III.34), we have

$$\text{div } \left(\frac{\nabla U_\varepsilon}{|u_\varepsilon|^2} \right) = 0 \text{ in } \Omega \backslash \cup_i B(a_i, \frac{\rho}{2}). \tag{III.37}$$

Let us then consider

$$\begin{aligned}
\int_\Omega \eta \frac{|\nabla(U_\varepsilon - \Phi_0)|^2}{|u_\varepsilon|^2} &= \int_\Omega \eta \nabla(U_\varepsilon - \Phi_0) \cdot \nabla \Phi_0 \left(1 - \frac{1}{|u_\varepsilon|^2}\right) + \int_\Omega \eta \nabla(U_\varepsilon - \Phi_0) \cdot \left(\frac{\nabla U_\varepsilon}{|u_\varepsilon|^2} - \nabla \Phi_0 \right) \\
&= \int_\Omega \eta \nabla(U_\varepsilon - \Phi_0) \cdot \nabla \Phi_0 \left(1 - \frac{1}{|u_\varepsilon|^2}\right) + \int_\Omega (U_\varepsilon - \Phi_0) \nabla \eta \cdot \nabla U_\varepsilon \left(1 - \frac{1}{|u_\varepsilon|^2}\right) \\
&\quad - \int_\Omega (U_\varepsilon - \Phi_0) \nabla \eta \cdot (\nabla U_\varepsilon - \nabla \Phi_0) - \int_\Omega \eta (U_\varepsilon - \Phi_0) \text{div } \left(\frac{\nabla U_\varepsilon}{|u_\varepsilon|^2} \right).
\end{aligned} \tag{III.38}$$

The first two terms on the right-hand side tend to zero thanks to the a priori bounds $\int_\Omega (1 - |u_\varepsilon|^2)^2 \leq C\varepsilon^2 |\log \varepsilon|$ and $\int_\Omega |\nabla U_\varepsilon|^2 \leq C |\log \varepsilon|$ combined with the fact that $|u_\varepsilon| \geq \frac{1}{2}$ on the support of η and $\nabla \eta$. The third term goes to zero from the strong L^p convergence of ∇U_ε to $\nabla \Phi_0$ (III.36). From (III.37), the last term vanishes. Finally we deduce that

$$\int_{\Omega \backslash \cup_i B(a_i, \rho)} |\nabla (U_\varepsilon - \Phi_0)|^2 \to 0 \text{ as } \varepsilon \to 0. \tag{III.39}$$

- *Step 3:* For the convergence of the modulus, we proceed as in [BBH]. Let us write locally in polar coordinates $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ outside of the zeroes of u_ε. Taking the scalar product of the equation (I.1) with u_ε, one finds

$$-\Delta \rho_\varepsilon + \rho_\varepsilon |\nabla \varphi_\varepsilon|^2 = \frac{1}{\varepsilon^2} \rho_\varepsilon (1 - \rho_\varepsilon^2), \tag{III.40}$$

but since $\nabla^\perp U_\varepsilon = (iu_\varepsilon, \nabla u_\varepsilon) = \rho_\varepsilon^2 \nabla \varphi_\varepsilon$, this becomes

$$-\Delta \rho_\varepsilon + \frac{|\nabla U_\varepsilon|^2}{\rho_\varepsilon^3} = \frac{1}{\varepsilon^2} \rho_\varepsilon (1 - \rho_\varepsilon^2) \tag{III.41}$$

in $\Omega \backslash \cup_i B(a_i, \frac{\rho}{2})$. Multiplying this equation by $\eta(1 - \rho_\varepsilon)$ and integrating, we are led to

(III.42)
$$\int_\Omega \eta |\nabla \rho_\varepsilon|^2 - \int_\Omega \eta \frac{(1 - \rho_\varepsilon)}{\rho_\varepsilon^3} |\nabla U_\varepsilon|^2 - \int_\Omega (1 - \rho_\varepsilon) \nabla \eta \cdot \nabla \rho_\varepsilon + \frac{1}{\varepsilon^2} \int_\Omega \eta \rho_\varepsilon (1 - \rho_\varepsilon)^2 (1 + \rho_\varepsilon) = 0.$$

But on the one hand,

$$\int_\Omega \eta \frac{(1-\rho_\varepsilon)}{\rho_\varepsilon^3} |\nabla U_\varepsilon|^2 = \int_\Omega \eta \frac{(1-\rho_\varepsilon)}{\rho_\varepsilon^3} |\nabla \Phi_0|^2 + \int_\Omega \eta \frac{(1-\rho_\varepsilon)}{\rho_\varepsilon^3} (|\nabla \Phi_0|^2 - |\nabla U_\varepsilon|^2),$$

and by Lebesgue's dominated convergence theorem, since $\rho_\varepsilon \to 1$ a.e., the first term tends to zero, while the second tends to zero by the strong L^2 convergence of ∇U_ε outside of the points a_i (see (III.36). The third term in the left-hand side of (III.42) tends to zero by the a priori estimates on ρ_ε, and finally we get that

$$\int_\Omega \eta |\nabla \rho_\varepsilon|^2 + \frac{1}{\varepsilon^2} \int_\Omega \eta \rho_\varepsilon (1-\rho_\varepsilon)^2 (1+\rho_\varepsilon) \to 0.$$

Using the fact that $\rho_\varepsilon(1-\rho_\varepsilon)^2(1+\rho_\varepsilon) \geq C(1-\rho_\varepsilon^2)^2$ where $\rho_\varepsilon \geq \frac{1}{2}$, we deduce that (III.5) holds. Finally (III.3) follows easily from this and (III.39). Indeed, to have (III.3) it suffices to prove that $\int_{\Omega \setminus \cup_i B(a_i,\rho)} |\rho_\varepsilon(\nabla \varphi_\varepsilon - \nabla^\perp \Phi_0)|^2 \to 0$, but this is smaller than $C \int_{\Omega \setminus \cup_i B(a_i,\rho)} |\rho_\varepsilon^2 \nabla \varphi_\varepsilon - \nabla^\perp \Phi_0 + (1-\rho_\varepsilon^2) \nabla^\perp \Phi_0|^2$ which tends to zero by (III.39) and Lebesgue's dominated convergence theorem. This completes the proof of the lemma. □

IV Proof of Theorem 3

Suppose, by contradiction, that a sequence (u_ε) of stable critical points exists, and has, up to extraction, k limiting vortices of nonzero degrees. Let $V \in \mathbb{R}^2$ be arbitrary, and as in the proof of Proposition III.1, define $a_1(t)$ for t small by the formula $a_1(t) = a_1 + tV$. Then in light of Proposition III.1 or the results (I.4) and (I.5) of Theorem 1, the criticality and stability of the sequence (u_ε) implies that

$$\text{(IV.1)} \qquad \frac{d}{dt}_{|t=0} W(a_1(t), a_2, \ldots, a_k) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}_{|t=0} W(a_1(t), a_2, \ldots, a_k) \geq 0.$$

Moreover, since we are allowing only the first vortex to vary with t, we find that the condition on the second derivative of W takes the form:

$$\text{(IV.2)} \qquad \sum_{i,j=1,2} \frac{\partial^2 W_1}{\partial x_i x_j}(x)_{|x=a_1} V^i V^j \geq 0,$$

where $V = (V^1, V^2)$ and

$$\text{(IV.3)} \qquad W_1(x) \equiv -\pi \sum_{j=2}^{k} d_1 d_j \log|x - a_j| - \pi d_1^2 R_1(x,x).$$

Here, recalling that we are in the Neumann case, $R_1(x,y) = \Phi(x,y) - \log|x-y|$ and Φ is the Green's function with singularity at y defined by

$$\Delta_x \Phi(x,y) = 2\pi \delta_y \text{ for } x \in \Omega, \ \Phi(x,y) = 0 \text{ for } x \in \partial\Omega.$$

Since $V \in \mathbb{R}^2$ is arbitrary, this says that the Hessian of W_1 is non-negative definite.

Now the function $R_1(x,x)$ has been well-studied and can in fact be written down explicitly in terms of a conformal map g from Ω to the unit disc (cf. [Ri], p. 323):

$$R_1(x,x) = \log|g'(x)| - \log(1-|g(x)|^2). \tag{IV.4}$$

A direct computation yields that $\Delta_x R_1(x,x) = 4e^{2R_1(x,x)}$. Hence,

$$\Delta W_1(a_1) = -4\pi e^{2R_1(a_1,a_1)} < 0,$$

contradicting (IV.2). Consequently, we deduce that if there is a sequence u_ε of stable solutions, necessarily their number of limiting vortices is 0, and $|u_\varepsilon| \geq \frac{1}{2}$ for ε small enough. We claim that this implies that u_ε are constant solutions. Indeed, since $|u_\varepsilon| \geq \frac{1}{2}$, u_ε can be written globally as $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ with ρ_ε and φ_ε real-valued functions, and $\rho_\varepsilon \geq \frac{1}{2}$. As seen in (III.33) and (III.34), we deduce from (I.1) that

$$\operatorname{div}\,(\rho_\varepsilon^2 \nabla\varphi_\varepsilon) = 0$$

with $\frac{\partial\varphi_\varepsilon}{\partial n} = 0$ and $\frac{\partial\rho_\varepsilon}{\partial n} = 0$ on $\partial\Omega$ from the Neumann boundary condition. Multiplying this relation by φ_ε and integrating by parts, we find $\int_\Omega \rho_\varepsilon^2|\nabla\varphi_\varepsilon|^2 = 0$ and thus φ_ε is a constant. On the other hand (III.40) holds and thus

$$-\Delta\rho_\varepsilon = \frac{1}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon^2)$$

Multiplying this equation by $(1-\rho_\varepsilon)$ and integrating, we are led, as in (III.42), to

$$\int_\Omega |\nabla\rho_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_\Omega \rho_\varepsilon(1-\rho_\varepsilon)^2(1+\rho_\varepsilon) = 0,$$

therefore, since $0 < \rho_\varepsilon \leq 1$, we have $\int_\Omega |\nabla\rho_\varepsilon|^2 = 0$, thus ρ_ε is also constant and u_ε is a constant. Finally, we conclude that if $\{u_\varepsilon\}$ is a family of stable critical points of (I.2), then necessarily u_ε is a constant for ε small enough, which completes the proof.

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